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ON THE SPECTRUM OF C_1 AS AN OPERATOR ON bv

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ABSTRACT

In 1985 John Reade determined the spectrum of C_1, the Cesàro Operator which is represented by the matrix:

$$C_1 = \begin{pmatrix} 1 & 0 & & 0\ldots \\ \frac{1}{2} & \frac{1}{2} & & 0\ldots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\ldots \\ & \ldots & & \end{pmatrix}$$

regarded as an operator on the space c_0 of all null sequences normed by $\|x\| = \sup_{n\geq 0} |x_n|$. It is the purpose of this paper to determine the spectrum of C_1 regarded as an operator on the space bv of all sequences x such that $\lim_{n\to\infty} x_n$ exists and $\|x\| = \lim_{n\to\infty} |x_n + \sum_{n=0}^{\infty} |x_{n+1}-x_n| < \infty$ We do so by proving that $(C_1 - \lambda I)^{-1} \in B$ (bv) for all $\lambda \in C$ such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$.

1. INTRODUCTION

In 1986 we determined the spectrum of the Cesàro Operator C_1 regarded as an operator on the space bv_0, the space of all sequences x such that $\lim_{n\to\infty} x_n = 0$ and $\|x\| = \sum_{n=0}^{\infty} |x_{n+1} - x_n| < \infty$. Using methods similar to those of John Reade in [6] we determine the spectrum of C_1 as an operator on bv.

1.1. Definition: (F, FK and BK spaces)

A Fréchet space F is a complete linear space. An FK–space is a Fréchet space with continuous coordinates. A normed FK–space is called a BK–space.

1.2. Theorem: bv is a BK–space with Schauder basis $(\delta, \delta^\circ, \delta^1, \ldots)$, where $\delta = (1, 1, 1, \ldots)$ and $\delta^k = (0, 0, \ldots, 0, 1, 0 \ldots)$.

Proof: bv is a BK–space by [10] page 110.

It is clear that lim $\in$ bv*, where bv* denotes the continuous dual of bv.

$$|\lim (x)| = \lim |x| \leq \lim_{n\to\infty} |x_n| + \sum_{n=0}^{\infty} |x_{n+1} - x_n| = \|x\|_{bv}$$

and so $\|\lim\| \leq 1$. Now

$$x = l\delta + \sum_{n=0}^{\infty} (x_n - l)\, \delta^{n'}$$

where $x \in$ bv and $l = \lim_{n\to\infty} x_n$ and if also $x = b\delta = \sum_{n=0}^{\infty} b_n \delta^n$,

then by the continuity of lim we have

$$\lim x = b \lim \delta + \sum_{n=0}^{\infty} b_n \lim \delta^n = b, \text{ therefore } b = l.$$

We also need to show that $b_n = x_n - l$ for all $n \geq 0$. So consider P_N: bv $\to$ C, then $P_N \in$ bv*

since $|P_N(x)| = |x_N|$ and $\|x\|_{bv} = \lim_{n\to\infty} |x_n| + \sum_{n=0}^{\infty} |x_{n+1} - x_n|$,

$$\|x\|_{bv} \geq \lim_{n\to\infty} |x_n| + \sum_{k=N}^{\infty} |x_{n+1} - x_n|, \text{ therefore}$$

$$\|x\|_{bv} \geq \lim_{n\to\infty} |x_n| + \lim_{m\to\infty} \sum_{n=N}^{m} |x_{n+1} - x_n| \geq \lim_{n\to\infty} |x_n| + \lim_{m\to\infty} |x_{m+1} - x_N|$$

that is, $\|x\|_{bv} \geq |l| + |l - x_N| \geq |x_N|$.

Hence we see that $|P_N(x)| \leq \|x\|_{by} \Rightarrow P_N \in$ bv*. So

$$P_N(x) = P_N\left(l\delta + \sum_{k=0}^{\infty} (x_n - l)\, \delta^n\right) = l\, P_N(\delta) + \sum_{n=0}^{\infty} (x_n - l)\, P_N(\delta^n)$$

$$= l + (x_N - l) = x_N.$$

But also $P_N(x) = P_N\left(b\delta + \sum_{n=0}^{\infty} b_n \delta^n\right) = b + b_N$ therefore

$x_N = b_N + b = b_N + l \Rightarrow b_n = x_n - l$. We therefore conclude that $(\delta, \delta^{\circ}, \delta^1, \ldots)$ is a Schauder basis for bv.

