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by

M.B. ZAMAN

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Absolute Summability by Series-to-Sequence Transformation Matrices

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M.B. ZAMAN

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SUMMARY

In this paper we define an absolute summability by a series–to–sequence transformation matrix. We obtain the necessary and sufficient conditions in order that every absolutely convergent series is absolutely summable by series-to-sequence transformation matrix. For this we find new classes of matrices-conservative series-to-sequence transformation matrices and regular series-to-sequence transformation matrices. We study the relation of these two new classes of matrices with K, β,T and γ-matrices. Finally we prove that the absolute summability of an absolutely convergent series by a matrix and the generalized limit of its absolute partial sum are equal under the suitable relation between the two matrices.

2. *Definitions.* By $\sum_k |u_k|$ we mean the series $\sum_{k=1}^{\infty} |u_k|$. If $s_k = \sum_{i=1}^{k} |u_i|$, s_k is said to be an *absolute partial* sum of the series $\sum_k u_k$. s is said to be an *abslute sum* of $\sum_k u_k$ if $\sum_k |u_k| = s$.

A series $\sum_k u_k$ is said to be *an absolutely sumable by a series –to–sequence transformation matrix* A if $z_n = \sum_{k=1}^{\infty} | a_{n,k} u_k | \rightarrow z$, as $n \rightarrow \infty$ In the above definition if $\sum_k | u_k | = s$, the transformation is called *conservative* or *regular* according as $z \neq s$ or $z = s$. *A matrix A is a K–matrix if it satisfies the following conditions*

(2.1) $\sum_{k=1}^{\infty} | a_{n,k} | \leq M$ *for every n,*

(2.2) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ *for eveey fixed* k

(2.3) $\operatorname{Lim}_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \alpha$ ([1], pp. 63).

A K–matrix A is *T–matrix* if $\alpha_k = 0$, $\alpha = 1$ ([1], pp. 64)

We write $\Delta a_{n,k} = a_{n,k} - a_{n,k+1}$, $\Delta |a_{n,k}| = | a_{n,k} | - |a_{n,k+1}|$

3. *Some Lemmas*. Our problem is to find the necessary and sufficient conditions in order that every absolutely convergent series may be absolutely summable by a series-to-sequence transformation matrix. We need the following lemmas.

Lemma 1 *The necessary and sufficient condition that a matrix A transforms all the null sepuences into null sequences are that*

(i) $\lim_{n\to\infty} a_{n,k} = 0$ *for every fixed* k, and

(ii) $\sum_{k=1}^{\infty} | a_{n,k} | \leq M$ *for every n, where M is independent of n*

For proof see [1], pp. 64 (4.1,II) and the remark is etalics concerning case $z=0$; also [2], pp. 49

Lemma 2. *The necessary and sufficient condition that* $\sum_{k=1}^{\infty} | a_{n,k} u_k |$ *exists for every n, whenever* $\sum_k u_k$ *is absolutely convergent, is that*

(3.1) $\lim_{k\to\infty} | a_{n,k} | \leq M$ *for every fixed n.*

Proof. We first observe that if (3.1) holds, there is a number M′ such that

$| a_{n,k} | \leq M'$ for all n and k.

Thus the condition is suffieient. for

$$\sum_{k=1}^{\infty} | a_{n,k} u_k | = \sum_{k=1}^{\infty} | a_{n,k} \| u_k | \leq M' \sum_{k=1}^{\infty} | u_k | \text{ exists}$$

for every fixed n, since $\sum_k u_k$ is absolutely convergent.

Conversely, we are to prove that if $\sum_{k=1}^{\infty} | a_{n,k} u_k |$ exists for every fixed n, whenever $\sum_k u_k$ is absolutely convergent, the the condition (3.1) is necessary, Suppose that (3.1) is false. Then there exists a sequence $\{ k_i \}$ of positive integers such that $| a_{n,k_i}] > i^2$ (i=1,2,3,.....,) for every fixed n.

Let $u_k=0$ for $k \neq k_i$ (i=1,2,3,....), and $u_{k_i}=1/i^2$ (i=1,2,3....,).

then $\sum_{k=1}^{\infty} | u_k | = \sum_{i=1}^{\infty} 1/i^2 = \pi/6$. But $\sum_{k=1}^{\infty} | a_{n,k} u_k | =$

$\sum_{i=1}^{\infty} | a_{n,ki}/i^2 | = \infty$, and hence the condition (3.1) is necessary.

