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Radii Of p-Valence Of Certain Analytic Functions With Negative Coefficients

by

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ABSTRACT

In this paper we determine the radii of p-valence of the function F(z) defined by

$$F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z), \quad z \in D$$

where $D = \{z: |z| < 1\}$, $\lambda \geq 0$ and the function f (z) belongs to certain subclasses of analytic p-valent functions with negative coefficients.

1. INTRODUCTION

Let T_p denote the class of functions $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ which are regular in the unit disc $D = \{z: |z| < 1\}$ and T_p^* denote that subclass of T_p whose members are p-valent in D. A function f(z) of T_p belongs to the class T_p^* (A, B) if $zf'(z)/f(z)$ is subordinate to $p(1 + Az)/(1 + Bz)$, $z \in D$, where $-1 \leq A < B \leq 1$. Equivalently $f(z) \in T_p^*$ (A, B) if and only if there exists a function $\omega(z)$ regular in D and satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in D$, such that

$$(1.1) \qquad \frac{zf'(z)}{f(z)} = p \, \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in D$$

It follows from (1.1) that $f(z) \in T_p^*$ (A,B) if and only if

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(\frac{Bzf'(z)}{f(z)} - Ap \right) \right| < 1, \quad z \in D.$$

Further f (z) is said to belong to the class C_p (A, B) if and only if $zf'(z)/p \in T_p^*(A, B)$. It is well known that the functions in T_p^* (A, B) and C_p (A, B) are p-valent starlike and p-valent convex respectively. Let P_p^* (A, B) denote the class obtained by replacing $zf'(z)/f(z)$ by $f'(z)/z^{p-1}$ in the definition of T_p^* (A, B). Clearly $f(z) \in P_p^*(A, B)$ implies $\text{Re}\{f'(z)/z^{p-1}\} > 0$, and hence the functions in P_p^* (A, B) are p-valent in D.

Recently Goel and Sohi [2] have established the following result for the class T_p^* (A, B).

Theorem A. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ belongs to T_p^* (A, B) if and only if

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |a_{n+p}| \leq (B-A)p. \tag{1.2}$$

The result is sharp with the extremal function

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)p}{(1+B)n + (B-A)p} z^{n+p}. \tag{1.3}$$

By using this result Goel and Sohi [2] obtained distortion and covering theorems and some other results for the classes T_p^* (A, B) and C_p (A, B). In the present paper we obtain some new results with the help of above theorem. Before using it we point out that the above theorem is valid only when $B \geq 0$. In fact in its proof Goel and Sohi [2] used the inequality

$$\left| \sum_{n=1}^{\infty} -n \, |a_{n+p}| z^{n+p} \right| - \left| (B-A)pz^p - \sum_{n=1}^{\infty} [nB + (B-A)p] \, |a_{n+p}| z^{n+p} \right|$$

$$\leq \sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |a_{n+p}| - (B-A)p.$$

We find that the above inequality holds only for $B \geq 0$. Since all the results except that of Theorem 2 of Goel and Sohi [2] have been obtained by using Theorem A, it is obvious that these are also valid only for $B \geq 0$.

Further we claim that the function f (z) given by (1.3) is not an extremal function for the purpose, since, in (1.2), equality does not hold for it. In fact for such a f (z)

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \mid a_{n+p} \mid$$

$$= \sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \left[\frac{(B-A)p}{(1+B)n+(B-A)p} \right]$$

$$= \infty$$

$$\neq (B-A)p.$$

We suggest that the function f (z) given by

$$f(z) = z^p - \frac{(B-A)p}{(1+B)n+(B-A)p} z^{n+p}$$

is a suitable extremal function, since, the equality holds in (1.2) for it.

We also need the following result for the class P_p^* (A, B), which is due to Shukla and Dashrath [3].

Theorem B. A function $f(z) = z^p - \sum_{n=1}^{\infty} \mid a_{n+p} \mid z^{n+p}$ belongs to P_p^* (A, B), $B \geq 0$, if and only if

$$\sum_{n=1}^{\infty} (n+p)(1+B) \mid a_{n+p} \mid \leq (B-A)p. \tag{1.4}$$

The result is sharp.

