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Some Fixed Point Theorems IV

by

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ABSTRACT

Results on fixed points have been proved for single-valued and multi-valued mappings satisfying a rational inequality.

I. INTRODUCTION

The well-known Banach fixed point theorem states that a contraction mapping of a complete metric space into itself has a unique fixed point. In recent years, this celebrated theorem has been extended and generalized in various way by putting conditions either on the mapping or on the space. For a quite upto date information, books by Singh [16] and Smart [17] are worth-mentioning.

More recently, Khan [8] has extended contraction principle through a symmetric rational expression and obtained the following result.

Theorem A. Let (X,d) be a complete metric space and T a selfmapping on X for which

$$(*) \qquad d(Tx,Ty) \leq K \left\{ \frac{d(x,Tx)\ d(x,Ty) + d(y,Ty)\ d(y,Tx)}{d(x,Ty) + d(y,Tx)} \right\}$$

holds for all $x,y \in X$, $0 < K < 1$. Then T has a unique fixed point.

The mapping T satisfying (*) has been extensively studied by various authors e.g. Khan [8], [9], [10], [11], [12]. Fisher and Khan [4], Ray and Singh [15] and Fisher [3].

It was later shown by Fisher [3] that the Theorem A was incorrect as it stood and needed the extra condition, $d(x,Ty) + d(y,Tx) = 0$ implies that $d(Tx,Ty) = 0$, for the theorem to hold. Fisher [3] also gave an example to support his result.

The purpose of this paper is to unify the results of Khan [8] and Banach under the observation of Fisher [3].

II. RESULTS FOR SINGLE-VALUED MAPPINGS

We first prove a fixed point theorem for a bi-metric space (X,d,∂) where d and ∂ are two metrics on the set X.

Definition 2.1 *(Ciric* [1]). A mapping T of a metric space X into itself is said to be orbitally continuous if $\lim_{i\to\infty} T^{n_i}x = u$ implies that

$\lim_{i\to\infty} T(T^{n_i}x) = Tu$ for each $x \in X$.

It is well-known that every continuous mapping of X into itself is orbitally continuous, but the converse is not true (e.g. Ciric [1]).

Definition 2.2 *(Jaggi* [7]). For $x_0 \in X$, let $O(x_0,T)$ denote the orbit of T at x_0 where T is a self-mapping of a metric space X. Then T is said to be x_0-orbitally continuous if $T: \overline{O(x_0,T)} \to X$, is continuous.

It is well-known that a mapping may be x_0-orbitally continuous for some $x \in X$ without being orbitally continuous (e.g. Jaggi [7]).

Theorem 2.3. Let T be a self-mapping of a bi-metric space (X, d, ∂) such that following hold:

(i) $d(x,y) \leq \partial(x,y)$, for all $x,y \in X$

(ii) there are non-negative numbers α,β with $\alpha+\beta < 1$ and for which T satisfies

$$\partial(Tx,Ty) \leq \alpha \left\{ \frac{\partial(x,Tx)\ \partial(x,Ty) + \partial(y,Ty)\ \partial(y,Tx)}{\partial(x,Ty) + \partial(y,Tx)} \right\} + \beta\ \partial(x,y),$$

for all $x,y \in X$, when $\partial(x,Ty) + \partial(y,Tx) \neq 0$.

Further, $\partial(Tx,Ty) = 0$ if $\partial(x,Ty) + \partial(y,Tx) = 0$;

(iii) there exists some point $x_0 \in X$ such that the sequence $\{T^n x_0\}$

of interates has a subsequence $\{T^{n_1} x_o\}$ converging to ξ with respect to d.

(iv) T is x_o-continuous with respect to d.

Then T has a unique fixed point.

Proof. Let $x_n = T^n x_o$, Then we have

$$\partial(x_n,x_{n+1}) = \partial(Tx_{n-1}, Tx_n)$$

$$\leq \alpha \left\{ \frac{(\partial(x_{n-1}, x_n)\partial(x_{n-1}, x_{n+1}) + \partial(x_n,x_{n+1})\partial(x_n,x_n)}{\partial(x_{n-1}, x_{n+1}) + \partial(x_n,x_n)} \right\} + \beta\, \partial(x_{n-1}, x_n)$$

$= (\alpha+\beta)\ \delta\ (x_{n-1},x_n)$ if $x_{n-1} \neq x_{n+1}$. However if $x_{n-1} = x_{n+1}$ then condition of theorem imply that $x_{n-1} = x_n = x_{n+1}$. Thus x_{n-1} would be a fixed point of T. Put $k = (\alpha+\beta)$. Then $k<1$ says that $\{T^n x_o\}$ is a Cauchy sequence with respect to ∂.

