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AN APPLICATION OF THE FIBRATION THEOREM OF EHRESMANN

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ABSTRACT

The main purpose of the paper is to prove that the map (7) and also its restriction to $GL^+(n, \mathbb{R})$ is a locally trivial fibration.

From the general theory of fiber bundles we know that a bundle map between two C^∞–differentiable manifolds is a surjective submersion. Here arise a natural problem: given M and N two C^∞ – differentiable manifolds and f: $M \to N$ a smooth surjective submersion, find sufficient conditions in order that f be a locally trivial fibration. A such condition is given by:

Theorem. (Ehresmann [3, Th. 8.12, p. 84]). If f: $M \to N$ is a proper surjective submersion then f is a locally trivial fibration.

We shall consider

$$GL(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}): \det X \neq 0\} \tag{1}$$

the real general linear group, which is a n^2 dimensional C^∞ – differentiable manifold, as an open subset of $M_n(\mathbb{R})$. It is known that $GL(n, \mathbb{R})$ has two connected components:

$$GL^+(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}): \det X \sim 0\} \text{ and}$$
$$GL^-(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}); \det X < 0\},$$

both open in $GL(n, \mathbb{R})$.

We also consider

$$S_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}); {}^tX = X\}, \tag{2}$$

the set of symmetric matrices. Clearly we can identify $S_n(\mathbb{R})$ with the Euclidean space $\mathbb{R}^{n(n+1)/2}$. In the following we shall denote by $S^+_n(\mathbb{R})$ the subset of $S_n(\mathbb{R})$ formed of all positive definite matrices.

Finally denote by

$$O_n(\mathrm{IR}) = \{X \in GL(n, \mathrm{IR}) : {}^tX . X = I_n\}, \tag{3}$$

the set of orthogonal matrices.

In the sequel we shall use the following two results:

(a) Diagonal form of symmetric matrices). For any $A \in S_n(\mathrm{IR})$ there exists $T \in O_n(\mathrm{IR})$ such that

$$ {}^tT\,A\,T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix A. (see, for example, [2, Th. 2, p. 83]).

(b) (Polar decomposition in $GL(n, \mathrm{IR})$). Any $X \in GL(n, \mathrm{IR})$ admits a unique decomposition in the form:

$$X = OS \tag{4}$$

with $0 \in O_n(\mathrm{IR})$ and $S \in S^+_n(\mathrm{IR})$. Moreover the application $O_n(\mathrm{IR}) \times S^+_n(\mathrm{IR}) \to GL(n, \mathrm{IR})$ given by

$$(O, S) \longrightarrow O.S \tag{5}$$

is a diffeomorfism.

In this paper, by using the above mentioned result of Ehresmann, we shall obtain a locally trivial fibration of $GL(n, \mathrm{IR})$ (and respectively of $GL^+(n, \mathrm{IR})$) and we will put in evidence an interesting connection with the trivial fibration given by det: $GL^+(n, \mathrm{IR}) \longrightarrow \mathrm{IR}^*_+$ (we denote by IR^*_+ the set of real positive numbers)

Let begin with the proof of two helping results:

Lemma 1. The set $S^+_n(\mathrm{IR})$ is open in $S_n(\mathrm{IR})$.

Proof: Observe that (a) supply us with the following relation:

$$S_n(\mathrm{IR}) = \bigcup_{T \in O_n(\mathrm{IR})} T\,\{D(\lambda_1, \ldots, \lambda_n) : \lambda_i \in \mathrm{IR}\}\,{}^tT$$

where

$$D(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

In this relation we have

$$S_n^+(\mathbb{R}) = \bigcup_{T \in O_n(\mathbb{R})} T \{D(\lambda_1, \ldots, \lambda_n) : \lambda_i > 0 \text{ for all } i = 1, \ldots, n)\} {}^tT$$

But every $T\{D(\lambda_1, \ldots, \lambda_n); \lambda_i > 0 \text{ for all } i = 1, \ldots, n\} {}^tT$ is clearly open in $T\{D(\lambda_1, \ldots, \lambda_n)\ \lambda_i \in \mathbb{R}\}\ {}^tT$; consequently $S_n^+(\mathbb{R})$ is open in $S_n(\mathbb{R})$. Lemma 1 is proved.