In this paper we determine the radius of p-valence of the function

$$F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z), \ \lambda \geq 0,$$

under the assumption that $B \geq 0$, when f (z) is in T_p^* (A, B), C_p (A, B) or P_p^* (A, B). All the results are sharp and generalize the recent results of Bhoosnurmath and Swamy [1].

Throughout this paper we assume that $B \geq 0$ and $\lambda \geq 0$.

2. MAIN RESULTS

Theorem 1. If $f(z) = z^p - \sum_{n=1}^{\infty} \mid a_{n+p} \mid z^{n+p} \in T_p^*$, then

$$\sum_{n=1}^{\infty} (n+p) \ |a_{n+p}| \leq p.$$

Proof. Suppose $\sum_{n=1}^{\infty} (n+p) \ |a_{n+p}| = p + \varepsilon$, where $\varepsilon > o$.

Then there exists an integer N such that

$$\sum_{n=1}^{N} (n+p) \ |a_{n+p}| > p + \frac{\varepsilon}{2}.$$

For z in the interval $[p/(p+\varepsilon/2)]^{\frac{1}{N}} < z < 1$, we have

$$G(z) = \frac{f'(z)}{z^{p-1}} \leq p - \sum_{n=1}^{N} (n+p) \ |a_{n+p}| \ z^n$$

$$\leq p - z^N \sum_{n=1}^{N} (n+p) \ |a_{n+p}|$$

$$< p - (p + \varepsilon/2) \ z^N$$

$$< 0.$$

Since $G(0) > 0$, there exists a real number z_0, $0 < z_0 < 1$, for which $G(z_0) = \frac{f'(z_0)}{z_0{}^{p-1}} = 0$. But this is contrary to the fact that f (z) is p-valent in D. Hence the required result follows.

Remark. For $p = 1$, our theorem generalizes Theorem 3 of Silverman [4].

Corollary 1. $T_p^* = T_p^*(-1, 1) = P_p^*(-1,1)$.

Theorem 2. Let $f(z) \in T_p^*(A,B)$ and $F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$ for $z \in D$. Then F (z) is p-valently starlike of order δ, $0 < \delta < 1$, for $|z| < r(p, \lambda, \delta, A, B)$, where

$$r(p, \lambda, \delta, A, B) = \inf_n \left[\frac{\{(1+B)\, n + (B-A)\, p\}\,(1-\delta)\, p}{(B-A)\,\{n + p\,(1-\delta)\}\,(p + n\,\lambda)}\right]^{\frac{1}{n}} \quad n = 1, 2, 3, \dots .$$

The result is sharp.

Proof. We have.

$$F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$$

$$= z^p - \sum_{n=1}^{\infty} \left(\frac{p+n\lambda}{p}\right) |a_{n+p}| z^{n+p}.$$

Now it suffices to show that the values of $\frac{zF'(z)}{F(z)}$ lie in a circle centered at p whose radius is p (1 – δ) for $|z| < r(p, \lambda, \delta, A, B)$.

We have

$$\left|\frac{zF'(z)}{F(z)} - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right)|a_{n+p}| z^n}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)|a_{n+p}| z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right)|a_{n+p}|\,|z|^n}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)|a_{n+p}|\,|z|^n}.$$

The last expression is bounded above by $p(1-\delta)$ if

$$\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right)|a_{n+p}|\,|z|^n \leq p(1-\delta)\left\{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)|a_{n+p}|\,|z|^n\right\}$$

or if

$$(2.1)\qquad \sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)\left(\frac{n+p(1-\delta)}{1-\delta}\right)|a_{n+p}|\,|z|^n \leq p.$$