So in view of (1) $\{T^n x_o\}$ is also a Cauchy sequence with respect to d. Due to (iv), it follows that $\{T^n x_o\}$ converges to ξ with respect to d. Now x_o-continuity of T with respect to d yields

$$T\xi = T(\lim_{n\to\infty} T^n x_o) = \lim_{n\to\infty} T^{n+1} x_o = \xi.$$

Thus ξ is a fixed point of T. For unicity of ξ, consider $\eta \neq \xi$ such that $\eta = T\eta$. Then $\partial(\xi,\eta) > 0$. Also,

$$\partial(\eta,\xi) = \partial(T\eta,T\xi) \leq \alpha \left\{ \frac{\partial(\eta,T\eta)\partial(\eta,T\xi) + \partial(\xi,T\xi)\partial(\xi,T\eta)}{\partial(\eta,T\xi) + \partial(\xi,T\eta)} \right\} + \beta\, \partial(\eta,\xi),$$

$$\leq \beta\ \partial(\xi,\eta).$$

Thus

$$(1-\beta)\ \partial(\eta,\xi) \leq 0,$$

implying thereby $\partial(\xi,\eta) = 0$. So $\xi = \eta$.

Remarks. (1) For $\alpha = o$, Theorem 2.3 reduces to that of Maia [13].

(ii) When $\beta = 0$ and $\partial = d$, Theorem 2.3 is the main theorem of Khan [8].

(iii) If X is equipped with n metrics $d_1, d_2, \ldots, d_n, \partial$ such that $d(x,y) \leq d_1(x,y) \leq d_2(x,y) \leq \ldots \leq d_{n-2} \leq \partial(x,y)$ for every $x,y \in X$, then the conclusion of Theorem 2.3 still holds.

Theorem 2.4. Let $T: X \to X$ be an orbitally continuous mapping on a metric space X such that

$$\text{(i)}\ d(Tx,Ty) < \alpha \left\{ \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} \right\} + \beta d(x,y)$$

for all $x,y \in X$, $\alpha+\beta = 1$ (α,β non-negative reals) whenever $d(x,Ty) + d(y,Tx) \neq 0$, and $d(Tx,Ty) = 0$ when $d(x,Ty) + d(y,Tx) = 0$.

(ii) For some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a cluster point $\xi \in X$. Then ξ is a unique fixed point of T.

Proof. If $T^{k-1} x_0 = T^k x_0$ for some $k \in N$, then $T^n x_0 = T^k x_0 = \xi$ for all $n \geq k$, so the result follows.

Assume now that $T^{k-1} x_0 \neq T^k x_0$ for all $k \in N$, and let $\lim_{i \to \infty} T^{n_i} x_0 = \xi$. Then for $T^{n-1} x_0$ and $T^n x_0$ in X we get

$$d(T^n x_0,T^{n+1} x_0)$$

$$\leq \alpha \left\{ \frac{d(T^{n-1}x_0,T^nx_0)d(T^{n-1}x_0,T^{n+1}x_0)+d(T^nx_0,T^{n+1}x_0)d(T^nx_0,T^nx_0)}{d(T^{n-1}x_0,T^{n+1}x_0)+d(T^nx_0,T^nx_0)} \right\} + \beta\ d(T^{n-1}x_0,T^nx_0).$$

If $d(T^{n-1} x_0,T^{n+1}x_0)+d(T^nx_0,T^nx_0)=0$, we find that $T(T^{n-1}x_0) = T(T^nx_0)$. So T^nx_0 is a fixed point of T.

Otherwise, above inequality reduces to

$$d(T^n x_0, T^{n+1} x_0) \leq (\alpha+\beta)\ d(T^{n-1} x_0, T^n x_0).$$

Hence

$$d(T^n x_0, T^{n+1} x_0) < d(T^{n-1} x_0, T^n x_0).$$

Therefore, the sequence $\{d(T^n x_0, T^{n+1} x_0)\}$ is a decreasing and hence is convergent sequence of positive real numbers. Further,

$$\lim_{i \to \infty} d(T^{n_i} x_0, T^{n_i+1} x_0) = d(\xi, T\xi),$$

and

$$\{d(T^{n_i} x_0, T^{n_i+1} x_0)\} \subseteq \{d(T^n x_0, T^{n+1} x_0)\}$$

implies that

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = d(\xi, T\xi).$$

Also, orbital continuity of T gives $\lim_{i\to\infty} T^{n_i+1} x_0 = T\xi$,

$\lim_{i\to\infty} T^{n_i+2} x_0 = T^2\xi$ and $\{d(T^{n_i+1} x_0, T^{n_i+2} x_0)\} \subseteq \{d(T^n x_0, T^{n+1} x_0)\}$.

