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ON THE CONGRUENCES OF LINES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH SOME SURFACES

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In this paper, the surfaces $\vec{z}$ and $\overset{\rightrightarrows}{z}$ in [2] have been considered as the base surface $\vec{x}$. And they are referred to their lines of curvature. The congruences generated by the instantaneous screwing axes of the moving trihedrons, moving along the lines of curvatire on $\vec{z}$ and $\overset{\rightrightarrows}{z}$, have been investigated. And some corresponds with [1] have been explained.

1. INTRODUCTION

1. Let a surface $\vec{x}$ be referred to its lines of curvature. In case the congruences $\vec{y}$ and $\overset{\rightrightarrows}{y}$, generated by the instantaneous screwing axes $\vec{G}$ and $\overset{\rightrightarrows}{G}$ of the moving trihedrons connected with these lines are normal congruences, the surfaces which generate them are consecutively,

$$\vec{z} = \vec{r} + \frac{1}{\bar{b}}\vec{g} \tag{1.1}$$

$$\vec{r} = \vec{x} + \frac{1}{r}\vec{\xi} = \vec{z} + \frac{1}{\bar{b}}\vec{n}\,,\quad \vec{g} = \frac{r\vec{x}_2 + q\vec{\xi}}{\sqrt{r^2 + \bar{q}^2}}$$

and

$$\overset{\rightrightarrows}{z} = \overset{\rightrightarrows}{r} + \frac{1}{\beta}\overset{\rightrightarrows}{g} \tag{1.2}$$

$$\overset{\rightrightarrows}{r} = \vec{x} + \frac{1}{\bar{r}}\vec{\xi} = \overset{\rightrightarrows}{z} + \frac{1}{\beta}\overset{\rightrightarrows}{n}\,,\quad \vec{g} = \frac{\bar{r}\vec{x}_1 + \bar{q}\vec{\xi}}{\sqrt{\bar{r}^2 + \bar{q}^2}}$$

([1] and [2]).

2. The Surfaces $\vec{z}$, $\vec{\bar{z}}$ as base surfaces.

1) The derivative formulas of the moving trihedron $(\vec{z}_{\bar{2}}, -\vec{z}_{\bar{1}}, \vec{n})$of the lines of curvature u = const. with tangent $\vec{z}_{\bar{2}}$ on the surface z, may be stated as

$$
\begin{aligned}
\vec{z}_{\bar{2}\bar{2}} &= -\bar{a}\vec{z}_{\bar{1}} + \bar{b}\vec{n} \\
\vec{z}_{\bar{1}\bar{2}} &= \bar{a}\vec{z}_{\bar{2}} \\
\vec{n}_{\bar{2}} &= -\bar{b}\vec{z}_{\bar{2}}
\end{aligned}
\tag{2.1}
$$

Here, $\bar{\tau}_{g_v} = (\vec{z}_{\bar{2}}\ \vec{n}\ \vec{n}_{\bar{2}}) = 0$, $\bar{a} = (\vec{n}\ \vec{n}_{\bar{2}}\ \vec{z}_{\bar{2}\bar{2}})$, $\bar{b} = -(\vec{z}_{\bar{2}} . \vec{n}_{\bar{2}})$.

And the derivative formulas of the moving trihedron $(\vec{z}_{\bar{1}}, \vec{z}_{\bar{2}}, \vec{n})$ of the lines of curvature v = const. on the surface $\vec{z}$ are

$$
\begin{aligned}
\vec{z}_{\bar{1}\bar{1}} &= -a\vec{z}_{\bar{2}} + b\vec{n} \\
\vec{z}_{\bar{2}\bar{1}} &= a\,\vec{z}_{\bar{1}} \\
\vec{n}_{\bar{1}} &= -b\vec{z}_{\bar{2}}
\end{aligned}
\tag{2.1}
$$

Here $\bar{\tau}_{gu} = (\vec{z}_{\bar{1}}\ \vec{n}\ \vec{n}_{\bar{1}}) = 0$, $-a = (\vec{n}_{\bar{1}}\ \vec{z}_{\bar{1}\bar{1}})$, $b = -(\vec{z}_{\bar{1}} . \vec{n}_{\bar{1}})$.

