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On L^1 Convergence Of A Walsh-sum

by

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On L^1 Convergence Of A Walsh-sum

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SUMMARY

It is shown that limit of the Walsh sum $g_n(x) = \sum_{k=o}^{n} \lambda^n_k \Psi_k(x)$, $0<x<1$,, exists and is integrable if and only if $\lim_{n\to\infty} \lambda_o^n < \infty$. Then g_n converges in L^1 norm to g and for $0 < p < 1$, $\lim \int_0^1 | g(x) - g_n(x) |^p dx = 0$.

1. The Rademacher functions are defined by

$$\varnothing_o(x) = 1 \ (0 \leq x < 1/2), \ \varnothing_o(x) = -1 \ (1/2 \leq x < 1)$$
$$\varnothing_o(x+1) = \varnothing_o(x), \ \ \varnothing_n(x) = \varnothing_o(2^n x) \ (n = 1, 2, \ldots\ldots).$$

Then the Walsh-functions are defined by

$\Psi_o(x) = 1$, $\Psi_n(x) = \varnothing_{n_1}(x) \varnothing_{n_2}(x) \varnothing_{n_3}(x) \ldots \varnothing_{n_r}(x)$, for $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$.

Let $\{\lambda_k^n\}$, $0 \leq k \leq n$ be a double sequence of real numbers satisfying the following conditions:

(1.1) $\lim_{n\to\infty} \lambda^n_n = 0$

(1.2) $\lim_{m\to\infty} (\lambda_k^m - \lambda^m_{n+1}) = \lambda^n_k$ if $k \leq n$

(1.3) $\lim_{n\to\infty} (\lambda_o^n - n\lambda^{n-1}_{n-1}) < \infty$ if and only if $\lim_{n\to\infty} \lambda_o^n < \infty$

(1.4) $\lim_{n\to\infty} (\Delta_k \lambda_k^n)_{k=n-1} = 0$ where $\Delta_k \lambda_k^n = \lambda_k^n - \lambda^n_{k+1}$

(1.5) $\Delta^2_k \lambda_k^{n+1} \geq \Delta^2_k \lambda_k^n \geq 0$ if $k \leq n\text{-}2$, where

$$\Delta^2_k \lambda^n_k = \Delta_k \lambda_k^n - \Delta_{k+1} \lambda^n_{k+1}$$

(1.6) $\Delta_k \lambda^n_k \geq 0$.

In the present paper, we prove the following theorems for Walsh sum $\Sigma^n_{k=o} \lambda^n_k \Psi_k (x)$.

Theorem 1. *The limit of the Walsh sum* $\Sigma^n_{k=o} \lambda^n_k \Psi_k (x)$, $0 < x < 1$, *exists and is integrable of and only if* $\lim_{n\to\infty} \lambda^n_o < \infty$.

Theorem 2. *If the limit* g *of* $g_n (x) = \sum_{k=0}^{n} \lambda_k^n \Psi_k(x)$ *is integrable, then* g_n *converges in* L^1 *norm to* g.

Theorem 3. *If the limit* g *of* $g_n (x) = \sum_{k=o}^{n} \lambda^n_k \Psi_k (x)$ *is integrable, then for* $0 < p < 1$ *we have*

$$\lim_{n\to\infty} \int_0^1 | g (x) - g_n (x) |^p dx = 0.$$

It may be remarked that Theorem 1 and 2 are analogous to the results of Garrett ([3], Theorems 1 and 2) for the genaralised Rees-Stanojevic cosine sum in the form $\frac{1}{2} \lambda^n_o + \sum_{k=1}^{n} \lambda^n_k \cos kx$ while Theorem 3 to a known result for trigonometric series ([1], p. 215).

2. We require the following Lemmas for the proof of our theorems:

Lemma 1 [2]. (i) *If* $D_n (x) = \sum_{k=o}^{n-1} \Psi_k (x)$, *then* $| D_n (x) | < 2/x$ *for* $0 < x < 1$.

(ii) *If* $K_n (x) = \frac{1}{n+1} \sum_{k=o}^{n} D_k X(x)$, *then* $| K_n (x) | < 2/x$ *for all* n.

