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THE SETS OF HOMOTHETIC MAPPINGS

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ABSTRACT

In this work, the homothetic Matrix Lie Group has been considered as an action group and the homothetic mapping sets have been obtained as a subset of mapping sets on E^n.

1. INTRODUCTION

Consider that G is a group and M is a differentiable manifold. As a consequence,

(a) The points on M coincide with elements of G

(b) $o : M x M \longrightarrow M$
$\quad (a,b) \longrightarrow aob^{-1}$

this operation is also differentiable in every where. (M, G) representation which has these two axioms is called a Lie Group [1].

If

$$\{[a_{ij}]_{nxn} \ | a_{ij} \in IR\}$$

is a submanifold of matrix space and a group with respect to matrix multiplication, then this group is defined as a matrix Lie Group [2].

Let M, $\bar{M}$ be n-dimentional C^∞ — manifolds and

$$\varphi : M \xrightarrow{\text{diffeomorphism}} \bar{M},$$

such that

$$\varphi_* : TM \longrightarrow T\bar{M}, \quad \forall\ x, y \in T_M(p)$$

and

$< \varphi_*(x), \varphi_*(y) > |_{\varphi(p)} = c^2 < x, y > |_p$, where c^2 is a constant. The transformation φ which satisfies above equality is defined as a Homothetic Transformation [3].

Since homothetic transformations are free of metric choice, there is no need to any specialization in the metric.

If A is an orthogonal nxn matrix and $k = cI_n$ is a scalar matrix, then

$$H = kA,$$

is called a homothetic matrix.

The set of homothetic transformations (H(M)) is a group with respect to the operation of composition of functions. The set of homothetic matrices ($\mathcal{H}$(M)) which corresponds to the set of homothetic transformations (H(M)) is also a group with respect to matrix multiplication. Thus, the set (H(M)) which corresponds to the set ($\mathcal{H}$ (M)) is a group isomorphism [4].

The set of homothetic matrices ($\mathcal{H}$ (M)) is also a Matrix Lie Group [4].

2. MAPPING ON $\mathcal{H}$ (E^n)

Definition (Homothetic mapping): Let E^n be an n-dimentional C^∞ — manifold and (U, ψ) be a coordinate neighborhood. Then, there exist such functions;

$$f_x = \{h_1|_x, h_2|_x, \ldots, h_n|_x ; x\}, \forall x \in \psi(U), f_x \in \mathcal{H} B(E^n),$$

$$h_i|_x = \sum_{k=1}^{n} c\, a_{ki} \frac{\partial}{\partial x_k}|_x .$$

The linear mapping (f_x) is called a homothetic mapping on E^n.

Theorem 1: $\{B(E^n)\ (E^n, GL\ (n, IR))\}$ is given as a main fibre set. Then, the following transformation exists:

$$V \subset E^n, \psi : \pi^{-1}(V) \longrightarrow V x GL(n, IR).$$

By means of above transformation, homothetic mapping converges to a homothetic matrix. In other words, every homothetic matrix indicates a homothetic mapping.

Proof: Let

$$f_x \in \mathcal{H} B(E^n) \ni \{h_1|_x, h_2|_x, \ldots, h_n|_x ; x\}$$

then, one obtains that

$$f_x \rightarrow \psi(f_x) = (x, [x_{ki}]),\ h_i|_x = \sum_{k=1}^{n} x_{ki} \frac{\partial}{\partial x_k}\Big|_x .$$

In fact,

$$\mathcal{H}\ B(E^n) \subset B(E^n).$$

Thus, one can say that $[x_{ki}] \in GL(n, IR)$.

$$f_x : IR^n{}_1 \xrightarrow{\text{Lineer}} T_{E^n}(x)$$

$$f_x P = [x_{ki}]\ p$$

$$f_x P = \begin{bmatrix} \sum_{i=1}^{n} x_{1i} p_i \\ \cdot \\ \cdot \\ \cdot \\ \sum_{i=1}^{n} x_{ni} p_i \end{bmatrix} = \left(\sum_{i=1}^{n} x_{1i} p_i\right) \frac{\partial}{\partial x_i}\Big|_x + \dots + \left(\sum_{i=1}^{n} x_{ni} p_i\right) \frac{\partial}{\partial x_n}\Big|_x$$

$$f_x P = \left(\sum_{i=1}^{n} ca_{1i} p_i\right) \frac{\partial}{\partial x}\Big|_x + \dots + \left(\sum_{i=1}^{n} ca_{ni} p_i\right) \frac{\partial}{\partial x_n}\Big|_x .$$