Above relations show that

$$d(T\xi, T^2\xi) = d(\xi, T\xi).$$

If $d(\xi, T\xi) > 0$, then one gets

$$d(T\xi,T^2\xi) < \alpha \left\{ \frac{d(\xi,T\xi)d(\xi,T^2\xi)+d(T\xi,T^2\xi)d(T\xi,T\xi)}{d(\xi,T^2\xi)+d(T\xi,T\xi)} \right\} + \beta d(T\xi,T^2\xi).$$

Then we have

$$d(T\xi,T^2\xi) < \left(\frac{\alpha}{1-\beta}\right) d(\xi,T\xi).$$

So

$$d(T\xi, T^2\xi) < d(\xi, T\xi),$$

which is a contradiction. Hence ξ is a fixed point of T which is clearly unique.

Remark. For $\alpha = 0$, our Theorem 2.4 extends a theorem of Edelstein [2].

Theorem 2.5. Let T be a continuous densifying mapping of a complete metric space X into itself such that for all $x, y \in X$ there are real constants α_i, $(i = 1,2,3,4)$, α and β satisfying $\alpha_1 + \alpha_2 + \alpha_3 \geq \alpha + \beta$, for which the inequality

$$\alpha_1 F(Tx,Ty)+\alpha_2 F(x,Tx)+\alpha_3 F(y,Ty)+\alpha_4 \min \{F(x,Ty),F(y,Tx)\}$$

$$< \alpha \left\{ \frac{F(x,Tx)\, F(x,Ty) + F(y,Ty)\, F(y,Tx)}{F(x,Ty) + F(y,Tx)} \right\} + \beta F(x,y).$$

holds for $x, y \in X$ whenever $F(x,Ty) + F(y,Tx) \neq 0$, and $F(Tx,Ty) = 0$, otherwise, a lower semi-continuous function $F: X \times X \to [0, \infty)$ with the property $F(x,y) = 0$ if and only if $x = y$. If for some $x_0 \in X$, the sequence of iterates $\{T^n x_0\}$ is bounded, then T has a fixed point.

Proof. For $y = Tx$, we have

$$\alpha_1 F(Tx,T^2 x)+\alpha_2 F(x,Tx)+\alpha_3 F(Tx,T^2 x)+\alpha_2 \min \{F(x,T^2 x),F(Tx,Tx)\}$$

$$< \alpha \left\{ \frac{F(x,Tx)F(x,T^2x)+F(Tx,T^2x)F(Tx,Tx)}{F(x,T^2x)+F(Tx,Tx)} \right\} + \beta F(x,Tx).$$

If $F(x,T^2 x) = 0$ then one gets $F(Tx,T^2 x) = 0$ which gives $T(Tx) = Tx$. So(Tx) is a fixed point of T.

If $F(x,T^2 x) \neq 0$, it is clear that $x \neq Tx$. So we get

$$F(Tx,T^2 x) < \left(\frac{\alpha+\beta-\alpha_2}{\alpha_1+\alpha_3}\right) F(x,Tx).$$

Hence

$$F(Tx,T^2 x) < F(x,Tx), \quad x \neq Tx.$$

Then from Theorem 5 of Iseki [6], we find that T has a fixed point.

Remark. Our Theorem 2.5 generalizes a fixed point Theorem of Furi and Vignoli [5] as well as Theorem 3 of Khan [11].

Theorem 2.4. Let X be a complete metric space and $\{T_n\}$ a sequence of mappings of X into itself. Suppose there are non-negative reals α,β with $\alpha+\beta < 1$ such that for all $x, y \in X$ the inequality

$$d(T^p_i x,T_j{}^q y) \leq \alpha \left\{ \frac{d(x,T_i{}^p x)d(x,T_j{}^q y)+d(y,T_j{}^q y)d(y,T_i{}^p x)}{d(x,T_j{}^q y)+d(y,T_i{}^p x)} \right\} + \beta d(x,y)$$

holds whenever $d(x,T_j{}^q y)+d(y,T_i{}^p x) \neq 0$, and further $d(T^p_i x, T_j{}^q y) = 0$ if $d(x,T_j{}^q y)+d(y,T_i{}^p x) = 0$, where p,q are some positive integers.

