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A Theorem on the Integrability of Power Series

by

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A Theorem on the Integrability of Power Series

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ABSTRACT

The aim of this paper is to obtain a most general generalization of Heywood's theorem [1] where a_n is not ultimately positive.

1. P. Heywood [(1), Theorem 3 (i)] has proved that if C_n is ultimately positive, that $\sum_{n=o}^{\infty} C_n = o$ and that $F(x) = \sum_{n=o}^{\infty} C_n x^n$ where the radius of convergence of power series is unity for $o \leq x < 1$, then $(1-x)^{-1} F(x)$ is $L(0,1)$ iff $\Sigma C_n \log n$ converges.

The object of this note is to prove more general result of this type, in which we do not assume that C_n is ultimately positive.

2. *Theorem*: If $f(x) = \sum_{n=o}^{\infty} C_n x^n$, where the series is convergent in $o \leq x < 1$. if $\sum_{n=o}^{\infty} C_n = o$, and if $\sum_{n=1}^{\infty} C_n \log n$ is convergent, then $\int_o^{1-\delta} \frac{f(x)}{1-x} dx$ tends to a unique finite limit as $\delta \to +0$, moreover if the series $\sum_{n=1}^{\infty} C_n \log n$ has bounded partial sums then $\int_o^{1-\delta} \frac{f(x)}{1-x} dx$ is bounded as $\delta \to +0$. Conversely, if $\int_o^{1-\delta} \frac{f(x)}{1-x} dx$ tends to a anlque finite limit as $\delta \to +0$ and if $S_n \log n \to \alpha$, as $n \to \infty$, then $\sum_{n=o}^{\infty} C_n \log n$ is

convergent where α is finite and $S_n = \sum_{r=o}^{\infty} Cr$. Moreover if $\int_o^{1-\delta} \frac{f(x)}{1-x}\, dx$ is bounded as $\delta \to +o$, and if $S_n \log n$ is bounded as $n \to \infty$, then the series has bounded partial sums.

Proof: We have

$$\int_o^{1-\delta} \frac{f(x)}{1-x}\, dx = \sum_{n=o}^{\infty} \int_o^{1-\delta} \frac{C_n x^n}{1-x}\, dx$$

$$= -\sum_{n=o}^{\infty} \int_o^{1-\delta} \frac{C_n(1-x^n)}{(1-x)}\, dx$$

$$= -\sum_{n=1}^{\infty} \int_o^{1-\delta} C_n (1 + x + \dots + x^{n-1})\, dx$$

$$= -\sum_{n=1}^{\infty} C_n \left[(1-\delta) + \frac{(1-\delta)^2}{2} + \dots + \frac{(1-\delta)^n}{n}\right]$$

Putting $1-\delta = e^{-A}$, we have

$$I_o = -\sum_{n=1}^{\infty} C_n \left[e^{-A} + \frac{e^{-2A}}{2} + \dots + \frac{e^{-nA}}{n}\right]$$

Also, if $e^{-A} = 1 - \frac{1}{N^2}$, where N is an positive integer, we have

$$\sum_{r=1}^{N+1} \frac{e^{-rA}}{r} = \sum_{r=1}^{N+1} \frac{1}{r}\left[1 - \frac{1}{N^2}\right]^r$$

$$= \sum_{r=1}^{N+1} \frac{1}{r}\left[1 - \frac{r}{N^2} + O\left(\frac{1}{N^2}\right)\right]$$

$$= \sum_{r=1}^{N+1} \frac{1}{r} + O\left(\frac{1}{N}\right)$$

$$= \text{Log } N + r^* + U_{N+1} + O\left(\frac{1}{N}\right)$$

where r^* is Euler's constant and u_n tends to zero as $n \to \infty$. It follows easily that, if N is fixed

$$\limsup_{M \to \infty} \frac{\sum_{r=1}^{M} \frac{e^{-r\Lambda}}{r}}{\operatorname{Log} M} = 0$$

