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TITLE: On Higher Curvatures Of A Curve

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PAGES: 0-0

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/1629219>

COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A₁: Mathématique

TOME 24

ANNÉE 1975

On Higher Curvatures Of A Curve

by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

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On Higher Curvatures Of A Curve

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(Received, January 1975)

ABSTRACT

The higher curvatures of a Curve in n -dimensional Euclidean space E^n are derived by Herman Gluck [1]. In this paper we give a new method to define and calculate the higher curvatures of a curve which is supposed a differentiable 1-manifold in E^n . We also prove that these curvatures are invariant under the rigid motions.

I. INTRODUCTION

In E^n n -dimensional Euclidean space, a curve α is a diffeomorphic image of an open segment I of a straight line. Therefore in differential geometry a curve of E^n can be regarded as a 1-manifold. Hence for a curve also we can define the Riemannian metric which is known for the submanifolds of E^n . This permits us to have the tangent space, at any point $\alpha(s)$ of the curve α , as an inner product space. We can also apply the theory of the vector fields, the vector-valued forms and the frame bundles over E^n . Thus we can accept that the properties of the equations of structure and the cross section are known also for the curves of E^n .

The Frenet frame of a curve in E^n is an oriented orthonormal frame which determines a cross section

$$C: \alpha(I) \longrightarrow \mathcal{F}_0$$

and hence we may consider the pull back 1-forms

$$\begin{aligned} C^*(\varnothing_i) &= \varnothing^c_i, \quad 1 \leq i \leq n \\ C^*(\varnothing_{ij}) &= \varnothing^c_{ij}, \quad 1 \leq i, j \leq n, \end{aligned}$$

where \mathcal{F}_0 denotes the orthonormal frame bundle over E^n , \varnothing_i , \varnothing_{ij} are the connection forms which appear in the equations of structure of the Lie groups.

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Because of the cotangent space to $\alpha(s)$ is spanned by ds for all $s \in I$ we may find that

$$\alpha^* (\varnothing^c_i) = f_i(s)ds, \quad 1 \leq i \leq n$$

$$\alpha^* (\varnothing^c_{ij}) = k_{ij}(s) ds, \quad 1 \leq i, j \leq n$$

where $k_{ji} = -k_{ij}$. We also show that the functions $f_i(s)$ and $k_{ij}(s)$ are differential geometric invariants of the curve α under the rigid motions.

In differential geometry the role of higher curvatures of a curve is important for the higher dimensions. To show this importance we can mention the papers [3] and [4]. The differential geometry of higher dimensions is about the p-forms, equations of structures on a manifold. These needs the new techniques of modern differential geometry. So it will be very valuable to treat the paper [1] of Herman Gluck in this new style.

II. BASIC CONCEPTS

a) Tangent and Cotangent Spaces

If M is an r -dimensional submanifold of E^n n -dimensional Euclidean space. Then we denote the tangent space to M at its point m by $T_M(m)$. We consider $T_M(m)$ as a linear subspace of the tangent space $T_{E^n}(m)$ of E^n at the same point m . Then, because of $T_{E^n}(m)$ has an Euclidean inner product, $T_M(m)$ inherits an inner product from $T_{E^n}(m)$. This inner product is called the induced Riemannian metric and denoted by ds^2 in classical books on differential geometry.

Let (Ψ, U) be a coordinate neighborhood for the submanifold M . This means that the mapping

$$\Psi : U \longrightarrow M$$

is a diffeomorphism. Then

$$(II. 1) \quad \Psi_* : T_{E^r}(u) \longrightarrow T_M(\Psi(u))$$

is a linear transformation which corresponds to the Jacobian matrix of Ψ . We denote the adjoint of Ψ_* by Ψ^* which is a transformation

$$(II. 2) \quad \Psi^*: T_M^*(\Psi(u)) \longrightarrow T_{E^r}^*(u)$$

where $T_M^*(\Psi(u))$ and $T_{E^r}^*(u)$ are the dual spaces of $T_M(\Psi(u))$ and $T_{E^r}(u)$, respectively. The vector spaces $T_M^*(\Psi(u))$ and $T_{E^r}^*(u)$ are the cotangent spaces at the corresponding points.

b) *Vector fields and forms*

Definition II. 1:

