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## Algebraic Structure Of The Continuum And The Factorial Representation


by

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# Algebraic Structure Of The Continuum And The Factorial Representation

Orhan Hamdi ALİSBAH

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## INTRODUCTION

In § 1 of this paper the Method of Factorial Representation is reviewed. § 2 deals with the concept of the Abelean Cluster $A_n$ and the Sequence of Abelean Clusters:

$A_1 \{0\}$, $A_2 \{1/2\}$, $A_3 \{1/6, 1/3, 2/3, 5/6\}$, ......

The element

$$x = \sum_{k=1}^{n} a_k e_k$$

of $A_n$ is characterized by the restrictions (1.6) and (1.7) of which the latter leads to

$$A_m \cap A_n = \varnothing, \text{ for } m \neq n.$$

The Sequence of Clusters represents a different version of the Cantor ordering of the rational numbers. § 3 contains the real and complex universe. § 4 deals with the minimal and maximal representatives of the Abelean Clusters and the Univers. § 5 is reserved for the approximation of the irrational numbers. § 6 contains the definition of the extended universe and a criterion for transcendence related to the matrix $T\infty = (a_m)_n$ of infinite rank.

In § 7 certain Diophantine Systems are formulated, which are of special interest in connection with the factorial representation. In § 8, Alef and C are symbolically described as additive infinity: Alef $= \Sigma\, N\,(l)$ and multiplicative infinity $C = \infty!$. In § 9 the factorial representation is being brought in relation with Dedekind Cut.

## § 1

This paragraph contains certain results, which were previously analized in a recent paper of the author[1]).

Throughout this article:

$$e_\lambda = \frac{1}{\lambda !} .$$

A. Every complex number $z = x + iy$ satisfying

$$(1.1) \qquad 0 \leq | z | \leq 1$$

has a unique factorial representation of the form

$$(1.2) \qquad z = x + iy = \sum_{\lambda=1}^{\infty} (a_\lambda + ib_\lambda) e_\lambda$$

whereby

$a_\lambda$'s as well as $b_\lambda$'s

are integers subject to

$$(1.3) \qquad 0 \leq a_\lambda \leq \lambda - 1 \text{ and } 0 \leq b_\lambda \leq \lambda - 1 .$$

B. Every rational number

$$x = \frac{p}{q} \qquad \text{with} \qquad [p, q] = 1$$

and subject to

$$(1.4) \qquad 0 \leq x < 1$$

has a unique finite representation of the form

$$(1.5) \qquad x = \frac{p}{q} = \sum_{\lambda=1}^{l} a_\lambda e_\lambda$$

where $l = l(q)$ is obtained by determining the least factorial multiple $l!$ of q and $a_\lambda$'s are integers satisfying

$$(1.6) \qquad 0 \leq a_\lambda \leq \lambda - 1 \qquad \text{for} \quad 1 \leq \lambda \leq l - 1$$

and

$$(1.7) \qquad 0 < a_l \leq l - 1 \qquad \text{for} \quad \lambda = l .$$

C. Any finite representation of the form (1.5) corresponds to a rational number x subject to (1.4).

D. The above defined function

$$l = l\ (q)$$

has the following elementary properties

(1.8) $$l = l\,(q) = q$$

only for q prime or $q = 4$.

(1.9) $$l\ (q) \leq \max\ l\ (p_\mu^{\alpha_\mu}) \leq q$$

$$\text{for} \quad q = \prod_{\mu=1}^{m} p_\mu^{\alpha_\mu} .$$

E. The inverse $l^{-1}$ (q) of (q) — with the exception of the cases stated in (1.8) — is multiple-valued.

We denote the number of integers q corresponding to the same $l$, in other words the degree of relative multiplicity of $l^{-1}$ (q) with

(1.10) $$I = I\ (l).$$

This concept plays an important role for the measurement of the density of the prime distribution.

## § 2

## ABELEAN CLUSTERS

We call the set $A_l$ consisting of elements of the form

$$x = \sum_{\lambda=1}^{l} a_\lambda\ e_\lambda$$

as defined by (1.5) and subject to (1.6) and (1.7) an Abelean cluster or Simplexoid of dimension $l$.