The surface $\vec{z}$ being considered as a base surface and during its motion on the lines of curvature u = const, if the axes of the moving trihedron $(\vec{z}_{\bar{2}}, -\vec{z}_{\bar{1}}, \vec{n})$ are denoted by $\vec{Z}_1 = \vec{z}_{\bar{1}} + \varepsilon\vec{z}_{\bar{1}0}$, $\vec{Z}_2 = \vec{z}_{\bar{2}} + \varepsilon\vec{z}_{\bar{2}0}$, $\vec{Z}_3 = \vec{n} + \varepsilon n_0$, $(\varepsilon^2 = 0)$, its instantaneous screwing axis $\vec{\bar{G}}^*$ becomes,

$$\bar{G}^* = \frac{\varepsilon \vec{Z}_2 + \bar{b}\vec{Z}_1 + \bar{a}\vec{Z}_3}{\sqrt{\bar{a}^2 + \bar{b}^2}}, \quad (\bar{a} \neq 0, \ \bar{b} \neq 0). \tag{2.2}$$

Seperating $\vec{G}^*$ into its real and dual parts, we get

$$\vec{G}^* = \frac{\bar{b}\vec{z}_{\bar{1}} - \bar{a}\vec{n}}{\sqrt{\bar{a}^2 + \bar{b}^2}} + \varepsilon \frac{\vec{z}_{\bar{2}} + \bar{b}\vec{z}_{\bar{1}0} + \bar{a}\vec{n}_0}{\sqrt{\bar{a}^2 + \bar{b}^2}} = \vec{g}^* + \varepsilon \vec{g}_0 \tag{2.2}$$

$$\vec{g}^* = \frac{\bar{b}\vec{z}_{\bar{1}} + \bar{a}\vec{n}}{\sqrt{\bar{a}^2 + \bar{b}^2}} \qquad \vec{g}_0{}^* = \frac{\vec{z}_{\bar{2}} + \bar{b}\vec{z}_{\bar{1}0} + \bar{a}\vec{n}_0}{\sqrt{\bar{a}^2 + \bar{b}^2}}.$$

From (2.1) and from the above definitions of $\bar{a}$, $\bar{b}$ we find,

$$\begin{aligned} \bar{b} &= \frac{1}{\lambda} \qquad \left(\lambda = \frac{1}{\bar{b}}\right) \\ \bar{a} &= \frac{q}{\lambda r} \qquad \left(\frac{\bar{a}}{\bar{b}} = \frac{q}{r}\right). \end{aligned} \tag{2.3}$$

If we substitute these into $\vec{g}^*$, we find

$$\vec{g}^* = \vec{\xi} \tag{2.4}$$

From this the following theorem may be stated:

2.1. Theorem

The instantaneous screwing axis $\vec{G}^*$ of the trihedron $(\vec{z}_{\bar{2}}, -\vec{z}_{\bar{1}}, \vec{n})$, moving along the lines of curvature u = const. on the base surface $\vec{z}$, coincides with the axis $\vec{X}_3$ carrying the surface normal $\vec{\xi}$ of the trihedron $(\vec{x}_1, \vec{x}_2, \vec{\xi})$ moving along the lines of curvature v = cons on the base surface $\vec{x}$.

The following theorem may be derived from (2.3).

2.2. Theorem

During the motion considered in Theorem 2.1, the ratio of the geodesic curvatures and the normal curvatures of the surface $\vec{x}$ and $\vec{z}$, are equal $\left(\frac{q}{r} = \frac{\bar{a}}{\bar{b}}\right)$. In other words, the angles between the surface normals and the principal normals of these surfaces are equal ($tg\theta = tg\gamma$).