Let U be a Euclidean neighborhood. A 1-form w is a mapping

$$w: U \longrightarrow \bigcup T_U^*(x), \quad x \in U$$

where the union is taken over all $x \in U$, such that $p \circ w: U \longrightarrow U$ is the identity mapping;

$$p: \bigcup T_U^*(x) \longrightarrow U, \quad t_x \in T_U^*(x),$$

$$p(t_x) = x.$$

Definition II. 2:

A vector field is a function

$$X: U \longrightarrow \bigcup T_U(m)$$

such that

$$p \circ X: U \longrightarrow U$$

is the identity mapping and

$$p: \bigcup T_U(x) \longrightarrow U,$$

$$p(t_x) = x, \quad t_x \in T_U(x).$$

Let $x = (x_1, \dots, x_n)$ be a Euclidean coordinate system in E^n then $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ is a basis of the vector space κ_p of all the parallel vector fields on E^n and $\{dx_1, \dots, dx_n\}$ is the dual basis of dual space Ω of κ_p . Let M be a r -dimensional submanifold of E^n with local coordinates (u_1, \dots, u_r) given by

$$x_i = x_i(u_1, \dots, u_r), \quad 1 \leq i \leq n.$$

Then a 1-form on M has the analytic expression

$$(II. 3) \quad w = \sum_{i,j} \frac{\partial x_i}{\partial u_j} du_j \otimes \frac{\partial}{\partial x_i}$$

where \otimes denotes the tensor product.

c) *Vector-Valued Forms*

Definition II.3 :

Let M be a submanifold of E^n and W be a q -dimensional vector space. A vector-valued form on M with values in W is a mapping A defined as

$$A(m) : T_M(m) \xrightarrow{\text{linear}} W.$$

According to Definition II. 3 the 1-form w in the definition II. 1 is a vector valued form. Now we need the following theorem [2]:

Theorem II.1:

Let A be a vector-valued form on M with values in W , and let w_1, \dots, w_q be a basis of W . Then, there exists a uniquely determined ordered set of q 1-forms $\theta_1, \dots, \theta_q$ on M such that

$$(II. 4) \quad A = \sum_{i=1}^q \theta_i \otimes w_i$$

which is defined by

$$(II. 5) \quad A(t_m) = \sum_{i=1}^q \langle t_m, \theta_i(m) \rangle w_i, \quad \forall t_m \in T_M(m).$$

d) *Reparameterization of a Curve in E^n*

Let $\alpha: [a, b] \rightarrow E^n$, where $[a, b]$ is the interval $a \leq t \leq b$, be a parameterized curve with parameter t such that $\alpha_* \left(\frac{\partial}{\partial t} \right) \neq 0$ for all $t \in [a, b]$ i. e. α is an immersion.

Definition II. 4:

Let $\alpha: [a, b] \rightarrow E^n$ be a parameterized curve with parameter t . Let $[a', b']$ be an interval with parameter s , and let

$$[a', b'] \xrightarrow{\eta} [a, b] \xrightarrow{\alpha} E^n,$$

where η has a nonvanishing Jacobian. Then, $\alpha \circ \eta$ is a parameterized curve with parameter s . The curve $\alpha \circ \eta$ is called a reparameterization of the curve α .

Definition II. 5:

Let $\alpha: [a, b] \longrightarrow E^n$ be a parameterized curve with parameter s . The parameterization of α is called the arc-length parameter if $\alpha^* \left(\frac{\partial}{\partial s} \right)$ has length one in $T_M(\alpha(s))$ for all $a \leq s \leq b$.

Now, we can give a well-known theorem [2].

Theorem II. 2:

A parameterized curve can always be reparameterized by an arc-length parameter.

e) Equations of Structure for \mathcal{F}_0

Let \mathcal{F}_0 be the orthonormal frame bundle over E^n . If (x_1, \dots, x_n) is a Euclidean coordinate system for E^n , we will choose $f_0 \in \mathcal{F}_0$

$$f_0 = \left(\frac{\partial}{\partial x_1} |_0, \dots, \frac{\partial}{\partial x_n} |_0 ; 0 \right).$$

Let r denotes a rigid motion in E^n , then r has the matrix representation

$$r = \begin{bmatrix} g & a \\ 0 & 1 \end{bmatrix}, \quad g = [g_{ij}],$$

where g is an orthogonal $n \times n$ matrix and a is a translation i. e. $n \times 1$ matrix.