A. Every $A_l$ contains

(2.1) $$N\ (l) = [\ (l-1)\,!\ ]\ .\ (l-1)$$

elements.

The statement A. follows from (1.5), (1.6), (1.7) immediately.

B. $$A_m \cap A_n = \varnothing \quad \text{for} \quad m \neq n$$

A result which follows from (1.7).

C. Every rational number

$$x = \frac{p}{q}, \qquad [p, q] = 1, \qquad 0 \leq x < 1$$

is element of only one $A_l$ with $l = l(q)$.

The statement C. is a simple conclusion drawn from B.

We shall refer to an abelean cluster $A_l$ whose elements are ordered according to their magnitude as an ordered abelean cluster $A_l^*$ .

In connection of B. and C. and with respect to this last definition we can now express:

D. The Sequence

$$(2.2) \qquad S: \qquad A_1^*, A_2^*, \ldots, A_l^*, \ldots$$

represents a different version of the Cantor ordering of the rational elements of the unit interval.

## § 3

### *The Real And Complex Universe.*

We call the set of elements of the form

$$x = \sum_{\lambda=1}^{\infty} a_\lambda e_\lambda$$

where $a_\lambda$'s are integers subject to

$$0 \leq a_\lambda \leq \lambda - 1$$

the Real Universe $A\infty$ and consequently the set of elements of *a* the form (1.2) and subjest to (1.3) the complex Universe $K\infty$. Obviously

$$A\infty \subset K\infty$$

It is easy to make the following observations concerning $A_l$ and $A\infty$ .

A. $\qquad A_l \subseteq A\infty$ .

B. $\qquad S = \bigcup_{\lambda=1}^{l} A_\lambda$

is an abelean group under addition mod. 1.

C. $$A\infty = \bigcup_{\lambda=1}^{\infty} A_\lambda \quad \text{and} \quad A\infty = \bar{A}\infty .$$

D. $A\infty$ is an abelean group addition mod. 1.

E. $A\infty$ is isomorphic to the closed unit interval.

These remarks can organisally be extended to $K_l$ and $K\infty$ .

§ 4

*The Minimal and the Maximal axis of $A_l$ and $A\infty$.*

We call

$$m_l = \sum_{\lambda=1}^{l} e_\lambda \tag{4.1}$$

the minimal axis

$$M_l = \sum_{\lambda=1}^{l} (\lambda - 1)\, e_\lambda \tag{4.2}$$

the maximal axis of $A_l$, and

$$m_\infty = \sum_{\lambda=1}^{\infty} e_\lambda = (e - 2) \tag{4.3}$$

minimal axis and

$$M_\infty = \sum_{\lambda=1}^{\infty} (\lambda - 1)\, e_\lambda = 1 \tag{4.4}$$

the maximal axis of the real universe $A\infty$ respectively.

We call $R_l$ defined by

$$M\infty - M_l = R_l = \sum_{\lambda=l+1}^{\infty} (\lambda - 1)\, e_\lambda \tag{4.5}$$

the residua complement of M.

Due to (4.4) and the identity

$$M_l + e_l = 1 \tag{4.6}$$

we observe elementarily the following property:

A.

(4.7) $$e_l = R_l \quad .$$

From the geometric point of view this result means that $e_l$ is a maximal element of the residual space $R_l$ .

## § 5

### *Approximation of the Irrational Numbers*

We call $x \in A$ with infinitely many non-zero components a proper element of A.

A. According to A., B., C., of § 1 with the exclusion of $x = 1$ all the other proper elements of A are irrational bumbers, satisfying $0 \leq x \leq 1$.

B. For every proper element x of A the following inequality holds:

(5.1) $$x = \sum_{\lambda=1}^{n} a_\lambda e_\lambda + \sum_{\lambda=n+1}^{\infty} a_\lambda e_\lambda \leq \sum_{\lambda=1}^{n} a_\lambda e_\lambda + R_n = x_n + R_n .$$

Hence:

C.