Since $f(z) \in T_p^*(A, B)$, we have from (1.2)

$$\sum_{n=1}^{\infty}\left[\frac{(1+B)n+(B-A)p}{B-A}\right]|a_{n+p}| \leq p.$$

Hence (2.1) holds if

$$\left(\frac{p+n\lambda}{p}\right)\left(\frac{n+p(1-\delta)}{1-\delta}\right)|a_{n+p}|\ |z|^n \leq \left[\frac{(1+B)n+(B-A)}{B-A}\right]|a_{n+p}|$$

or if

$$|z| \leq \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{(B-A)(p+n\lambda)\,\{n+p(1-\delta)\}}\right]^{\frac{1}{n}},\ n = 1, 2, 3, \ldots.$$

The result is sharp for the function

$$f(z) = z^p - \frac{p(B-A)}{(1+B)n+(B-A)p}\,z^{n+p},\ n = 1,2,3, \ldots\ .$$

Corollary 2.1. Let $f(z) \in T_p^*$ and $F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$ for $z \in D$. Then $F(z)$ is p-valently starlike of order δ, $0 \leq \delta < 1$, for $|z| < r(p, \lambda, \delta, -1, 1)$ where

$$r(p, \lambda, \delta, -1, 1) = \inf_n \left[\frac{p(n+p)(1-\delta)}{(p+n\lambda)\,\{n+p(1-\delta)\}}\right]^{\frac{1}{n}},\ n=1, 2, 3, ..$$

The result is sharp.

Corollary 2.2. Let $f(z) \in T_p^*(A, B)$. Then $f(z)$ is p-valently starlike of order δ, $0 \leq \delta < 1$, in

$$|z| < r(p,0,\delta,A,B) = \inf_n \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{p(B-A)\,\{n+p(1-\delta)\}}\right]^{\frac{1}{n}},\ n=1,2,3,\ldots$$

The result is sharp.

Corollary 2.3. Let $f(z) \in T_p^*(A, B)$. Then $f(z)$ is p-valently convex of order δ, $0 \leq \delta < 1$ in

$$|z| < r(p,1,\delta,A,B) = \inf_n \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{(p+n)(B-A)\,\{n+p(1-\delta)\}}\right]^{\frac{1}{n}},\ n=1,2,3,\ldots$$

The result is sharp.

Corollary 2.4. Let $f(z) \in T_p^*(A,B)$ and $c > -p$, then

$$F(z) = \frac{\{z^c f(z)\}'}{(p+c)\,z^{c-1}},$$ for $z \in D$, is p-valently starlike of order δ, $0 \leq \delta < 1$, in

$$|z| < r\left(p, \frac{p}{p+c}, \delta, A, B\right) = \inf_n \left[\frac{\{(1+B)\,n + (B-A)\,p\}\,(1-\delta)(p+c)}{(B-A)(p+c+n)\{n+p\,(1-\delta)\}}\right]^{\frac{1}{n}}$$

$n = 1, 2, 3, \ldots.$

The result is sharp.

Theorem 3. Let $f(z) \in C_p(A, B)$ and $F(z) = (1-\lambda)\, f(z) + \frac{\lambda}{p}\, zf'(z)$ for $z \in D$. Then $F(z)$ is p-valently close-to-convex in D if $\lambda < \frac{1+B}{B-A}$ and $F(z)$ is p-valently convex of order δ, $0 \leq \delta < 1$, in $|z| < r(p, \lambda, \delta, A, B)$ where $r(p, \lambda, \delta, A, B)$ is as stated in Theorem 2. The result is sharp.

Proof. We have

$$F'(z) = (1-\lambda)\, f'(z) + \frac{\lambda}{p}\, \{zf'(z)\}'.$$

Therefore

$$(2.2)\quad \mathrm{Re}\left\{\frac{F'(z)}{f'(z)}\right\} = 1-\lambda + \frac{\lambda}{p}\, \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

Since $f(z) \in C_p(A, B)$, we can easily prove that

$$(2.3)\qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq p\, \frac{1+A}{1+B}.$$

By using (2.3) in (2.2) we have

$$\mathrm{Re}\left\{\frac{F'(z)}{f'(z)}\right\} \geq 1-\lambda + \frac{\lambda}{p} \cdot p\, \frac{1+A}{1+B}$$

$$\geq 1-\lambda+\lambda\, \frac{1-A}{1+B}.$$

Now $\mathrm{Re}\left\{\frac{F'(z)}{f'(z)}\right\} > 0$ if $1-\lambda+\lambda\, \frac{1+A}{1+B} > 0$ or if $\lambda < \frac{1+B}{B-A}$.