2) As the derivarative formulas of the trihedron $(\vec{z}_{1}, \vec{z}_{2}, \vec{n})$ of the lines of curvature $v = \text{const}$. On the surface $\vec{z}$, we may write.

$$
\begin{aligned}
\vec{z}_{11} &= -\alpha \vec{z}_{2} + \beta \vec{n} \\
\vec{z}_{21} &= \alpha \vec{z}_{1} \\
\vec{n}_{1} &= -\beta \vec{z}_{1}
\end{aligned}
\qquad (2.5)
$$

Here, $\tau_{gu} = (\vec{z}_{1}\, \vec{n}\, \vec{n}_{1}) = 0$, $-\alpha = (\vec{n}\, \vec{z}_{1} \vec{z}_{11})$, $\beta = -(\vec{z}_{1} . \vec{n}_{1})$.

Also from (2.5) the derivative formulas of the trihedron $(\vec{z}_{2}, -\vec{z}_{1}, \vec{n})$ of the lines of curvature $u = \text{const}$. on $\vec{z}$, are

$$
\begin{aligned}
\vec{z}_{22} &= -\bar{\alpha} \vec{z}_{1} + \bar{\beta} \vec{n} \\
\vec{z}_{12} &= \bar{\alpha} \vec{z}_{2} \\
\vec{n}_{2} &= -\bar{\beta} \vec{z}_{2}.
\end{aligned}
\qquad (2.5)
$$

Here, $\bar{\tau}_{gv} = (\vec{z}_{2}\, \vec{n}\, \vec{n}_{2}) = 0$, $\alpha = (\vec{n}\, \vec{z}_{2}\, \vec{z}_{22})$, $\beta = -(\vec{z}_{2} . \vec{n}_{2})$.

The surface $\vec{z}$ being considered as a base surface and during its motion on the lines of curvature $v = \text{const.}$, if the axes of the moving trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$ are denoted by $\vec{Z}_1 = \vec{z}_1 + \varepsilon \vec{z}_{10}$, $\vec{Z}_2 = \vec{z}_2 + \varepsilon \vec{z}_{20}$, $\vec{Z}_3 = \vec{n} + \varepsilon \vec{n}_0$, its instantaneous screwing axis $\vec{G}^{**}$ becomes

$$\vec{G}^{**} = \frac{\varepsilon \vec{Z}_1 - \beta \vec{Z}_2 - \alpha \vec{Z}_3}{\sqrt{\alpha^2 + \beta^2}}, \quad (\alpha \neq 0, \ \beta \neq 0). \tag{2.6}$$

Separating $\vec{G}^{**}$ into its real and dual parts, we get

$$\vec{G}^{**} = -\frac{\beta \vec{z}_2 + \alpha \vec{n}}{\sqrt{\alpha^2 + \beta^2}} + \varepsilon \frac{\vec{z}_1 \;\; \beta \vec{z}_{20} \;\; \alpha \vec{n}_0}{\sqrt{\alpha^2 + \beta^2}} = \vec{g}^{**} + \varepsilon \vec{g}^{**}_{o} \tag{2.6)'$$

$$\vec{g}^{**} = -\frac{\beta \vec{z}_2 + \alpha \vec{n}}{\sqrt{\alpha^2 + \beta^2}}, \quad \vec{g}^{**}_{o} \frac{\vec{z}_1 - \beta \vec{z}_{20} - \alpha \vec{n}_0}{\sqrt{\alpha^2 + \beta^2}}.$$

From (2.5) and from the above definitions of α and β, we find

$$\begin{aligned} \beta &= \frac{1}{\bar{\lambda}} \quad \left(\bar{\lambda} = \frac{1}{\beta}\right) \\ \alpha &= -\frac{1}{\bar{\lambda} \bar{r}} \quad \left(\frac{\alpha}{\beta} = -\frac{\bar{q}}{\bar{r}}\right). \end{aligned} \tag{2.7}$$

If we substitute these into $\vec{g}^{**}$, we find

$$\vec{g}^{**} = -\vec{\xi}. \tag{2.8}$$

Therefore, the following theorem may be stated:

2.3. Theorem

The instantaneous screwing axis $\vec{\bar{G}}^{**}$ of the trihedron $(\vec{\bar{z}}_{\bar{1}}, \vec{\bar{z}}_{2}, \vec{\bar{n}})$, moving along the lines of curvature $v =$ const. on the base surface $\vec{\bar{z}}$ concides with $-\vec{X}_3$ axis carrying the surface normal $\vec{\xi}$ of the trrihedon $(\vec{x}_2, -\vec{x}_1, \vec{\xi})$, moving along the lines of curvature $u =$ const. on the base surface $\vec{x}$.