(5.2) $$x - x_n \leq R_n$$

or according to (4.7)

(5.3) $$x - x_n \leq e_n .$$

We call

$$x_n = \sum_{\lambda=1}^{n} a_\lambda e_\lambda$$

of (5.1) the proper n–th representative of the irrational number x.

D. Due to (1.6), $x_n$ is the largest element of $A_n$ satisfying

(5.4) $$0 < x_n < x \ .$$

The concept proper n—th representative can also be applied for

(5.5) $$y = \sum_{\lambda=1}^{l} a_\lambda e_\lambda \, , \; y \in A_l \, .$$

For $n < l$ we call

(5.6) $$y_n = \sum_{\lambda=1}^{n} a_\lambda e_\lambda$$

correspondingly the proper n-th representative of y. (5.3) remains also valid for this interpretation.

## § 6

### *The extended Universe and The Criterion of Transcendence*

We call the sequence

(6.1) $$E: e_1, e_2, \dots , e_n , \dots$$

the fundamental base of the Universe and the sequence

(6.2) $$Y: y_1, y_2 , \dots , y_n , \dots$$

introduced by

(6.3) $$y_n = \sum_{\lambda=1}^{\infty} b_{n\lambda} e_\lambda \qquad \text{for } n = 1, 2, \dots$$

the **regular** base of the Extenden Universe $U_\infty$, provided the infinite matrix of the transformation

(6.4) $$T_\infty = (b_{n\lambda}), \; (n, \lambda) = 1, 2, \dots$$

is of infinite rank and the components are finite complex numbers. Obviously:

A. $$A_\infty \subset K_\infty \subset U_\infty \, .$$

We refer to x as a transcendental element of $U_\infty$, whenever the power sequence

(6.5) $$P: \quad x, x^2, \dots , x^n, \dots$$

is a regular base of $U_\infty$.

*Examble 1.* the power sequence

$$(6.6) \qquad E_1 : e^m = \sum_{\lambda=0}^{\infty} m^\lambda e_\lambda ,$$

with the infinite matrix :

$$(6.7) \qquad T_\infty = (m^\lambda)$$

$$m = 1, 2, \ldots ; \lambda = 0, 1, 2, \ldots$$

is a regular base.

*Examble 2.* The power sequence

$$(6.8) \qquad E : (e - 2), (e - 2)^2, \ldots , (e - 2)^n, \ldots$$

is a regular base.

*Remark.*

$$(6.9) \qquad (e - 2)^n = \sum_{\lambda=0}^{n} (\lambda^n) (- 2)^{n-\lambda} . e^\lambda$$

is a linear combination of

$$(6.10) \qquad 1, \quad e^2 , \ldots , e^n .$$

B. If x is transcendental with repect to $U_\infty$ then $x - [x]$ is trascendental with respect to A. $[x]$ is the integral part of x.

We call $(c_\lambda z^\lambda)$ — c and z complex — local contraction and $c_\lambda$ the $\lambda$ — th contractor.

C. If $\quad f(z) = \sum_{\lambda=0}^{\infty} c_\lambda z^\lambda e_\lambda$

is analytic over $\quad |z| < R \leq \infty$

then $\quad f(z) \in U_\infty$ .

## § 7

### *Certain Diophantine Systems*

A close study of linear independence of the fundamental and regular bases leads to certain mathematical questions, which are attractive for their own sake.

The following Diophantine System is a typical example of this kind:

$$(7.1) \qquad s.\,(\lambda - 1) = \sum_{k=1}^{m} t_k \ a_{k\lambda}$$

$$\lambda = 1, 2, \ldots, n, \ldots$$

where $t_k$' s are integers and $a_{k\lambda}$'s are integers subject to

$$(7.2) \qquad 0 \leq a_{k\lambda} \leq \lambda - 1$$

$$(7.3) \qquad m \leq \infty .$$

*Examble 1.* $x = \sum_{\lambda=1}^{\infty} a_{1\lambda}\, e_\lambda \in A_\infty$, $\lambda = \sum_{\lambda=1}^{\infty} a_{2\lambda}\, e_\lambda \in A_\infty$ and $x + y = 1$

$s=1$, $t_1 = t_2 = 1$, $m = 2$ and $a_{1\lambda} + a_{2\lambda} = \lambda - 1$ .