Consider the following sets: for any $A \in GL(n, \mathbb{R}) \subset S_n(\mathbb{R})$ put

$$O_n(A, \mathbb{R}) = \{X \in GL(n, \mathbb{R});\ {}^tXX = A\}$$

and if $\det A > 0$

$$O^+_n(A, \mathbb{R}) = \{X \in O_n(A, \mathbb{R}) : \det X = \sqrt{\det A}\}$$

$$O^-_n(A, \mathbb{R}) = \{X \in O_n(A, \mathbb{R}); \det X = -\sqrt{\det A}\}$$

Clearly $O_n(I_n, \mathbb{R}) = O_n(\mathbb{R})$ and $O^+_n(I_n, \mathbb{R}) = SO_n(\mathbb{R})$ where $SO_n(\mathbb{R})$ represents the special orthogonal group.

Lemma 2. (i) The map $\varphi : S^+_n(\mathbb{R}) \longrightarrow S^+_n(\mathbb{R})$

$$X\varphi(\chi) = \chi^2 \tag{6}$$

is a proper bijection.

(ii) We have the following chain of equivalences:

$O_n(A, \mathbb{R}) \neq \varnothing \Leftrightarrow O_n^+(A, \mathbb{R}) \neq \varnothing \Leftrightarrow A \in S^+_n(\mathbb{R})$.

Proof: (i) The fact that φ is one–to–one is an immediate consequence of (a). Let's show that φ is proper: for $K \subset S^+_n(\mathbb{R})$ compact we have to prove that $\varphi^{-1}(K)$ is bounded.

A very useful norm on $M_n(\mathbb{R})$, equivalent with the Euclidean norm is

$\| A \| = [\max \{ |\lambda_i| : \lambda_i \text{ eigenvalue of } {}^tAA\}]^{1/2}$.

But if $A \in S^+_n(\mathbb{R})$ then

$\| A \| = [\max \{\lambda_i^2 : \lambda_i \text{ eigenvalue of } A\}]^{1/2}$,

so that $\| \varphi^{-1}(A) \| = \sqrt{\| A \|}$. Because K is bounded, $\varphi^{-1}(K)$ is bounded, too.

(ii) The first equivalence holds because $\det(J_nA) = -\det A$,

where $J_n = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \in O_n(\mathbb{R})$.

Assume $O_n(A, \mathbb{R}) \neq \varnothing$. Therefore $A = {}^tXX$, thus A is symmetric. If λ is an eigenvalue of A and $x \in \mathbb{R}^n$ is an eigenvector corresponding to λ, then

$$\lambda = \frac{\| X\, x \|}{\| x \|^2} > 0.$$

It follows that $A \in S^+_n(\mathbb{R})$.

Conversely, if $A \in S^+_n(\mathbb{R})$, by the surjectivity of φ one obtains that $O_n(A, \mathbb{R}) \neq \varnothing$.

Now, we are in position to state the main result of this paper.

Theorem (i) The map $f\colon GL(n, \mathbb{R}) \to S^+_n$ given by

$$X\ f(\chi) = {}^t\chi\chi \tag{7}$$

is a fibration of $GL(n, \mathbb{R})$ with the type fiber $O_n(\mathbb{R})$.

(ii) The restriction $f \big|_{GL^+(n, \mathbb{R})} : GL^+(n, \mathbb{R}) \to S^+_n(\mathbb{R})$

is also a fibration, with the type fiber $SO_n(\mathbb{R})$.