1.3. Theorem: Let $T \in B(X)$, where X is any Banach space, $T \in B(X)$ denotes a bounded operator on X, then the spectrum of T^* is identical with the spectrum of T.

Furthermore, $R_\lambda(T^*) = (R_\lambda(T))^*$ for $\lambda \in \rho(T) = \rho(T^*)$, where $R_\lambda(T) = (T-\lambda I)^{-1}$ and $\rho(T) = \{\lambda \in \mathbb{C}: (T-\lambda I)^{-1} \text{ exists}\}$ and T^* denotes the adjoint operator of T.

Proof: The proof of this is given in [1] page 568 and [2] page 71.

1.4. Lemma: Let $Z_n = \prod_{\nu=0}^{n} \left(1 - \frac{1}{\lambda(\nu+1)}\right)$, $\lambda \neq 0$, $\lambda \in \mathbb{C}$.

Then the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded iff

$$\text{Re}\left(\frac{1}{\lambda}\right) \geq 1, \quad \lambda \neq 1$$

Proof: Let C be a constant depending only on λ which may be different at each occurrence, A a non-zero constant and O denotes capital order. We have that:

$$\log_e(1-u) = -u + O(u^2)$$

Uniformly in $|u| \leq \frac{1}{2}$, $u \in \mathbb{C}$. Now given $\lambda \neq 0$ there is a ν_0 such that $|\lambda|(\nu+1) > z$ for $\nu \geq \nu_0$ hence for $n \geq \nu_0$

$$\log_e Z_n = \sum_{\nu=0}^{n} \log\left(1 - \frac{1}{\lambda(\nu+1)}\right)$$

$$= C - \frac{1}{\lambda} \sum_{\nu=\nu_0}^{n} \frac{1}{\nu+1} + \sum_{\nu=\nu_0}^{n} t_\nu$$

where $t_\nu = O\left(\frac{1}{\nu^2}\right)$.

$$\text{Now } \sum_{\nu=\nu_0}^{n} t_\nu = \sum_{\nu=\nu_0}^{\infty} t_\nu - \sum_{\nu=n+1}^{\infty} t_\nu = C + O\left(\frac{1}{n}\right)$$

Also $\sum_{\nu=\nu_0}^{n} \frac{1}{\nu+1} = C + \log n + O\left(\frac{1}{n}\right)$. If $C_n = \sum_{\nu=0}^{n} \frac{1}{\nu+1} - \log n$, then

$$C_{n+1} - C_n = \frac{1}{2+n} - \log\left(\frac{n+1}{n}\right) = O\left(\frac{1}{n^2}\right)$$

Therefore $C_{n+1} = C + \sum_{\nu=0}^{n} (C_{\nu+1} - C_\nu) = C + O\left(\frac{1}{n}\right)$

Hence as $n \to \infty$ $\log Z_n = C - \frac{1}{\lambda} \log n + O\left(\frac{1}{n}\right)$

So that $Z_n = An^{-\frac{1}{\lambda}} \left(1 + O\left(\frac{1}{n}\right)\right)$

$$= An^{-\frac{1}{\lambda}} + O\left(n^{-\mathrm{Re}\left(\frac{1}{\lambda}\right) - 1}\right)$$

If $\mathrm{Re}\left(\frac{1}{\lambda}\right) \geq 1$, $\lambda \neq 1$, the partial sums of

$\sum_{n=1}^{\infty} n^{-\frac{1}{\lambda}}$ are bounded and $\sum_{n=1}^{\infty} n^{-\mathrm{Re}\left(\frac{1}{\lambda}\right) - 1} < \infty$

so that the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded.