Then the sequence $\{T_n\}$ has a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Construct a sequence $\{x_n\}$ as follows:

$x_1 = T_i{}^p x_0,\ x_2 = T_j{}^q x_1,\ x_3 = T_i{}^p x_2, \ldots$

i.e.

$$x_n = T_n^p (x_{n-1}), \text{ when n is odd}$$

and

$$x_n = T_n^q (x_{n-1}), \text{ when n is even},$$

Then, by a routine calculation, it follows that $\{x_n\}$ is a Cauchy sequence which has a limit u, (say) in X.

It is not hard to see that u is a unique common fixed point of the sequence $\{T_n\}$. This completes the proof.

Definition 2.7. A self-mapping T on a metric space (X,d) is said to be non-expansive if

$$d(Tx,Ty) \leq d(x,y), \text{ for all } x, y \in X.$$

It is well-known (e.g., Smart [17] or Singh [16]) that a non-expansive mapping on a complete metric space need not fix any point of the space. For such mappings, however, we have the following common fixed point theorem.

Theorem 2.8. Let T, T_1,T_2 be three self-mappings of a complete metric space (X,d) where T is non-expansive. Also for all x, y $\in$ X, and non-negative numbers α, β with $\alpha+\beta < 1$, we have

(i) $d(T_1^p x, T_2^q y)$

$$\leq \alpha \left\{ \frac{d(Tx,TT_1^p x)d(Tx,TT_2^q y)+d(Ty,TT_2^q y)d(Ty,TT_1^p x)}{d(x,T_2^q y)+d(y,T_1^p x)} \right\} + \beta d(Tx,Ty),$$

whenever $d(x,T_2^q y) + d(y,T_1^p x) \neq 0$, and $d(T_1^p x, T_2^q y) = 0$, whenever $d(x,T_2^q y) + d(y,T_1^p x) = 0$, for some positive integers p,q;

(ii) T commutes with T_2^q.

Then there is a unique common fixed point of T, T_1 and T_2.

Proof. Follows from Theorem 2.6 once we use the non-expansiveness of T in (i). So T_1 and T_2 have a unique common fixed point say ξ. Then to show that ξ is also a fixed point of T, consider

$$\begin{aligned} d(\xi,T\xi) &= d(T_1^p\xi TT_2^q\xi) \\ &= d(T_1^p\xi,T_2^q(T\xi)) \end{aligned}$$

$$\leq \alpha \left\{ \frac{d(T\xi,TT_1{}^p\xi)\, d(T\xi,T_2{}^q(T^2\xi)) + d(T^2\xi,T_2{}^qT^2\xi)\, d(T^2\xi,TT_1{}^p\xi)}{d(T\xi,T_2{}^qT^2\xi) + d(T^2\xi,TT_1{}^p\xi)} \right\}$$

$$+ \beta\, d(T\xi,T^2\xi) = \beta\, d(T\xi,T^2\xi).$$

Again using non-expansive property of T and the fact $\beta < 1$, we find that $T\xi = \xi$. Hence ξ is a unique common fixed point of T, T_1 and T_2.

This complettes the proof.

Remarks. (i) If T is the identity map, Theorem 2.8 reduces to Theorem 2.6. This would mean that T may have more than one fixed point, but the common fixed point of T, T_1 and T_2 is unique.

(ii) As remarked above, only non-expansiveness of T by itself would not ensure a fixed point for T.

(iii) In Theorem 2.8 one can take a sequence of self-mappings $\{T_n\}$ of X so as to prove that T, T_1, T_2 ... have a unique common fixed point.

III. RESULTS FOR MULTI-VALUED MAPPINGS

Lastly, we prove multi-valued version of several results obtained previously. Throughout this section, we follow the notations of Nadler [14]. For a metric space (X, d), $A \subset X$, $B \subset X$, and $\varepsilon > o$, we write

(i) $CB(X) = \{A$: A is a non-empty closed and bounded subset of X);

(ii) $N(A,\varepsilon) = \{x \in X: d(x, a) < \varepsilon \text{ for some } a \in A\}$;

(iii) $D(A,B) = \inf \{d(a,b): a \in A,\ b \in B\}$;

(iv) $H(A,B) = \inf \{\varepsilon > o: N(B,\varepsilon) \subset A \text{ and } N(A,\varepsilon) \supset B\}$.