Lemma 3. *Let s be any positive number and* $\{ x_k \}$ *be any arbitrary null sequences, then there exists an absolutely convergent series*

$\sum_k u_k$ *such that* $s-s_k=x_k$, *where* $s_k= \sum_{i=1}^{k} | u_i |$.

$s-s_1=x_1$ or, $s_1=s-x_1$ or $| u_1 | = s-x_1$;

$s-s_2 = x_2$ Or, $s_2 = s- x_2$

or, $| u_1 | + | u_2 | = s- x_2$

or $| u_2 | = s- x_2 - | u_1 |$.

$= s-x_2-s+ x_1$

$= x_1-x_2$;

and so on.

Now $\sum_{i=1}^{k} | u_i | = | u_1 | + | u_2 | + | u_3 | + ...+ | u_k |$

$= (s-x_1) + (x_1 - x_2) + ... + (x_{k-1}-x_k)$

$= s- x_k$

Therefore $\sum_{i=1}^{\infty} | u_i | = s$, since $x_k \to 0$ as $k \to \infty$.

This proves the lemma.

Lemma 4. *A necessary condition that* $\lim_{n\to\infty} \sum_{k=1}^{\infty} | a_{n,k} u_k |$ *exists, whenever* $\sum_k u_k$ *is absolutely convergent, is that*

$$(3.2) \quad \sum_{k=1}^{\infty} \| a_{n,k} | - | a_{n,k+1} \| \leq M \text{ for every } n.$$

Proof. since $\lim_{n\to\infty} \sum_{k=1}^{\infty} | a_{n,k} u_k |$ exists,

$\sum_{k=1}^{\infty} | a_{n,k} u_k |$ exists for every n and hence by Lemma 2

$$(3.3) \quad \lim_{k\to\infty} | a_{n,k} | \leq G \text{ for every fixed } n.$$

Take any positive integer r and put $u_r=1$, $u_k = 0$ for $k\neq r$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} | a_{n,k} u_k | = \lim_{n\to\infty} | a_{n,r} |.$$

Therefore

$$(3.4) \quad \lim_{n\to\infty} | a_{n,r} | \text{ exists.}$$

Let s be any positive number and $\{ x_k \}$ be any orbitrary null sequence; then, by Lemma 3, $\sum_k u_k$ is an absolutely convergent series such that $s-s_k = x_k$, where $s_k = \sum_{i=1}^{k} | u_i |$. Now we have $| u_k | = s_k - s_{k-1} = x_{k-1} - x_k$. Also

$$(3.5)\quad \sum_{k=1}^{m} |a_{n,k}\, u_k| = \sum_{k=1}^{m} |a_{n,k}\| u_k|$$

$$= \sum_{k=k}^{m} |a_{n,k}| \,(x_{k-1}-x_k)$$

$$= a_{n,1}\, x_o - \sum_{k=1}^{m-1} \left(|a_{n,k}| - |a_{n,k+1}| \right) x_k - a_{n,m}\, x_m.$$

From (3.3) we get

$$(3.6)\quad \lim_{m\to\infty} |a_{n,m}\, x_m| = 0, \text{ since } x_m \to 0 \text{ as } m \to \infty.$$

Now (3.5) and (3.6) togather imply that

$$(3.7)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(|a_{n,k}| - |a_{n,k+1}| \right) x_k$$

$$= \lim_{n\to\infty} |a_{n,1}|\, x_o - \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k}\, u_k|.$$

By the hypothesis and (3.4), the right-hand side of (3.7) exists and therefore

$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left(|a_{n,k}| - |a_{n,k+1}| \right) x_k$ exist for an arbitrary null sequence $\{ x_k \}$

Hence the necessity follows from Lemma.

4. *Conservative Transformation and* $|\beta|$ *– matrix.* In this section we study the problem of the absolute summability by a method of conservative series-to-sequence transformation matrix.