*Examble 2.* For $e - 2 = x_n + r_n$ , $n = 1, 2, \ldots.$ the residual sequence

$$r_1 , r_2 , \ldots , r_n , \ldots$$

is a regular base and

$$t_k = 1 \text{ for } k = 1, 2, \ldots , n , \ldots$$

$$a_{k\lambda} = 1 \text{ for } \lambda \geq k \text{ and } a_{k\lambda} = 0 \text{ for } \lambda < k .$$

## § 8

### *Symbolic Interpretetion of Alef and C*

A. *The power of the countable set*

$$(8.1) \quad \text{Alef} = \sum_{\lambda=1}^{\infty} N(\lambda) \text{ (additive infinity)}$$

and in connection with (1.2) and (1.3).

B. *The power of the Continuum*

$$(8.2) \qquad C = 1.2.3.\ldots.n.\ldots = \infty \quad \text{(mutiplicative infinity).}$$

## § 9

### *The Dedekind Cut and Factorial Representation*

For every $x \in A_{\infty,}$ we wcall the finite or infinite sequence

(9.1) $\quad A_{k_1}, A_{k_2}, \ldots, A_{k_n}, \ldots$

which is obtained from the sequence

(9.2) $\quad A_1, A_2, \ldots, A_n, \ldots$

by the omission of those clusters $A_n$ in which there is no-representative of x, The Reduced Sequence of the Representatives of x. Obviously (9.1) might as well be identical with (9.2).

A. Every $x \in A_{\infty}$ is a Dedekind Cut represented by the finite or infinite sequence

(9.3) $\quad x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots$

$x_{k_n} \in A_{\infty}$ and maximal relative to x.

B. $x_{k_n} = \varnothing_x (A_{k_n})$ is the relative choice function of Zermelo.

C. Every subset of (9.3) has a maximal element, x is a maximal element (Zorin ' s Lemma).

**REFERENCES**

z1e *Orhan Hamdi Alisbah*, M.E.T.U. Jour. of Pure And Appl. Sc. Vol. 1, No 2, 1968, p. 71–77.

z2e On the suggestion of the author *Uluğ Çapar* of M.E.T.U has made an interesting survey concerning the nature of l (q) in 1967 (unpublished). His observetions are not included.

z3e On the suggsetion of the auther *Ellen Fenwick* of *Rutgers* (with the help of *Rabert Turkelson* of *Rutgers*) has conducted a programed survey concerning the related distribution of $l^{-1}$ (g), in 1968. The method they used is different in nature from the one applied in the previous paper of the author, (unpublished).

## ÖZET

*Continuum'un Cebirsel Yapısı ve Faktöryel Temsil*

Bu makale aslında bundan önce Orta Doğu Teknik Üniversitesi Temel ve Uygulamalı Bilimler Dergisi Cilt 1, Sayı 2, yıl 1968 Sayfa 71 de yayımlanan bir araştırmanın devamıdır. § 1 de temel konuya değinilmiştir. § 2 de Abel Kümeleri tanımlanmıştır. Düzenlenmiş Abel Kümeleri yardımıyle rasyonel sayıların sıralanabilirliği Cantor metodundan farklı bir şekilde gösterilmiştir. § 3 de reel ve komplex universe tanımlanmış, § 4 te Abel Kümelerinin ve universe'in minimal ve maximal eksenleri ifade olunmuştur. § 5 irrasyonel sayıların yaklaşık değerlerinin hesaplanması ve hata tahminine dairdir. § 6 da genelleştirilmiş universe tanımlanmakta ve bir transcendence kriteri verilmektedir. Aynı paragrafta e ve (e–2) örnekleri incelenmektedir. § 7 de faktöryel temsil ile bazı Diophant denklem sistemleri ve o tipten iki örnek irdelenmektedir. § 8 de sembolik olarak Alef $= \Sigma\ N\ (l)$, $C = \infty$ ! eşitlikleri ileri sürülmektedir. § 9 da faktöryel temsil Dedekind kesiti ile karşılaştırılmaktadır.

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