Lemma 2 [4]. *If* $K_n(x) = \frac{1}{n+1} \sum_{k=o}^{n} D_k (x)$, *then*

$$\int_0^1 | K_n(x) | \, dx \leq 2, \text{ for all } n.$$

3. *Proof of Theorem* 1. Let

$$g_n(x) = \sum_{k=o}^{n} \lambda^n_k \Psi_k(x)$$

Applying Abel's transformation twice we have

$$(3.1) \quad g_n(x) = \sum_{k=o}^{n-1} \Delta_k \lambda^n_k D_{k+1}(x) + \lambda^n_n D_{n+1}(x)$$

$$= \sum_{k=o}^{n-2} (k+1) \Delta^2_k \lambda^n_k K_k(x) + n [\Delta_k \lambda^n_k]_{k=n-1} K_{n-1}(x) + \lambda^n_n D_{n+1}(x).$$

Since $| D_n(x) | < 2/x$, $| K_n(x) | \leq 2/x$ $(0 < x < 1)$, $\lim_{n\to\infty} \lambda^n_n = 0$ and $\lim_{n\to\infty} [\Delta_k \lambda^n_k]_{k=n-1} = 0$; the last two terms in (3.1) tend to zero in the limit. Finally,

$$\lim_{n\to\infty} \sum_{k=o}^{n-2} (k+1) \Delta^2_k \lambda^n_k K_k(x)$$

$$\leq \lim_{n\to\infty} \frac{2}{x} \sum_{k=o}^{n-2} (k+1) \Delta^2_k \lambda^n_k$$

$$= \lim_{n\to\infty} \frac{2}{x} \Big[\sum_{k=o}^{n-2} \Delta_k \lambda^n_k - (n-1) \Delta_{n-1} \lambda^n_{n-1} \Big]$$

$$= \lim_{n\to\infty} \frac{2}{x} [\lambda^n_o - \lambda^n_{n-1} - (n-1)(\lambda^n_{n-1} - \lambda^n_n)]$$

$$= \lim_{n\to\infty} \frac{2}{x} [\lambda^n_o - n(\lambda^n_{n-1} - \lambda^n_n) - \lambda^n_n]$$

$$= \lim_{n\to\infty} \frac{2}{x} [\lambda^n_o - n \lim_{m\to\infty} (\lambda^m_{n-1} - \lambda^m_{n+1} - \lambda^m_n + \lambda^m_{n+1}) - \lambda^n_n]$$

$$= \lim_{n\to\infty} \frac{2}{x} [\lambda^n_o - n \lambda^{n-1}_{n-1} - \lambda^n_n]$$

$$= \lim_{n\to\infty} \frac{2}{x} (\lambda^n_o - n\, \lambda^{n-1}_{n-1}) < \infty \text{ if and ony if}$$

$$\lim_{n\to\infty} \lambda^n_o < \infty.$$

It follows therefore, that $\lim_{n\to\infty} g_n(x)$ exists if and only if $\lim_{n\to\infty} \lambda^n_o < \infty$.

Now let $g(x) = \lim_{n\to\infty} g_n(x)$. Then $g(x) = \lim_{n\to\infty}$

$$\sum_{k=o}^{n-2} (k+1)\, \Delta^2_k\, \lambda^n_k\, K_k(x).$$

Since $\sum_{k=o}^{n-2} (k+1)\, \Delta^2_k\, \lambda^n_k\, K_k(x)$ is nondecreasing, using Lemma 2, we have

$$\int_0^1 |\, g(x)\, |\, dx = \lim_{n\to\infty} \int_0^1 \sum_{k=o}^{n-2} (k+1)\, \Delta^2_k \lambda^n_k\, K_k(x)\, dx$$

$$\leq 2 \lim_{n\to\infty} \sum_{k=o}^{n-2} (k+1)\, \Delta^2_k\, \lambda^n_k$$

$$= 2 \lim_{n\to\infty} [\lambda^n_o - n\lambda^{n-1}_{n-1} - \lambda^n_n]$$

$$< \infty \text{ if and only if } \lim_{n\to\infty} \lambda^n_o < \infty$$

Proof of Theorem 2. Since g_n is integrable, therefore by Theorem 1, we have $\lim_{n\to\infty} \lambda^n_o < \infty$. Then it follows from $\lambda_o^{k-1} = \lim_{n\to\infty} [\lambda^n_o - \lambda^n_k]$ that $\lim_{n\to\infty} \lambda^n_k < \infty$ for all $k \geqslant 0$. We denote $\lim_{n\to\infty} \lambda^n_k$ by λ_k^∞. Then

$$g_n(x) = \sum_{k=o}^{n} \lambda^n_k\, \Psi_k(x)$$

$$= \sum_{k=o}^{n} (\lambda_k^\infty - \lambda_{n+1}^\infty)\, \Psi_k(x)$$

$$= \sum_{k=o}^{n} \lambda_k^\infty\, \Psi_k(x) - \lambda_{n+1}^\infty\, D_{n+1}(x).$$