Hence $F(z)$ is p-valently close-to-convex in D if $\lambda < \frac{1+B}{B-A}$.

We now prove that F (z) is p-valently convex of order $\delta, 0 < \delta < 1$ in $|z| < r(p, \lambda, \delta, A, B)$. We have

$$(2.4)\quad \frac{zF'(z)}{p} = (1-\lambda)\frac{zf'(z)}{p} + \frac{\lambda z}{p}\left\{\frac{zf'(z)}{p}\right\}' \quad \text{for } z \in D.$$

Since $f(z) \in C_p(A,B)$ it follows that $\dfrac{zf'(z)}{p} \in T_p^*(A,B)$

Applying Theorem 2 with $\dfrac{zf'(z)}{p}$ in place of f (z), it follows from (2.4) that $\dfrac{zF'(z)}{p}$ is p-valently starlike of order δ in $|z| < r(p,\lambda,\delta, A,B)$, equivalently, F (z) is p-valently convex of order δ in $|z| < r(p,\lambda,\varrho,A,B)$. The result is sharp for the function.

$$f(z) = z^p - \frac{p^2(B-A)}{(n+p)\{(1+B)n + (B-A)p\}} z^{n+p}, \quad n = 1,2,3,\ldots.$$

Theorem 4. Let $f(z) \in P_p^*(A, B)$ and $F(z) = (1-\lambda) f(z) + \dfrac{\lambda}{p} zf'(z)$ for $z \in D$. Then $\operatorname{Re}\left\{\dfrac{F'(z)}{z^{p-1}}\right\} > p\delta, \ 0 \leq \delta < 1$ for $|z| < r(p, \lambda, \delta, A, B)$, where

$$r(p, \lambda, \delta, A, B) = \inf_n \left[\frac{p(1+B)(1-\delta)}{(p+n\lambda)(B-A)}\right]^{\frac{1}{n}}, \quad n = 1, 2, 3, \ldots.$$

The result is sharp.

Proof. To prove the result it is sufficient to show that the values of $\dfrac{F'(z)}{z^{p-1}}$ lie in a circle centered at p whose radius is $p(1-\delta)$ for $|z| < r(p, \lambda, \delta, A, B)$. We have

$$\left|\frac{F'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty} \frac{(n+p)(p+n\lambda)}{p} |a_{n+p}| z^n\right|$$

$$\leq \sum_{n=1}^{\infty} \frac{(n+p)(p+n\lambda)}{p} |a_{n+p}|\ |z|^n.$$

Hence $\left| \frac{F'(z)}{z^{p-1}} - p \right| \leq p(1-\delta)$ if

$$\sum_{n=1}^{\infty} \left\{ \frac{(n+p)(p+n\lambda)}{p(1-\delta)} \right\} |a_{n+p}| \; |z|^n \leq p.$$

Since $f(z) \in P_p^*(A, B)$, we have from (1.4)

$$\sum_{n=1}^{\infty} \left\{ \frac{(n+p)(1+B)}{B-A} \right\} |a_{n+p}| \leq p.$$

Now proceeding as in the proof of Theorem 2, we can easily obtain the required result.

The result is sharp for the function

$$f(z) = z^p - \frac{p(B-A)}{(1+B)(n+p)} z^{n+p}, \; n = 1, 2, 3, \ldots \; .$$

Remark: Putting $p = 1$ and taking $A = (2\alpha - 1)$, $B = 1$, where $0 \leq \alpha < 1$, in the above theorems we get the results obtained by Bhoosnurmath and Swamy [1].

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