(4.1). *The necessary and sufficient conditions in order that every absolutely convergent series is absolutely summable by a series–to–sequence transformation matrix A are that*

$$(4.1)\quad \sum_{k=1}^{\infty} \|a_{n,k}| - |a_{n,k+1}| \leq M \textit{ for every } n, \textit{ qnd}$$

$$(4.2)\quad \lim_{n\to\infty} |a_{n,k}| = \lambda_k \text{ for every fixed } k.$$

Moreover, under these conditions

$$(4.3)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k}\, u_k| = \lambda_1 s + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1})(s_k - s)$$

Whenever $s_k = \sum_{i=1}^{k} |u_i| \to s$ as $k \to \infty$

Proof.

Sufficiency. We have

$$(4.4)\quad \sum_{k=1}^{\infty} |a_{n,k}\, u_k| = \lim_{m\to\infty} \sum_{k=1}^{m} |a_{n,k}\, u_k|$$

$$= \lim_{m\to\infty} \sum_{m-1}^{m} |a_{n,k}| \{ (s_k - s) - (s_{k-1} - s) \}$$

$$= \lim_{m\to\infty} \sum_{k=1}^{m-1} (|a_{n,k}| - |a_{n,k+1}|) (s_k - s) + |a_{n,1}|\, s$$

$$+ (s_m - s)\, |a_{n,m}|.$$

Also

$$(4.5)\quad |a_{n,m}| = |a_{n,1}| - \sum_{k=1}^{m-1} (|a_{n,k}| - |a_{n,k+1}|)$$

Now (4.1) and (4.5) togather imply that $\lim_{m\to\infty} |a_{n,m}| \leq G$

for every fixed n, since, by (4.2), $|a_{n,1}|$ is bounded for every fixed n. Hence

(4.6) $|a_{n,m}| (s_m - s) \to 0$ as $m \to \infty$, since $s_m \to s$ as $m \to \infty$.

From (4.4) and (4.6) we get

$$(4.7)\quad \sum_{k=1}^{\infty} |a_{n,k}\, u_k| = |a_{n,1}|\, s + \sum_{k=1}^{\infty} (|a_{n,k}| - |a_{n,k+1}|) (s_k - s).$$

It follows from (4.1) and $s_k \to s$ that the right-hand side of (4.7) exists for every fixed n. Hence the left-hand side exists for every fixed n.

Take any $\epsilon > 0$. Choose N such that $|s_k - s| > \epsilon/M$ for all $k > N$ and write (4.7) in the form

$$(4.8)\quad \sum_{k=1}^{\infty} |a_{n,k} u_k| = |a_{n,1} s| + \left(\sum_{k=1}^{N} + \sum_{k=n+1}^{\infty} \right) (|a_{n,k}| - |a_{n,k+}|)(s_k - s).$$

Then, by the condition (4.1),

$$(4.9)\quad \sum_{k=N+1}^{\infty} (|a_{n,k}| - |a_{n,k+1}|)(s_k - s) \leq M.\ \epsilon/M = \epsilon \text{ for}$$

every n, and, by (4.2),

$$(4.10)\quad \sum_{k=1}^{N} (|a_{n,k}| - |a_{,k+1}|)(s_k - s) \to \sum_{k=1}^{N} (\lambda_k - \lambda_{k+1})(s_k - s)$$

and $|a_{n,1}| s \to \lambda_1 s$ as $n \to \infty$.

From (4.8), (4.9) and (4.10) we get

$$(4.11)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} u_k| = \lambda_1 s + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1})(s_k - s).$$

Hence the conditions are sufficients.

Necessity. Suppose $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} u_k|$ exists

Whenever $\sum_k u_k$ is absolutely convergent.

Let $u_k = 1$ for $k=p$ and $u_k = 0$ for $k \neq p$, then

$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} u_k| = \lim_{n\to\infty} |a_{n,p}|$; and hence the condition (4.2) is necessary.

The necessity of condition (4.1) follows from Lemma 4.

This completes the proof of the theorem.

Definition 1. If a matrix A satisfies the conditions (4.1) and (4.2), it will be called $|\beta|$ – *matrix* and λ_k will be called its *characteristic* number.

Definition 2. A matrix A is a β–*matrix* if and only if

(4.12) $\sum_{k=1}^{\infty} \mid a_{n,k} - a_{n,k+1} \mid \leq M$ for every n, and

(4.13) $\lim_{n\to\infty} a_{n,k} = \beta_k$ for every fixed k,

where β_k is its characteristic number ([1], pp. 66).