The following theorem may be derived from (2.7):

2.4. Theorem

During the motion considered in Theorem 2.3, the ratio of the geodesic curvatures and the normal curvatures of the surfaces $\vec{x}$ and $\vec{\bar{z}}$, are equal with a minus sign $\left(\frac{\alpha}{\beta} = -\frac{\bar{q}}{\bar{r}}\right)$. In other words, the angles between the surface normals and the principal normals of these surfaces are equal with a minus sign ($\mathrm{tg}\omega = -\mathrm{tg}\,\bar{\theta}$).

3. The congruences generated by the instantaneous screwing axes connected with the moving trihedrons of the surfaces $\vec{z}$, $\vec{\bar{z}}$.

1) Considering $z_{gv} = 0$, $\rho_{gv} = \bar{a}$, $\rho_{nv} = \bar{b}$, the instantaneous screwing axis of the moving trihedron $(\vec{z}_{\bar{2}}, -\vec{z}_{\bar{1}}, \vec{n})$, moving along the lines of curvature $u=$const. on the surface $\vec{z}\,(u, v)$ is seen in (2.2) and (2.2)′. Therefore, the congruence generated by $\vec{\bar{G}}^{*}$ may be expressed as

$$\vec{\bar{y}}^{*} = \vec{r} + t^{*}\,\vec{\bar{g}}^{*}\ ,\ (\vec{\bar{g}}^{*} = 1). \tag{3.1}$$

Here, the reference surface $\vec{r}$.

$$\vec{r} = \vec{z} + \frac{1}{\bar{b}}\,\vec{n} \tag{3.2}$$

is the center surface belonging to the line curvature $u =$ const. on the surface $\vec{z}$ [2].

Therefore, the vectorial equation of the congruence which will be investigated may be stated as

$$\vec{y}^* = \vec{y}^* (u, v, \bar{t}^*) = z (u, v) + \frac{1}{\bar{b} ((u, v)} \vec{n} (u, v) +$$

$$+ \bar{t}^* \frac{\bar{b} (u, v) \vec{z}_{\bar{1}} (u, v) + \bar{a} (u, v) \vec{n} (u, v)}{\sqrt{\bar{a}^2 (u, v) + \bar{b}^2 (u, v)}}$$

2. Considering $\tau_{gu} = 0$, $\rho_{gu} = -\alpha$, $\rho_{nu} = \beta$, the instantaneous screwing axis of the trihedron $(\vec{z}_{\bar{1}}, \vec{z}_{\bar{2}}, \vec{n})$, moving along the lines of curvature $v =$ const. on the surface $z (u, v)$ is seen in (2.6) and (2.6)′. Therefore, the congruence generated by $\vec{G}^{**}$ may be expressed as

$$\vec{y}^{**} = \vec{r} + \bar{t}^* \vec{g}^{**}, \quad (\vec{g}^{**2} = 1). \tag{3.3}$$

Here, the reference surface $\vec{r}$,

$$\vec{r} = \vec{z} + \frac{1}{\beta} \vec{\bar{n}} \tag{3.4}$$

is the central surface belonging to the lines of curvature $v =$ const. on the surface $\vec{z}$ [2].