$$\text{Let } f = r(f_0) = (e_1, \dots, e_n ; a)$$

where

$$e_i = \sum_{j=1}^n g_{ji} \frac{\partial}{\partial x_j} |_a.$$

A vector field X_i over \mathcal{F}_0 is defined by

$$X_i(f) = e_i.$$

Then a vector-valued form defined by

$$(II. 6) \quad dX_i = \sum_{k=1}^n \varnothing_{ik} \otimes X_k$$

can be expressed in the form [2]

$$(II. 7) \quad dX = \sum_{k=1}^n \varphi_k \otimes X_k$$

where

$$1 \leq i, k \leq n$$

$$(II. 8) \quad dX_i = \sum_{k=1}^n \varphi_{ik} \otimes X_k$$

$$(II.9) \quad \begin{cases} d\varphi_i = \sum_{s=1}^n \varphi_s \wedge \varphi_{si} \\ d\varphi_{ij} = \sum_{s=1}^n \varphi_{is} \wedge \varphi_{sj}, \quad 1 \leq i, s, j \leq n, \\ \varphi_{ij} = -\varphi_{ji}. \end{cases}$$

Formulas II.6, II. 7, II.8 and II.9 are called the equations of structure of \mathcal{F}_0 .

III. HIGHER CURVATURES OF A CURVE IN E^n

a) Frenet Frame Along A Curve.

Let (α, S) be a local coordinat system for an arc-length curve of E^n with the arc-length s as parameter, such that S is an open interval, i. e.,

$$S = \{s: a < s < b, s \in \mathbb{R}\}.$$

We further restrict our attention to curves

$$\alpha: S \longrightarrow E^n$$

which are immersions, i.e, which satisfy $\alpha_* \left(\frac{\partial}{\partial s} \right) \neq 0$. Then clearly α has the property that

$$\| \alpha_* \left(\frac{\partial}{\partial s} \right) \| = 1, \quad \forall s \in S.$$

Let define the vector field X_1 as follows

$$(III.1) \quad X_1 = \alpha_* \left(\frac{\partial}{\partial s} \right).$$

Then X_1 is the unit tangent vector to the curve α in the direction of increasing s .

If $\alpha(s) = (x_1(s), \dots, x_n(s))$ is Euclidean coordinate system of E^n then from [2, Theorem II. 6. 4] we have that

$$(III.2) \quad X_1 = \sum_{i=1}^n \frac{dx_i}{ds} \frac{\partial}{\partial x_i},$$

and the corresponding vector-valued form is

$$(III.3) \quad dX_1 = \sum_{i=1}^n d\left(\frac{dx_i}{ds}\right) \otimes \frac{\partial}{\partial x_i}$$

where \otimes denotes the tensor product. By means of Theorem II.1 we may write that

$$\begin{aligned} dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) &= \sum_{i=1}^n \left\langle \alpha_* \left(\frac{\partial}{\partial s} \right), d\left(\frac{dx_i}{ds}\right) \left(\frac{\partial}{\partial s} \right) \right\rangle \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^n \left\langle \alpha_* \left(\frac{\partial}{\partial s} \right), \frac{d^2 x_i}{ds^2} ds \left(\frac{\partial}{\partial s} \right) \right\rangle \frac{\partial}{\partial x_i} \end{aligned}$$

or since we have that $ds \left(\frac{\partial}{\partial s} \right) = 1$

$$= \sum_{i=1}^n \left\langle \alpha_* \left(\frac{\partial}{\partial s} \right), \frac{d^2 x_i}{ds^2} \right\rangle \frac{\partial}{\partial x_i}.$$

or since we have that

$$dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \alpha^* \left(dX_1 \left(\frac{\partial}{\partial s} \right) \right)$$

and α^* is a linear transformation

$$dX_1 \left(\frac{\partial}{\partial s} \right) = \sum_{i=1}^n (\alpha^*)^{-1} \left\langle \alpha_* \left(\frac{\partial}{\partial s} \right), \frac{d^2 x_i}{ds^2} \right\rangle \frac{\partial}{\partial x_i}$$

$$(III.4) \quad dX_1 = \sum_{i=1}^n (\alpha^*)^{-1} \frac{d^2 x_i}{ds^2} ds \otimes \frac{\partial}{\partial x_i}.$$