Remarks :– The condition (4.12) implies the condition (4.1) but (4.1) may or may not imply (4.12). It is obvious that (4.13) implies (4.2). Hence

every β –matrix is a $|\beta|$ –matrix.

Take $a_{n,k} = (-1)^{k-1} \frac{n+k}{nk}$. Now we have

$$| a_{n,k} | - | a_{n,k+1} | = \frac{n+k}{nk} - \frac{n+k+1}{n(k+1)} = \frac{nk+n+k^2+k-nk-k^2-k}{nk\ (k+1)}$$

$$= \frac{n}{nk\,(k+1)} = \frac{1}{k(k+1)} < \frac{1}{k^2}$$

Therefore

$$\sum_{k=1}^{\infty} \| a_{n,k} | - | a_{n,k+1} \| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\Pi^2}{6}\ .$$

This implies that

$$\sum_{k=1}^{\infty} \| a_{n,k} | - | a_{n,k+1} \| \leq M \text{ for every n.}$$

Thus (4.1) is satisfied.

Again $\lim_{n\to\infty} | a_{n,k} | = \frac{1}{k}$ for every fixed k, and thus (4.2) is satisfied.

Consequently $A = (a_{n,k})$ is a $|\beta|$ – matrix.

Also we have

$$|a_{n,k}-a_{n,k+1}| = \frac{n+k}{nk} + \frac{n+k+1}{n(k+1)} = \frac{(n+k)(k+1)+(n+k+1)}{nk\ (k+1)}$$

$$= \frac{2nk+2k^2+2k+n}{nk(k+1)} > \frac{n(k+1)}{nk(k+1)} = \frac{1}{k}\ .$$

Therefore

$$\sum_{k=1}^{\infty} |\ a_{n,k}-a_{n,k+1}\ | > \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

This implies that $\sum_{k=1}^{\infty} |\ a_{n,k}-a_{n,k+1}\ |$ is not bounded.

Thus the condition (4.12) is not satisfied. Hence $A = (a_{n,k})$ is not β–matrix and we obtain the following result:

(4,II). Every *β–matrix is a $|\beta|$ –matrix but the converse is not true.*

(4,III). *The suffient condition in order that*$\alpha\lambda$ $|\beta|$ *-matrix* A *should be a β– matrix is that* $a_{n,k} \geq 0$ *for every n and* k.

Proof. The condition is sufficient, for

$|a_{n,k}| = a_{n,k}$ and $|a_{n,k}| - |a_{k,k+1}| = a_{n,k} - a_{n,k+1}$ and thus the $|\beta|$ –matrix satisfies the conditions (4.12) and (4.13).

(4,IV). *Every Absolutely convergent series is absolutely summable by a β matrix.*

This follows from (4,1) and (4,II).

5. *Regular Transformation and $|\gamma|$ –matrix.* In this section we study the absolute summability by a method of regular series-to-sequence transformation matrix.

(5,1). *The necessary and suffient conditions that every absolutely convergent series* $\sum_{k} u_k$ *is absolutely summable to s by a series–to–sequence transfotmation matrix* A, *whenever* $\sum_{k} |u_k| = s$, *are that*

(5.1) $\sum_{k=1}^{\infty} \| a_{n,k}| - |a_{n,k+1}\| \leq M$ *for every* n, *and*

(5.2) $\lim_{n\to\infty} | a_{n,k} | = |$ for every fixed k.

Proof.

Sufficiency. Putting $\lambda_k = 1$ in (4,1), the conditions (5.1) and (5.2) are immediately seen to be sufficient.

Necessity. Take $u_k = 1$ $(k=p)$

$= O$ $(k\neq p)$, p being a fixed integer.

Then $\sum_k | u_k | = 1$.

But $\lim_{n\to\infty} \sum_{k=1}^{\infty} | a_{n,k} u_k | = \lim_{n\to\infty} | a_{n,p} |$.

Hence (5.2) is a necessary condition.

Since $\lim_{n\to\infty} \sum_{k=k}^{\infty} | a_{n,k} u_k | = s$, whenever $\sum_k | u_k | = s$,

the necessity of the condition (5.1) follows from Lemma 4.