If $0 < \mathrm{Re}\left(\frac{1}{\lambda}\right) < 1$ or $\lambda = 1$ then the partial sums of

$\sum_{n=1}^{\infty} n^{-\mathrm{Re}\left(\frac{1}{\lambda}\right) - 1} < \infty$ are unbounded but still we have

$$\sum_{n=1}^{\infty} n^{-\mathrm{Re}\left(\frac{1}{\lambda}\right) - 1} < \infty.$$

If $\operatorname{Re}\left(\frac{1}{\lambda}\right) \leq 0$, then $\sum_{n=1}^{N} n^{-\frac{1}{\lambda}} \asymp N^{-\frac{1}{\lambda}} \Big/ \left(1 - \frac{1}{\lambda}\right)$

where $a_n \asymp b_n$ means that there exist

$m, M \in \mathbb{R}^+$ such that $mb_n \leq a_n \leq Mb_n$

$$\text{Now } \sum_{n=1}^{N} n^{-\operatorname{Re}\left(\frac{1}{\lambda}\right) - 1} = \begin{cases} O\left(N^{-\operatorname{Re}\left(\frac{1}{\lambda}\right)}\right), & \operatorname{Re}\left(\frac{1}{\lambda}\right) < 0 \\ O(\log N), & \operatorname{Re}\left(\frac{1}{\lambda}\right) = 0 \end{cases}$$

Hence we see that the partial sums of

$\sum_{n=1}^{\infty} n^{-\frac{1}{\lambda}}$ are unbounded although

$\sum_{n=1}^{\infty} n^{-\operatorname{Re}\left(\frac{1}{\lambda}\right) - 1} < \infty$ hence we conclude

that the partial sums of $\sum_{n=1}^{\infty} Z_n$ are bounded iff $\operatorname{Re}\left(\frac{1}{\lambda}\right) \geq 1$.

2. Determination of the Spectrum of c_1 on bv

2.1. Lemma: Let C_1: bv $\to$ bv, then
C_1^*: bv* $\to$ bv* and $\| C_1 \|_{(bv,\, bv)} = \| C_1 \|_{(bv_0,\, bv_0)} =$
$\| C_1^* \|_{(bv^*,\, bv^*)} = 1$ so that C_1^* is bounded, where

$$\| C_1 \|_{(bv,\, bv)} = \sup_{n \geq 1} \sum_{j=0}^{\infty} \left| \sum_{k=n}^{\infty} a_{jk} - \sum_{k=n}^{\infty} a_{j-1,\, k} \right|, \quad a_{jk} = C_{jk}.$$

Proof: Let T: bv $\to$ bv be given by the matrix $A = (a_{nk})$ then we show that T*: bv* $\to$ bv* is given by the matrix:

$$T^* = \begin{bmatrix} \bar{\chi} & v_0 - \bar{\chi} & v_1 - \bar{\chi} & v_2 - \bar{\chi} & \dots \\ a_0 & a_{00} - a_0 & a_{10} - a_0 & a_{20} - a_0 & \dots \\ a_1 & a_{01} - a_1 & a_{11} - a_1 & a_{21} - a_1 & \dots \\ & \dots & \dots & \dots & \end{bmatrix}$$

We then choose $A = (a_{nk})$ to be C_1 and conclude the lemma.

It is clear that bv^* is equivalent to $\mathbb{C} \oplus$ bs via the map $h(f) = (\bar{\chi}, t_0, t_1 .)$ where $\oplus$ denotes the direct sum and bs denotes the space of sequences

x such that $\sup_{n \geq 0} \left| \sum_{k=0}^{n} x_k \right| < \infty$

Define $W = hoT^* \, oh^{-1}: \mathbb{C} \oplus bs \to \mathbb{C} \oplus bs$, that is,

$$W: \mathbb{C} \oplus bs \to \mathbb{C} \oplus bs, \ h: bv^* \to \mathbb{C} \oplus bs$$

is an isometry, where

$$\| (l, x) \|_{\mathbb{C} \oplus bs} = \max \left(| l |, \sup_{n \geq 0} \left| \sum_{k=0}^{n} x_k \right| \right) \text{ and}$$

$$h(\lim) = (\lim \delta, \lim \delta^1, \lim \delta^2, \ldots) = (1, \theta) \tag{2.1}$$

where $\lim \in bv^*$, i.e. lim is a functional and

θ is the zero sequence. Thus the zero column of W is:

$$\begin{aligned} W(1, \theta) &= hoT^* \, oh^{-1} (1, \theta) \\ &= hoT^* \, oh^{-1} h (\lim) \\ &= hoT^* \, o\lim = (\lim \ o \ T) = \\ &= (\lim \ oT) (\delta), (\lim \ o \ T) (\delta^\circ), \ldots) \\ &= (\bar{\chi}, a_0, a_1, a_2, \ldots) \end{aligned}$$

where $\bar{\chi} = (\lim \ o \ T)(\delta) = \lim_{n \to \infty} \sum_{\nu=0}^{\infty} a_{n\nu}$ and