From (III.4) we that

$$dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \left(\sum_{i=1}^n (\alpha^*)^{-1} \frac{d^2 x_i}{ds^2} ds \otimes \frac{\partial}{\partial x_i} \right) \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right)$$

$$\begin{aligned}
&= \sum_{i=1}^n (\alpha^*)^{-1} \frac{d^2 x_i}{ds^2} ds \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \frac{d^2 x_i}{ds^2} (\alpha^*)^{-1} (ds) \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \frac{\partial}{\partial x_i}
\end{aligned}$$

$\alpha_* \left(\frac{\partial}{\partial s} \right)$ and $(\alpha^*)^{-1} (ds)$ are dual basis for each other we have that

$$(\alpha^*)^{-1} (ds) \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = 1$$

and so

$$(III.5) \quad dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \sum_{i=1}^n \frac{d^2 x_i}{ds^2} \frac{\partial}{\partial x_i}.$$

The vector field $\frac{dX_1}{ds}$ along the curve α is completely determined by the curve and the concept of parallel displacement in E^n .

If we let

$$(III.6) \quad Y_2(\alpha(s)) = dX_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right)$$

Y_2 is a vector field on $\alpha(S)$ and then by means of (III.5) we have that

$$(III.7) \quad Y_2(\alpha(s)) = \sum_{i=1}^n \frac{d^2 x_i}{ds^2} (s) \frac{\partial}{\partial x_i} \Big|_{\alpha(s)}.$$

Let Y_j be defined by

$$(III. 8) \quad Y_j = \sum_{i=1}^n \frac{d^j x_i}{ds^j} \frac{\partial}{\partial x_i}, \quad 1 < j \leq n.$$

Then we suppose that the system

$$\{X_1, Y_2, \dots, Y_n\}$$

of vector fields is linearly independent. Since $T_{E^n}(\alpha(s))$ is an inner-product vector space with a basis $\{X_1, Y_2, \dots, Y_n\}$, there exists an algorithm, called the Gram-Schmidt process, for converting $\{X_1, Y_2, \dots, Y_n\}$ into a coherently oriented orthonormal frame $\{X_1, X_2, \dots, X_n\}$ which is an orthonormal basis of $T_{E^n}(\alpha(s))$.

Since $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ is also another positive oriented orthonormal basis of $T_{E^n}(a(s))$ these two bases $\{X_i\}$ and $\left\{ \frac{\partial}{\partial x_i} \right\}$ are related by

$$(III. 9) \quad X_i = \sum_{j=1}^n k_{ji} \frac{\partial}{\partial x_j}, \quad 1 \leq i \leq n.$$

We have thus proven the following theorem.

Theorem III.1:

Let α be a curve in E^n with arc - length parameter s and

$$X_1 = \alpha_* \left(\frac{\partial}{\partial s} \right). \text{ If } \frac{dX_1}{ds} = dX_1 \left(\frac{\partial}{\partial s} \right) \neq 0, \text{ where } dX_1 \text{ is}$$

the vector - valued form determined by the vector field X_1 and Euclidean parallelism, at any point of α , we have a positive oriented orthonormal frame X_1, X_2, \dots, X_n at each point of α .

Definition III. 1:

Using the above notation, $\{X_1, \dots, X_n\}$ is called the Frenet frame along α .

b) Frenet Formulas And The Higher Curvatures.

Definition III. 2:

A mapping

$$C_f : \alpha(S) \longrightarrow \mathcal{F}_\alpha^+$$

defined by

$$C_f(\alpha(s)) = (X_1(\alpha(s)), \dots, X_n(\alpha(s)) : \alpha(s)) \in \mathcal{F}_\alpha^+$$

is called the Frenet cross section.

If $\varnothing_{ij}, \varnothing_i, 1 \leq i, j \leq n$, are the 1-forms defined on \mathcal{F}_α^+ then the 1-forms defined on S are

$$(III. 10) \quad \begin{cases} (C_f)^* (\varnothing_{ij}) = \varnothing_{ij}^{C_f} \\ (C_f)^* (\varnothing_i) = \varnothing_i^{C_f} \end{cases}$$

The basis of the vector space of all 1-forms on S is $\{ds\}$. Therefore there are differentiable functions

$$t_{ij} : S \longrightarrow \mathbb{R}$$

such that

$$(III.11) \quad \alpha^*(\varnothing_{ij}^{C_f}) = t_{ij} ds, \quad 1 \leq i, j \leq n.$$

On the other hand from (II.8) we have that

$$(III.12) \quad dX_i = \sum_{j=1}^n \varnothing_{ij}^{C_f} \otimes X_j \quad 1 \leq i \leq n.$$