Proof: First, we will show that f is a submersion. Using the well-known result concerning the equality of the Frechet and Gateaux differentials for smooth maps, we obtain:

$$(df)_B(C) = \lim_{\lambda \to 0} \frac{1}{\lambda} [f(B + \lambda C) - f(B)] =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} [{}^t(B + \lambda C)(B + \lambda C) - {}^tBB] = {}^tBC + {}^tCB.$$

It follows that, for every $B \in GL(n, \mathbb{R})$ the differential $(df)_B\colon M_n(\mathbb{R}) \to S_n(\mathbb{R})$ is surjective. Let $D \in S_n(\mathbb{R})$ and choose $C = {}^t(B^{-1})D/2$. Then

$$(df)_B(C) = {}^tB\,{}^t(B^{-1})\,D/2 + {}^tDB^{-1}B/2 = D/2 + D/2 = D.$$

We prove now that f is a proper map. To this end, let's observe that, by using the polar decomposition (b) it follows that $f(X) = S^2$. So that $f = \varphi \circ h$, where φ is given by (6) and $h\colon GL(n, \mathbb{R}) \to S^+_n(\mathbb{R})$ is given by

$$h(X) = S. \tag{8}$$

By Lemma 2, (i), it remains to show that h is proper. If $K \subset S^+_n(\mathbb{R})$ is compact then by (5) the set $h^{-1}(K)$ is diffeomorphic to $K \times O_n(\mathbb{R})$, so that is compact (don't forget that $O_n(\mathbb{R})$ is compact).

Now, we can apply Ehresmann's theorem and deduce that $f: GL(n, \mathbb{R}) \to S^+_n(\mathbb{R})$ is a locally trivial fibration. The fiber along the identic matrix I_n is, as we have already seen, $O_n(\mathbb{R})$.

The assertion (ii) follows observing that the restriction of f to the open set $GL^+(n, \mathbb{R})$ is a proper submersion and applying then Ehresmann's theorem.

Remarks 1. Notice that both fibrations obtained in Theorem are in fact trivial, since the polar decomposition gives allways a diffeomorphism. In addition, we can give an explicit formula for the fiber along a matrix $A \in S^+_n(\mathbb{R})$ for both fibrations:

if $X_0 \in O^+_n(A, \mathbb{R}) \subset O_n(A, \mathbb{R})$ then the fiber is $O_n(\mathbb{R})\, X_0 = \{XX_0 : X \in O_n(\mathbb{R})\}$ for the first fibration, respectively $SO_n(\mathbb{R})\, X_0$ for the second one.

2. It is worth to mention the following interesting connection between the fibration (ii) and the bundle map

$$g: GL^+(n, \mathbb{R}) \to \mathbb{R}^*_+, \quad g(X) = (\det X)^2.$$

The correspondence $X \to \left((\det X)^2, \dfrac{1}{\sqrt[n]{\det X}}\, X\right)$

gives a diffeomorphism between $GL^+(n, \mathbb{R})$ and $\mathbb{R}^*_+ \times SL(n, \mathbb{R})$ which shows that g is a trivial fibration of the fiber $SL(n, \mathbb{R})$. We denote the fiber along $\alpha \in \mathbb{R}^*_+$ by $SL^{(\alpha)}(n, \mathbb{R})$; so that

$$SL^{(\alpha)}(n, \mathbb{R}) = \{X \in GL^+(n, \mathbb{R}) : \det X = \sqrt{\alpha}\}$$

Denoting $S_n^{(\alpha)}(\mathbb{R}) = \{A \in S^+_n(\mathbb{R}) : \det A = \alpha\}$ the restriction $f\big|_{SL^{(\alpha)}(n, \mathbb{R})} : SL^{(\alpha)}(n, \mathbb{R}) \to S_n^{(\alpha)}(\mathbb{R})$

will be a proper submersion. We obtain a fibration of the fiber $SL^{(\alpha)}(n, \mathbb{R})$ with the fiber along $A \in S_n^{(\alpha)}(\mathbb{R})$

$\{X \in SL^{(\alpha)}(n, \mathbb{R}) : {}^tXX = A\} = \{X \in GL^+(n, \mathbb{R}) : {}^tXX = A$ and $\det X = \sqrt{\det A}\} = SO_n(A, \mathbb{R})$,

the same fiber as in (ii).

3. There are serious reasons to believe that some of the results presented here remain valid in the case when $GL(n, \mathbb{R})$ is replaced by the automorphism group of an infinite dimensional Hilbert space.