$a_n = (\lim \ o \ T)(\delta^n) = \lim_{k \to \infty} a_{kn}$ by [9]

Also $\lim_{n \to \infty} a_{nk} = \lim_{n \to \infty} c_{nk} = 0$ for each $k \geq 0$

since $a_{nk} = c_{nk} = \dfrac{1}{1+n}$ and $v_k = (P_k oT)(\delta) = 1$ for T represented by

$a_{nk} = \dfrac{1}{1+n}$ hence $C^*{}_1$ has the representation as the infinite matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \dots \\ & \dots & \dots & \dots & \end{bmatrix}$$

acting on $\mathbb{C} \oplus bs \simeq bv^*$ ($\mathbb{C} \oplus bs$ is isomorphic to bv^*) which is bounded since

$$\| C_1 \|_{(bv, bv)} = \| C_1 \|_{(bv^*, bv^*)} = 1 \text{ by [5].}$$

2.2. Theorem: Let $C_1: s \to s$, where s is the space of all sequences,

then $\lambda = \dfrac{1}{1+m'}$,

$m \geq 0$ are the only eigenvalues of C_1

where $x^{(m)} = (x_n^{(m)})_{n=1}^{\infty}$ the eigenvectors

corresponding to λ are given by:

$$x_n^{(m)} = \begin{cases} \binom{n}{m}, & n \geq m \\ 0, & 1 \leq n < m \end{cases}$$

Note that when $m = 0$, $\lambda = 1$ and the eigenvector corresponding to this eigenvalue is:

$$x^{(m)} = x^{(0)} = (x_n^{(0)})_{n=0}^{\infty} = (1, 1, 1, \dots) = \delta$$

When $m \geq 1$ none of the eigenvectors corresponding to

$\lambda = \dfrac{1}{1+m}$ is bounded.

Proof: See [4]

3.2. Corollary: $C_1 \in B(c)$, where c is the space of all convergent sequences has only one eigenvalue, namely $\lambda = 1$ corresponding to the eigenvector $x^{(0)} = \delta$.

Proof: The proof follows immediately from Theorem 2.2 since

$C_1: s \to s$ has countably many eigenvalues $\lambda = \dfrac{1}{1+m}$, $m \geq 0$ corresponding to $x^{(m)}$; $\lambda = \frac{1}{1+m}$, $m \geq 1$ gives rise to unbounded sequences which cannot be in $c \subset s$. Hence $\lambda = 1$ is the only eigenvalue of $C_1 \in B(c)$.

2.4. Corollary: The only eigenvalue of $C_1 \in B$ (bv) is $\lambda = 1$.

Proof: The proof follows from Theorem 2.3 since $bv \subset c$ and both bv and c are BK–spaces with $(\delta, \delta^{\circ}, \delta^1, \delta^2, \ldots)$ as Schauder basis.

2.5. Theorem: The eigenvalues of

$C^*_1 \in B\,(bv^*) = B\,(\mathbb{C} \oplus bs)$ are all $\lambda \in \mathbb{C}$ satisfying

$|\lambda - \frac{1}{2}| \leq \frac{1}{2}$

Proof: Suppose $C^*_1\, x = \lambda\, x$, $x \in \mathbb{C} \oplus bs$, $x \neq \theta$, then solving the system of equations:

$$x_0 = \lambda\, x_0$$

$$x_1 + \tfrac{1}{2} x_2 + \tfrac{1}{3} x_3 + \ldots = \lambda\, x_1$$

$$\tfrac{1}{2} x_2 + \tfrac{1}{3} x_3 + \ldots = \lambda\, x_2$$

We obtain:

$$x_0 = 0 \text{ or } \lambda = 1$$

$$x_2 = \left(1 - \frac{1}{\lambda}\right) x_1$$

$$x_3 = \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right) x_1$$

$$\ldots$$

$$x_N = \prod_{n=2}^{N} \left(1 - \frac{1}{(n-1)\,\lambda}\right) x_1$$

$$x_{N+1} = \left[\prod_{n=2}^{N} \left(1 - \frac{1}{(n-1)\,\lambda}\right)\right]\left(1 - \frac{1}{N\lambda}\right) x_1$$

therefore $x_N / x_{N+1} = \dfrac{1}{1 - \dfrac{1}{N\lambda}} = 1 + \dfrac{1}{N\lambda - 1}$.