Replacing (III.11) in (III.12) we obtain that

$$(III.13) \quad dX_i = \sum_{j=1}^n (\alpha^*)^{-1}(t_{ij} ds) \otimes X_j$$

and so applying the same computation for (III.5) we obtain from (III.13) that

$$(III.14) \quad dX_i(\alpha_* \left(\frac{\partial}{\partial s} \right)) = \sum_{j=1}^n t_{ij}(s) X_j.$$

If we have the notation

$$dX_i \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = X'_i$$

according to Herman Gluck, the expressions (III.13) and (III.14) are the Frenet Formulas. Since we know [1] that

$$(III.15) \quad \begin{cases} X'_i(s) = -t_{i-1}(s) X_{i-1}(s) + t_i(s) X_{i+1}(s), & 2 \leq i \leq n-1 \\ X'_n(s) = -t_{n-1}(s) X_{n-1}(s) \end{cases}$$

then the matrix $[t_{ij}]$ has the form

$$(III.16) \quad \begin{bmatrix} 0 & t_{12} & 0 & \dots & 0 & 0 & 0 \\ -t_{12} & 0 & t_{23} & \dots & 0 & 0 & 0 \\ 0 & -t_{23} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_{(n-2)(n-1)} & 0 & t_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 & t_{(n-1)n} & 0 \end{bmatrix}.$$

The 1-forms $\alpha^*(\varnothing_{ij}^{C_f})$ have the expressions [1, pp: 105, formula V]:

$$(III.17) \quad \alpha^*(\varnothing_{ij}^{C_f}) = \sum_{\lambda=1}^n k_{\lambda j} \frac{dk_{\lambda i}}{ds} ds.$$

Equations (III.11) and (III.17) give us that

$$(II.18) \quad t_{ij} = \sum_{\lambda=1}^n k_{\lambda j} \frac{dk_{\lambda i}}{ds}.$$

Definition III.3:

Using the above notation the coefficients t_{ij} are called the higher curvatures of the curve α .

c) *Special Case:*

If $n = 3$ then the matrix (III.16) becomes

$$\begin{bmatrix} 0 & t_{12} & 0 \\ -t_{12} & 0 & t_{23} \\ 0 & -t_{23} & 0 \end{bmatrix}$$

where t_{12} is from (III.18)

$$t_{12} = \sum_{\lambda=1}^3 k_{\lambda 2} \frac{dk_{\lambda 1}}{ds}.$$

On the otherhand, from (III.2), we may write that

$$X_1 = \sum_{\lambda=1}^3 k_{\lambda 1} \frac{\partial}{\partial x_{\lambda}} = \sum_{\lambda=1}^3 \frac{dx_{\lambda}}{ds} \frac{\partial}{\partial x_{\lambda}}$$

and from (III.8).

$$X_2 = \sum_{\lambda=1}^3 k_{\lambda 2} \frac{\partial}{\partial x_{\lambda}} = \frac{1}{\|Y_2\|} \sum_{\lambda=1}^3 \frac{d^2 x_{\lambda}}{ds^2} \frac{\partial}{\partial x_{\lambda}}.$$

Thus we can write that

$$\begin{aligned} t_{12} &= \sum_{\lambda=1}^3 \frac{1}{\|Y_2\|} \frac{d^2 x_{\lambda}}{ds^2} \frac{d^2 x_{\lambda}}{ds^2} \\ &= \frac{1}{\|Y_2\|} \sum_{\lambda=1}^3 \left(\frac{d^2 x_{\lambda}}{ds^2} \right)^2 \\ &= \frac{1}{\|Y_2\|} \cdot \|Y_2\|^2 \\ (III.19) \quad t_{12} &= \sqrt{\sum_{\lambda=1}^3 \left(\frac{d^2 x_{\lambda}}{ds^2} \right)^2} \end{aligned}$$

Then t_{12} is the first curvature of the curve. In a similar way we see that

$$(III.20) \quad |t_{23}| = \left\| dx_3 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \right\| = \| X'_3 \|$$

and so t_{23} is the second curvature (torsion) of the curve.

d) The Invariance Of The Curvatures t_{ij} .

The curvatures t_{ij} of the curve α are invariant under the rigid motions of E^n .

The distance-preserving mappings of E^n are called rigid motions. The set of all rigid motions of E^n will be denoted by $R(n)$.