Applying summation by parts we hawe

$$g_n(x) = \sum_{k=o}^{n-1} \Delta_k \lambda^\infty{}_k D_{k+1}(x) + \lambda^\infty{}_n D_{n+1}(x) - \lambda^\infty{}_{n+1} D_{n+1}(x)$$

$$= \sum_{k=o}^{n} \Delta_k \lambda_k{}^\infty D_{k+1}(x)$$

$$= \sum_{k=o}^{n-1} (k+1) \Delta^2{}_k \lambda^\infty{}_k K_k(x) + (n+1) \, [\Delta_k \lambda^\infty{}_k]_{k=n} K_n(x).$$

Since $\lim_{n\to\infty} (\Delta_k \lambda_k{}^\infty)_{k=n} = \lim_{n\to\infty} (\lambda_n{}^\infty - \lambda^\infty{}_{n+1}) = \lim_{n\to\infty} \lambda^n{}_n = 0,$

we have

$$g(x) = \lim_{n\to\infty} g_n(x) = \sum_{k=o}^{\infty} (k+1) \Delta^2{}_k \lambda^\infty{}_k K_k(x).$$

Then

$$\lim_{n\to\infty} \int_0^1 | \, g(x) - g_n(x) \, | \, dx$$

$$= \lim_{n\to\infty} \int_0^1 | \sum_{k=n}^{\infty} (k+1) \Delta^2{}_k \lambda^\infty{}_k K_k(x) - (n+1)(\Delta_k \lambda^\infty{}_k)_{k=n} K_n(x) \, | dx$$

$$= \lim_{n\to\infty} [\sum_{k=n}^{\infty} (k+1) \Delta^2{}_k \lambda^\infty{}_k \int_0^1 K_k(x) dx + (n+1)(\Delta_k \lambda^\infty{}_k)_{k=n} \int_0^1 K_n(x) \, dx]$$

$$\leq 2 \lim_{n\to\infty} [\sum_{k=n}^{\infty} (k+1) \Delta^2{}_k \lambda^\infty{}_k + (n+1) (\Delta_k \lambda^\infty{}_k)_{k=n}]$$

$$= 2 \lim_{n\to\infty} [\sum_{k=n}^{\infty} \Delta_k \lambda^\infty{}_k + n(\Delta_k \lambda^\infty{}_k)_{k=n} + (n+1)(\Delta_k \lambda^\infty{}_k)_{k=n}]$$

$$= 2 \lim_{n\to\infty} [\sum_{n=k}^{\infty} \Delta_k \lambda_k{}^\infty + (2n+1) (\Delta_k \lambda_k{}^\infty)_{k=n}] = 0.$$

since $\sum_{k=1}^{\infty} \Delta_k \lambda_k{}^\infty = \lambda_1{}^\infty < \infty$ and $\Delta^2{}_k \lambda_k{}^\infty = \lim_{m\to\infty} \Delta^2{}_k \lambda^m{}_k \geq 0.$

Proof of Theorem 3. As in the proof of Theorem 2, we have by using Lemma 1

$$\lim_{n\to\infty} \int_0^1 | g(x) - g_n(x) |^p dx$$

$$= \lim_{n\to\infty} \int_0^1 | \sum_{k=n}^{\infty} (k+1)\Delta^2{}_k\lambda_k{}^{\infty} K(x)-(n+1) (\Delta_k\lambda^{\infty}{}_k)_{k=n}K_n(x) |^p dx$$

$$\leq \lim_{n\to\infty} 2^p \left[\sum_{k=n}^{\infty} | (k+1)\Delta^2{}_k\lambda^{\infty}{}_k | + (n+1) (\Delta_k\lambda_k{}^{\infty})_{k=n} \right] \int_0^1 x^{-p} dx$$

$$= 2^p \lim_{n\to\infty} \left[\sum_{k=n}^{\infty} \Delta_k\lambda^{\infty}{}_k + (2n+1) (\Delta_k\lambda^{\infty}{}_k)_{k=n} \right] \int_0^1 x^{-p} dx = 0.$$

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ÖZET

Gösteriliyor ki $g_n(x) = \sum_{k=0}^{n} \lambda^n{}_k \Psi_k(x)$, $0< x< 1$, Walsh toplamı, mevcuttur ve integrallenebilir yalnız ve yalnız $\lim_{n\to\infty} \lambda_0{}^n < \infty$. Bu taktirde g_n , L^1 normunda g ye yakımsaktır ve $0 < p < 1$ ise $\lim_{n\to\infty} \int_0^1 | g(x) - g_n(x)|^p dx = 0.$

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