Also, we have $\sum_{n=N+2}^{\infty} \frac{e^{-n\Lambda}}{n \log n} = I_1 + I_2 + I_3$

$$I_1 = \sum_{n=N+2}^{\overline{N}} \frac{e^{-n\Lambda}}{n \log n}, \quad I_2 = \sum_{n=\overline{N}+1}^{N^2} \frac{e^{-n\Lambda}}{n \log n}$$

$$I_3 = \sum_{n=N^2+1}^{\infty} \frac{e^{-n\Lambda}}{n \log n}, \quad \overline{N} = \left[\frac{N^2}{\operatorname{Log}\operatorname{Log} N}\right],$$

where $[x]$ denotes the greatest integer not exceeding x. It follows easily that $I_1 \to \log 2$, $I_2 \to 0$, $I_3 \to 0$, as $n \to \infty$. Now, we have

$$\sum_{n=1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r}\right] = \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r}\right] +$$

$$+ \sum_{n=N+1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r}\right]$$

$$= I_{1,1} + I_{1,2} \text{ say.}$$

$$I_{1,1} = \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r} \left[1 - \frac{1}{N^2}\right]^r\right]$$

$$= \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r} \left(1 - \frac{r}{N^2} + O\left(\frac{1}{N^2}\right)\right)\right]$$

$$= \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r}\right] - \frac{1}{N^2} \sum_{n=1}^{N} n\, C_n + O\left(\frac{1}{N}\right)$$

$$= \sum_{n=1}^{N} C_n (\log n + r^* + u_n) + O(1)$$

$$= \sum_{n=1}^{N} C_n \log n + \sum_{n=1}^{N} C_n u_n - r^* C_0 + O(1)$$

$$= \sum_{n=1}^{N} C_n \log n + H + O(1)$$

where $H = - r^* C_0 + \sum_{n=1}^{\infty} C_n u_n$, r^* is Euler's constant and u_n tends steadily to zero as $n \to \infty$ Also since by hypothesis, $S_n \operatorname{Log} n \to \alpha$ as $n \to \infty$, we have

$$I_{2,2} = \sum_{n=N+1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \sum_{N+1}^{M} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \sum_{N+1}^{M} (S_n - S_{n-1}) \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \left(S_M \sum_{r=1}^{M} \frac{e^{-r\Lambda}}{r} \right) - \lim_{M \to \infty} \sum_{N+2}^{M} S_{n-1} \frac{e^{-n\Lambda}}{n}$$

$$- S_N \sum_{r=1}^{N+1} \frac{e^{-\Lambda r}}{r}$$

$$= \lim_{M \to \infty} \left[(S_M \operatorname{Log} M) \frac{\sum_{r=1}^{M} \frac{e^{-\Lambda r}}{r}}{\operatorname{Log} M} \right]$$

$$- \lim_{M \to \infty} \sum_{N+2}^{M} (S_{n-1} \operatorname{Log} n) \frac{e^{-n\Lambda}}{n \log n}$$

$$- (S_N \operatorname{Log} N) \frac{\sum_{r=1}^{N+1} \frac{e^{-r\Lambda}}{r}}{\operatorname{Log} N}$$

and since we have proved that

$\sum_{n+2}^{\infty} \frac{e^{-n\Lambda}}{n \log n} \longrightarrow \text{Log } 2$, as $N \to \infty$ then by (2.1) it follows that $\sum_{n=1}^{\infty} C_n \text{ Log } n$ converges or oscillates finitely, according as $\int_0^{1-\delta} \frac{f(x)}{1-x} dx$ converges or oscillates finitely as $\delta \to +0$

Since I_0 is bounded as $\delta \to +0$, therefore $I_{2,2}$ is bounded because by hypothesis $S_n \text{ Log } n$ is bounded as $n \to \infty$. Since $I_{1,1} + I_{1,2}$ it follows that $\sum_{n=1}^{\infty} C_n \log n$ has a bounded partial sum because $\sum_{n=1}^{N} C_n \log n$ is bounded for all N.

This completes the proof of our theorem.

I am thankful those persons who helped me in preparing this paper.

ÖZET

Bu çalışmada, a_n bir yerden itibaren sürekli olarak pozitif olmamak üzere, [1] de verilen Heywood Teoreminin en geniş anlamda bir genelleştirilmesi elde edilmiştir.

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