Remark. (4,1) and (5,1) are also true if we replace the integer n by the continous variable w and in the conditions (4.1) and (5.1) we write $w > w_0$ in place of every n.

Definition 1. If a matrix A satisties the conditions (5.1) and (5.2), we shall call it $| \gamma |$-matrix.

Definition 2. The matrix A is a γ *-matrix* if and only)if

(5.3) $\sum_{k=1}^{\infty} | a_{n,k}-a_{n,k+1} | \leq M$ for every n, and

(5.4) $\lim_{n\to\infty} a_{n,k} = 1$ for every fixed k. (|1], pp. 68).

The condition (5.3) implies the condition (5.1) but (5.1) may or may not imply (5.3), and (5.4) obviously implies (5.2).

Hence *every* γ *-matrix* is a $| \gamma |$ *-matrix.*

Example. Take $a_{n,k} = (-1)^{k-1} \frac{nk+1}{nk}$, then $| a_{n,k}-a_{n,k+1} |$

$$= \frac{nk+1}{nk} + \frac{n(k+1)+1}{n(k+1)} = \frac{nk^2+k+nk+1+nk^2+nk+k}{nk(k+1)} >$$

$$\frac{nk+n}{nk(k+1)} = \frac{1}{k}$$

Therefore

$$\sum_{k=1}^{\infty} | a_{n,k}-a_{n,k+1} | > \sum_{k=1}^{\infty} \frac{1}{k}$$

This implies that $\sum_{k=1}^{\infty} | \Delta a_{n,k} |$ is not bounded,since $\sum_{k=1}^{\infty} \frac{1}{k}$

is divergent; thus the condition (5.3) is not satisfied. Hence A is not γ - matrix.

Again $|a_{n,k}| - |a_{n,k+1}| = \frac{nk+1}{nk} - \frac{n(k+1)+1}{n(k+1)} = \frac{1}{nk(k+1)} < \frac{1}{k^2}$

Therefore

$$\sum_{k=1}^{\infty} \| a_{n,k} | - | a_{n,k+1} \| < \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

This implies that

$$\sum_{k=1}^{\infty} \| a_{n,k} | - | a_{n,k+1} \| \leq M \text{ for every n.}$$

Thus the condituon (5.1) is satisfied.

Again $\lim_{n\to\infty} | a_{n,k} | = 1$ for every fixed k.

Hence A is a $|\gamma|$ -matrix.

Now we obtain the following result:

(5,II). *Every γ -matrix is a $|\gamma|$ -matrix but the converse is not true.*

(5,III) *The sufficient condition in order that a $|\gamma|$ -matrix should be a γ -matrix is that*

$a_{n,k} \geq 0$ *for every* n *and* k.

The condition is sufficient for $| a_{n,k} | = a_{n,k}$ and $| a_{n,k} | - | a_{n,k+1} | = a_{n,k} - a_{n,k+1}$ and consequently $| \gamma |$ -matrix satisfies the conditions (5.3) and (5.4).

(5,IV). *Every absolutely convergent series is absolutely summable by* γ *-matrix*

This follows from (5.5., II) and (5.5,I).

6. *Absolute Summability and Generalized Limit.* By the definition the absolute sum of an infinite series $\sum_k u_k$ is the limit of the sequence $s_k = | u_1 | + | u_2 | + \dots + | u_k |$ which is its absolute partial sum. Now the question arises as to whether the absolute summability of $\sum_k u_k$ and the generalized limit of the sequence of its absolute partial sum are equal under suitable relations between two matrices. In the proof of our results we require the following lemmas.

Lemma 5. *If* $| g_{n,k} | = \sum_{i=k}^{\infty} a_{n,i}$, *to every* K*-matrix* $A=(a_{n,k})$ *corresponds* $| \beta |$ *-matrix* $G = (g_{n,k})$ *and to every T-matrix A corresponds* $| \gamma |$ *-matrix* G.

Proof. If A is a K-matrix, A satisfies the conditions (2.1), (2.2) and (2.3). Since $| g_{n,k} | = \sum_{i=k}^{\infty} a_{n,i}$, $\Delta | g_{n,k} | = a_{n,k}$;

and therefore, by using (2.1). $\sum_{k=1}^{\infty} \| g_{n,k} | - | g_{n,k+1} \| =$

$\sum_{k=1}^{\infty} | a_{n,k} | \leq M$ for every n.