The space CB(X) is a metric space with respect to the distance function H(A,B) called the Hausdorrf metric.

Theorem 3.1. Let X be a complete metric space and $F:X \to CB(X)$ a continuous multi-valued mapping. Suppose that F satisfies the inequality

$$H(Fx,Fy) \leq \alpha \left\{ \frac{D(x,Fx)\, D(x,Fy) + D(y,Fy)\, D(y,Fx)}{D(x,Fy) + D(y,Fx)} \right\} + \beta\, d(x,y)$$

for $x, y \in X$, $0 \leq \alpha, \beta$ with $\alpha + \beta < 1$, whenever $D(x,Fy) + D(y,Fx) \neq 0$, and $H(Fx,Fy) = 0$ when $D(x,Fy) + D(y,Fx) = 0$. Then F has a fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in FX_0$. We may assume that $H(Fx_0, Fx_1) > 0$, since otherwise $x_1 \in Fx_1$, which implics that x_1 is a fixed point of F.

Let a be any real number with $0 < a < 1$ and $K = \alpha+\beta$. Since $H(Fx_0, Fx_1) < K^{-a} H(Fx_0,Fx_1)$ and $x_1 \in Fx_0$, by the definition of H, there exists $x_2 \in Fx_1$ such that

$$d(x_1,x_2) \leq K^{-a} H(Fx_0,Fx_1).$$

Let $H(Fx_1,Fx_2) > 0$. Then $H(Fx_1,Fx_2) < K^{-a} H(Fx_1,Fx_2)$, which implies the existence of $x_3 \in Fx_0$ with the property

$$d(x_2,x_3) \leq K^{-a} H(Fx_1,Fx_2).$$

Continuing in this fashion, we produce a sequence $\{x_n\}$ of points of X such that

$$x_{n+1} \in Fx_n \text{ and } d(x_n, x_{n+1}) \leq K^{-a} H(Fx_{n-1}, Fx_n).$$

Now we shall prove that $\{x_n\}$ is actually a Cauchy sequence in X. For this consider the inequality

$$d(x_n, x_{n+1}) \leq K^{-a} H(Fx_{n-1}, Fx_n)$$

$$\leq K^{-a} \left[\alpha \left\{ \frac{D(x_{n-1},Fx_{n-1})D(x_{n-1},Fx_n)+D(x_n,Fx_n)D(x_n,Fx_{n-1})}{D(x_{n-1},Fx_n) + D(x_n,Fx_{n-1})} \right\} + \beta\, d(x_{n-1},x_n) \right]$$

$\leq K^{-a} (\alpha+\beta)\, d(x_{n-1}, x_n) \leq K^{1-a} d(x_{n-1}, x_n)$, when $D(x_{n-1}, Fx_n) \neq 0$. Clearly, $x_n \in Fx_{n-1} = Fx_n$ when $D(x_{n-1}, Fx_n) = 0$, This implies therefore that x_n is a fixed point of F.

From $K^{1-a} < 1$ and $d(x_n, x_{n+1}) \leq K^{1-a} d(x_{n-1}, x_n)$, we observe that $\{x_n\}$ is a Cauchy sequence in X and has a limit z, say, Now

$$D(z,Fz) \leq d(z,x_{n+1}) + D(x_{n+1}, Fz)$$

$$\leq d(z, x_{n+1}) + H(Fx_n, Fz)$$

$$\leq d(z,x_{n+1})+\alpha \left\{ \frac{D(x_n,Fx_n)\, D(x_n,F_z) + D(z,Fz)\, D(z,Fx_n)}{D(x_n,Fz) + D(z,Fx_n)} \right\} + \beta\, d(x_n,z).$$

$$\leq d(z,x_{n+1}) + \alpha \left\{ \frac{d(x_n,x_{n+1})\, D(x_n,Fz) + D(z,Fz)\, d(z,x_{n+1})}{D(x_n,Fz) + D(z,Fx_n)} \right\} + \beta d(x_n,z).$$

Letting n tending to infinity; we get $D(z,Fz) = 0$,

As Fz is a closed subset of X, it follows that $z \in Fz$. Thus z is a fixed point of F, and the proof is complete.

Remarks.

(i) For $\alpha = 0$, Theorem 3.1 reduces to a result of Nadler [14].

(ii) Where $\beta = 0$, we get a multivalued version of the main theorem of Khan [8].

(iii) We observe that the continuity requirement of the mapping F in Theorem 3.1 can be waived if $\alpha = 0$.

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