Using the above equality, we can write

$$[x_{ki}] = [ca_{ki}],\ 1 \leq i, k \leq n$$

$$[x_{ki}] \in \mathcal{H}\ (E^n)$$

or, in other way,

$[ca_{ki}] \in \mathcal{H}\ (E^n)$ is given

if

$$[ca_{ki}] \in \mathcal{H}\ (E^n)$$

then

$$[ca_{ki}] \in G\ L\ (n, IR) \quad .$$

Thus,

$$\exists\, f'_x \in B\ (E^n) \ni f'_x = \{h'_1\ |_x, \dots, h'_n\ |_x;\ x\}$$

where

$$h_i\ |_x = \sum_{k=1}^{n} ca_{ki} \frac{\partial}{\partial x_k}\ |_x \quad .$$

Finally, we can write

$$f'_x \in \mathcal{H}\ B(E^n) \quad .$$

Theorem 2: Let x be an any point on the n-dimentional Enclidean space E^n. If φ is a homothetic transformation of E^n then there is a radial transformation r of E^n and a rotation g around x and a sliding t(or another sliding t′) of E^n, such that

$$\varphi = torog \text{ or } \varphi = rogot'.$$

Proof: Let an orthogonal system with initial point x at E^n be

$$\{x_1, x_1, \ldots, x_n\}$$

and a homothetic transformation be φ. Using the orthogonal system, homothetic motion, with matrix representation, will be,

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} kA & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}, \ k = cI_n \in \sigma(n), \ A \in O(n), \ B \in \mathcal{R}^n{}_1$$

and using the fact that $D = \frac{1}{c} A^{-1} B$ one can obtain

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} cI_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{c}k & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

In the above equality, the first left matrix represents a scalar matrix $k = cI_n$, which gives us a radial transformation r. Second matrix defines a rotation around the point x and the third matrix indicates a sliding of E^n which is defined by

$D = \frac{1}{c} A^{-1} B$. So we can write that $\varphi = rogot$.

One can shows that the set of homothetic motions $\mathcal{H}(n)$ is a group with respect to the matrix multiplication.

Theorem 3:

For x, y $\in E^n$ and $f_x, f_y \in \mathcal{H} B(E^n)$ there is only one homothetic motion φ such as

$$\varphi(f_x) = f'_y.$$

Proof: Let

$$f_x = \{h_1|_x, h_2|_x, \ldots, h_n|_x; x\} \ ; \ f_y' = \{h'_1|_y, \ldots, h'_n|_y; y\} \in \mathcal{H}B(E^n)$$

where φ denotes the homothetic motion,
r denotes the radial transformation,
g denotes the orthogonal transformation,
t denotes the sliding motion.

By using the theorem 2, one can write

$$\varphi = \text{torog}.$$

On the other hand, by using the technique given in [1], one obtains (in the following figure)

$$t(x) = y, \text{ when } t \in T(n),$$

similarly, for only one rog,

$$t_*^{-1}(h'_i) = (rog)_* (h_i)$$

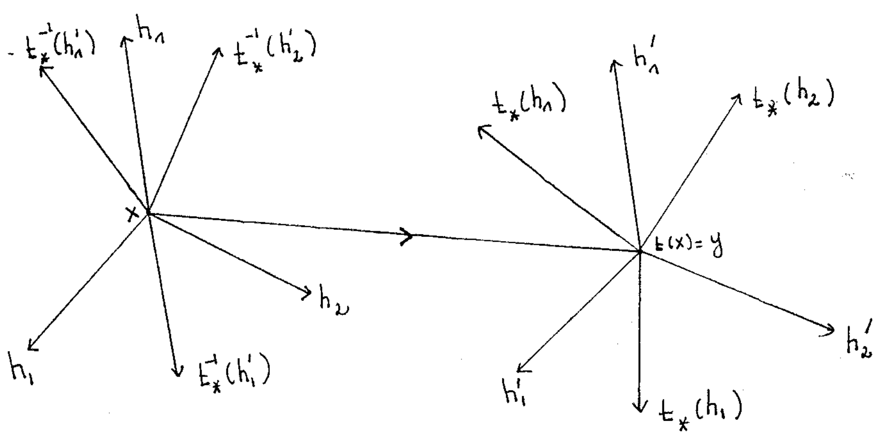
or

$$h'_i = t_*(rog)_* (h_i)$$
$$h'_i = (torog)_* (h_i)$$
$$h'_i = \varphi_* (h_i).$$

Thus we can write that

$$\varphi(\{h_1|_x, \ldots, h_n|_x;\ x\}) = \{\varphi_*(h_1|\varphi(x)), \ldots, \varphi_*(h_n|\varphi(x));\ \varphi(x)\}$$
$$= \{h'_1|_y, \ldots, h'_n|_y; y)\}$$
$$\varphi(f_x) = f'_y.$$

This result shows us the availability of a homothetic motion φ and its singularity.



(figure)

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