Hence the condition (4.1) is satisfied.

Again $\lim_{n\to\infty} | g_{n,k} | = \lim_{n\to\infty} \left[\sum_{k=1}^{\infty} a_{n,k}k - \sum_{i=1}^{k-1} a_{n,i} \right]$

$= \alpha - \alpha_1 - \alpha_2 - \dots - \alpha_{k-1} = \lambda_k$ (say)

Where α_k and α are the characteristic numbers of A. Thus the condition (4.2) is also satisfied and hence G is a $|\beta|$-matrix.

Also we have

$$(6.1) \begin{cases} \lambda_1 = \alpha \\ \lambda_2 = \alpha - \alpha_1 \\ \lambda_3 = \alpha - \alpha_1 - \alpha_2 \\ \cdots\cdots\cdots\cdots \end{cases}$$

This implies that

$$(6.2)\ \lambda_1 = \alpha_1,\ \lambda_k - \lambda_{k+1} = \alpha_k \ (k \geq 2).$$

If A is a T–matrix, $\alpha_k = 0$ and $\alpha = 1$ so that

$\lim\limits_{n\to\infty} |g_{n,k}| = 1$. Thus G is a $|\gamma|$-matrix.

Lemma 6. *If* G *is a* $|\beta|$ *–matrix, the necessary and sufficient condition that* $a_{n,k} = |g_{n,k}| - |g_{n,k+1}|$ *should be a* K*–matrix is that*

$$(6.3)\ \ |g_n| = \lim_{k\to\infty} |g_{n,k}| \text{ should tend to a limit as } n\to\infty.$$

Proof.

Sufficiency. If G=$(g_{n,k})$ is a $|\beta|$-matrix, it satisfies the conditions.

$$(6.4)\ \ \sum_{k=1}^{\infty} \| g_{n,k} | - | g_{n,k+1} \| \leq M \text{ for every n, and}$$

$$(6.5)\ \ \lim_{n\to\infty} g_{n,k} | = \lambda_k \text{ for every fixed k.}$$

Now it follows from (6.4), (6.5) and $a_{n,k} = |g_{n,k}| - |g_{n,k+1}|$ that

$$\sum_{k=1}^{\infty} |a_{n,k}| \leq M \text{ for every n, and } \lim_{n\to\infty} a_{n,k} = \lambda_k - \lambda_{k+1} \text{ for}$$

everyfixed k.

Thus the conditions (2.1) and (2.2) are satisfied.

Also we have

$$(6.6) \quad \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} (| g_{n,k} | - | g_{n,k+1} |)$$

$$= | g_{n,1} | - \lim_{k \to \infty} | g_{n,k} | .$$

From (6.3), (6.5) and (6.6) we get $\sum_{k=1}^{\infty} a_{n,k}$ tends to a limit as $n \to \infty$ and thus the condition (2.3) is also satisfied.

Hence A is a K–matrix.

Necessity. Suppose that A is a K–matrix, then $\sum_{k=1}^{\infty} a_{n,k}$ tends to a a limit as $n \to \infty$. Now it follows from (6.6) that the condition (6.3) is necessary is order that $\sum_{k=1}^{\infty} a_{n,k}$ tends to a limit as $n \to \infty$, since $| g_{n,1} | \to \lambda_1$ as $n \to \infty$.

This completes the proof of the lemma.

Lemma 7. *If* G *is a* $| \gamma |$*–matrix and* $a_{n,k} = | g_{n,k} | - | g_{n,k+1} |$, *the necessary and sufficient condition that A should be a T–matrix is that*

$\lim_{k \to \infty} | g_{n,k} | = | g_n | \to o$ *as* $n \to \infty$.

Proof. Since G is a $| \gamma |$ –matrix, $\lim_{n \to \infty} | g_{n,k} | = 1$.

Put $\lambda_k = 1$ in Lemma 6 then the condition is immediately seen to be sufficient and necessary.