By Lemma 1.4., $(x_N)_{N=1}^{\infty} \in bs$ iff Re $\left(\frac{1}{\lambda}\right) \geq 1, \lambda \neq 1$

i.e. $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$. Thus the eigenvalues of

$C^*_1 \in B(bs)$ are all $\lambda \in \mathbb{C}$ such that $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$.

2.6. Corollary: Let $C_1 : bv \to bv$, then the spectrum of C_1 is given by

$$\sigma(C_1) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{1}{2}| \leq \tfrac{1}{2}\}$$

Proof: By virtue of Theorem 2.5 and the fact that $\sigma(C_1) = \sigma(C^*_1)$ (see Theorem 1.3), it is enough to prove that $(C_1 - \lambda I)^{-1} \in B(bv)$ for all λ such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. Solving the equation $(C_1 - \lambda I)\, x = y$ for x in terms of y we obtain:

$$x_0 = \frac{1}{1-\lambda}\, y_0$$

$$x_1 = -\frac{1}{(1-\lambda)(1-2\lambda)}\, y_0 = \frac{2}{1-2\lambda}\, y_1$$

$$x_2 = \frac{2\lambda}{(1-\lambda)(1-2\lambda)(1-3\lambda)}\, y_0 - \frac{2\lambda}{(1-2\lambda)(1-3\lambda)} + \frac{3}{1-3\lambda}\, y_2$$

$$\ldots \qquad \ldots \qquad \ldots$$

therefore $(C_1 - \lambda I)^{-1} = B = \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0\ldots \\ -\frac{1}{(1-\lambda)(1-2\lambda)} & \frac{2}{1-2\lambda} & 0 & 0\ldots \\ \ldots & \ldots & & \ldots \end{bmatrix}$

that is, $B = (b_{nk})$, where

$$b_{nk} = \begin{cases} -1/(1+n)\,\lambda^2 \prod_{\nu=k}^{n} \left(1 - \frac{1}{(\nu+1)\lambda}\right), & 0 \leq k < n \\ \frac{1+n}{1-(1+n)\lambda}, & n = k \end{cases}$$

Now $\|B\|_{(bv_0,\, bv_0)} = \|B\|_{(bv,\, bv)} < \infty$, $B = (b_{nk}) \in (bv, bv)$

by M. Stieglitz and H. Tietz [7]. Also

$$\lim_{n\to\infty} b_{nk} = -\frac{1}{(1+n)\lambda^2 \prod_{\nu=k}^{n}\left(1-\frac{1}{(1+\nu)\lambda}\right)} = 0 \text{ by}$$

Reade [6] Lemma 7.

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} b_{nk} = \lim_{n\to\infty} \sum_{k=0}^{n} b_{nk} \text{ exists since each row}$$

of (b_{nk}) is finite and $\sum_{k=0}^{n} b_{nk} = \frac{1}{1-\lambda}$ hence

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_{nk} = \frac{1}{1-\lambda},\ \lambda \neq 1, \text{ therefore } B \in B(bv).$$

2.7. Remark: $C_1: l_p \to l_p$ $(1 \leq p < \infty)$, where l_p is the space of all sequences x such that

$$\sum_{k=0}^{\infty} |x_k|^p < \infty \text{ normed by } \|x\| = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}$$

has no eigenvalues and the spectrum of C_1 acting on l_p is given by:

$$\sigma(C_1) = \left\{ \lambda \in \mathbb{C} : \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2} \right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Since $l_p \subset c_0$ $(1 \leq p < \infty)$ and

$C_1: c_0 \to c_0$ has no eigenvalues by [6]

$C_1: l_p \to l_p$ has no eigenvalues either.

$\sigma(C_1) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{q}{2}\right| \leq \frac{q}{2}$ follows from Leibowitz [4].

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