Let $r \in R(n)$ and the Frenet frame of the curve $r(\alpha(S))$ be the frame $\{X^r_1, X^r_2, \dots, X^r_n\}$. Thus Equation (III.14), for the curve $r(\alpha(S))$, becomes

$$(III.14)' \quad dX^r_1 \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) = \sum_{j=1}^n t^r_{ij} X^r_j.$$

We want to show that $t_{ij} = t^r_{ij}$.

Let the local coordinat system of $r(\alpha(S))$ be $(r\alpha, S)$. Then for this new curve (III.1) reduces to

$$(III.1)' \quad X^r_1 = (r\alpha)_* \left(\frac{\partial}{\partial s} \right).$$

Indeed $\|X^r_1\| = 1$. To show this we use the fact that

$$(r\alpha)_* = r_* \circ \alpha_*$$

and so since we know that $r \in R(n)$

$$\|X^r_1\| = \left\| r_* \left(\alpha_* \left(\frac{\partial}{\partial s} \right) \right) \right\| = \left\| \alpha_* \left(\frac{\partial}{\partial s} \right) \right\| = 1.$$

Since $r \in R(n)$ has the matrix form

$$r = \begin{bmatrix} g & a \\ 0 & 1 \end{bmatrix}$$

as we see in the paragraph II we may have that

$$\begin{aligned}
(III.8)' \quad Y_j &= \sum_{i=1}^n \frac{d^j \left(\sum_{\lambda=1}^n g_{i\lambda} x_\lambda \right)}{ds^j} \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{\lambda=1}^n g_{i\lambda} \frac{d^j x_\lambda}{ds^j} \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \left(\sum_{\lambda=1}^n g_{i\lambda} \frac{d^j x_\lambda}{ds^j} \right) \frac{\partial}{\partial x_i} \\
&= r^* (Y_j)
\end{aligned}$$

and so

$$(X_1^r, \dots, X_n^r) = (r_*(X_1), \dots, r_*(X_n)).$$

Definition III.2 gives us the following Frenet Cross section for the curve $r(a(S))$:

$$C_f^r : (roa)(S) \longrightarrow \mathcal{F}_0$$

which is defined by

$$C_f^r(s) = (r_*(X_1(s)), \dots, r_*(X_n(s)); r(a(s))).$$

Hence Equation (III.11), for this new curve, reduces to

$$(III.11)' \quad (roa)^* (\varnothing_{ij}^{C_f^r}) = a^* (r^* (\varnothing_{ij}^{C_f^r}) = t_{ij}^r ds.$$

And finally as a similar way for Equation (III.17) we have

$$\begin{aligned}
(roa)^* (\varnothing_{ij}^{C_f^r}) &= \sum_{\lambda=1}^n \left(\sum_{t=1}^n g_{\lambda t} k_{tj} \right) \frac{d \left(\sum_{\mu=1}^n g_{\lambda \mu} k_{\mu i} \right)}{ds} ds \\
&= \sum_{\lambda=1}^n \sum_{t=1}^n g_{\lambda t} k_{tj} \sum_{\mu=1}^n g_{\lambda \mu} \frac{dk_{\mu i}}{ds} ds \\
&= \sum_{t=1}^n k_{tj} \left(\sum_{\mu, \lambda} g_{\lambda t} g_{\lambda \mu} \right) \frac{dk_{\mu i}}{ds} ds \\
&= \sum_{t=1}^n k_{tj} \delta_{\lambda \mu} \frac{dk_{\mu i}}{ds} ds
\end{aligned}$$

$$= \sum_{i=1}^n k_{tj} \frac{dk_{ti}}{ds} ds$$

$$(III.21) \quad (\text{roa})^* (\varnothing_{ij}^{C_f}) = \alpha^* (\varnothing_{ij}^{C_f}).$$

Using the Equations (III.11), (III.11)' and (III.21) we have that

$$t_{ij}^r = t_{ij}$$

which completes the proof.

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ÖZET

n-boyutlu Öklid uzayında bir eğrinin yüksek mertebeden eğrilikleri Herman Gluck tarafından hesaplanmıştır. Bu çalışmada aynı eğriliklerin hesaplanması için yeni bir metot verilmektedir. Ayrıca bu eğriliklerin katı hareketler altında değişmez kaldıkları da bu yeni metodun bir sonucu olarak gösterilmektedir.

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