(6.1). *If* $| g_{n,k} | = \sum_{i=1}^{\infty} a_{n,i}$ *and* A *is a K–matrix,*

K–limit of $s_k = | u_1 | + | u_2 | + ... + | u_k |$ *wheever* $s_k \to s$ *as* $k \to \infty$ *is equal to* $\lim_{n \to \infty} \sum_{k=1}^{\infty} | g_{n,k} u_k |$.

Proof. If A is a K–matrix, it follows from (4,1,I) of [1], pp. 63 that

(6.7) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{n,k}\, s_k = \alpha\, s + \sum\limits_{k=1}^{\infty} \alpha_k\, (s_k - s).$

Since $| g_{n,k} | = \sum\limits_{i=k}^{\infty} a_{n,i}$, then, by Lemma 5, $G = (g_{n,k})$ is a $|\beta|$ –matrix.

Therefore, by (4,1), we have

(6.8) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} | g_{n,k}\, u_k | = \lambda_1 s + \sum\limits_{k=1}^{\infty} (\lambda_k - \lambda_{k+1})\, (s_k - s).$

Now it follows from (6.7) (6.8) and (6.2) that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} | g_{n,k}\, u_k |.$$

Corollary. If $| g_{n,k} | = \sum\limits_{i=k}^{\infty} | a_{n,i} |$ *and* A *is a T–matrix,*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} | g_{n,k}\, u_k |.$$

Whenever $s_n = \sum\limits_{i=1}^{k} | u_i | \to s$ *as* $k \to \infty$.

Proof. If A is a T–matrix, by Lemma 5, $G = (g_{n,k})$ is a $|\gamma|$ –matrix. The result follows from (6,I) with $\alpha = 1$, $\alpha_k = 0$.

(6,II). *Let* $a_{n,k} = | g_{n,k} | - | g_{n,k+1} |$ *and* G be a $|\beta|$ *–matrix satisfying the condition.*

(6.9). $\lim\limits_{n\to\infty} \left(\lim\limits_{k\to\infty} | g_{n,k} | \right) = 0$; then

(6.10). $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{n,k}\, s_k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} | g_{n,k}\, u_k |.$

Whenever $s_k = \sum\limits_{i=1}^{k} | u_i | \to s$ *as* $k \to \infty$.

Proof. If $a_{n,k} = |g_{n,k}| - |g_{n,k+1}|$ and $G = (g_{n,k})$ is a $|\beta|$ –matrix, it follows from

(6.9) and Lemma 6 that $A = (a_{n,k})$ is K–matrix. Then by (4.1, I) of [1], pp. 63.

$$(6.11)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, s_k = \alpha s + \sum_{k=1}^{\infty} \alpha_k (s_k - s),$$

Where $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \alpha$, $\lim_{n\to\infty} a_{n,k} = \alpha_k$ for every fixed k,

Since G is a $|\beta|$ –matrix and

$a_{n,k} = |g_{n,k}| - |g_{n,k+1}|$, by (6.9)

$$(6.12)\quad \alpha = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty} [\,|g_{n,k}| - \lim_{k\to\infty} |g_{n,k}|\,] = \lambda_1$$

and also

$$(6.13)\quad \alpha_k = \lim_{n\to\infty} a_{n,k} = \lambda_k - \lambda_{k+1}.$$

Therefore, from (6.11), (6.12) and (6.13),

$$(6.14)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, s_k = \lambda_1 s + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1})(s_k - s).$$

Since $\sum_k u_k$ is an absolutely convergent series and G is a $|\beta|$ –matrix, by (4,II)

$$(6.15)\quad \lim_{n\to\infty} \sum_{k=1}^{\infty} |g_{n,k}\, u_k| = \lambda_1 s + \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1})(s_k - s).$$

Consequently it follows from (6.14) and (6.15) that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} |g_{n,k}\, u_k|.$$

Corollary. Let $a_{n,k} = |g_{n,k}| - |g_{n,k+1}|$ *and* G *be* $|\gamma|$ *–matrix satisfying the condition*

(6.16) $\lim_{n\to\infty} (\lim_{k\to\infty} | g_{n,k} |) = 0,$

then

(6.17) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} | g_{n,k} u_k | = s$

Whenever $s_k = \sum_{i=1}^{k} | u_i | \to s$ *as* $k \to \infty$.

The corollary immediately follows from Lemma 7 and (6,II) with $\lambda_k = 1$.

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