The vectorial equation of the line congruence to be investigated may be written as

$$\vec{y}^{**} = \vec{y}^{**} (u, v, \bar{t}^{**}) = \vec{z} (u, v) + \frac{1}{\beta (u, v)} \vec{\bar{n}} (u \; v)$$

$$-\bar{t}^{**} \frac{\beta (u, v) \vec{z}_{\bar{2}} (u, v) + \alpha (u, v) \vec{n} (u, v)}{\sqrt{\alpha^2 (u, v) + \beta^2 (u, v)}}$$

4. Properties of the congruences $\vec{y}^*$, $\vec{y}^{**}$.

In order to investigate the properties of the congruences $\vec{y}^*$, $\vec{y}^{**}$ before calculating the first and second fundamental forms, in the KUMMER sense, according to (2.4) and (2.8), (Theorems 2.1 and 2.3), the below theorem may be stated:

4.1. Theorem

The congruences $\vec{y}^*$ and $\vec{y}^{**}$ are normal congruences.

1) The first and second fundamental forms of the congruence $\vec{y}^*$ are calculated

$$\bar{\mathcal{E}}^* = \vec{g}_u^{*2} = \vec{\xi}_u^2 = \vec{\xi}_{\bar{1}}^2 \, \bar{E} = \vec{\xi}_1^2 \quad E = r^2 E$$

$$\bar{\mathcal{F}}^* = \vec{g}_u^{*2} . \vec{z}_u^* = \vec{\xi}_u . \vec{\xi}_v = \vec{\xi}_{\bar{1}} . \vec{\xi}_{\bar{2}} \sqrt{\bar{E}\,\bar{G}} = \vec{\xi}_1 . \vec{\xi}_2 \sqrt{EG} = 0 \quad (4.1.)$$

$$\bar{\mathcal{G}}^* = \vec{g}_v^{*2} = \vec{\xi}_v^{*2} = \vec{\xi}_{\bar{2}}^2 \, \bar{G} = \vec{\xi}_2^2 \quad G = \bar{r}^2 \, G ,$$

$$d\bar{\sigma}^{*2} = r^2 \, Edu^2 + \bar{r}^2 \, Gdu^2 = [\bar{I}^*] \qquad (4.2.)$$

and

$$\bar{e}^* = \vec{r}_u . \vec{g}_u^* = \vec{r}_u . \vec{\xi}_u = \vec{r}_{\bar{1}} . \vec{\xi}_{\bar{1}} \, \bar{E} = \vec{r}_1 . \vec{\xi}_1 \, \bar{E} = 0$$

$$\bar{f}^* = \vec{r}_v . \vec{g}_u^* = \vec{r}_v . \vec{\xi}_u = \vec{r}_{\bar{2}} . \vec{\xi}_{\bar{1}} \sqrt{\bar{E}\bar{G}} = \vec{r}_2 . \vec{\xi}_1 \sqrt{EG} = 0 \qquad (4.3)$$

$$\bar{f}'^* = \vec{r}_u . \vec{g}_v^* = \vec{r}_u . \vec{\xi}_v = \vec{r}_{\bar{1}} . \vec{\xi}_{\bar{2}} \sqrt{\bar{E}\bar{G}} = \vec{r}_1 . \vec{\xi}_2 \sqrt{EG} = 0$$

$$\bar{g}^* = \vec{r}_v . \vec{g}_v^* = \vec{r}_v . \vec{\xi}_v = \vec{r}_{\bar{2}} . \vec{\xi}_{\bar{2}} \, \bar{G} = \vec{r}_2 . \vec{\xi}_2 \, G = -\left(\frac{1}{r}\right)_2 \frac{K}{q} \; G,$$

$$d\vec{r} . d\vec{g}^* = d\vec{r} . d\vec{\xi} = -\left(\frac{1}{r}\right)_2 \frac{K}{q} \; Gdv^2 = [\bar{II}^*] \qquad (4.4.)$$

are found.

2. The first and second fundamental form of the congruence $\vec{y}^{**}$,

$$\bar{\mathcal{E}}^{**} = \vec{g}_u^{**2} = (-\vec{\xi}_u)^2 = \vec{\xi}^2_{\underset{1}{=}} \bar{\bar{E}} = \vec{\xi}_1^2 E = r^2 E$$

$$\bar{\mathcal{F}}^{**} = \vec{g}_u^{**} \cdot \vec{g}_v^{**} = (-\vec{\xi}_u)\cdot(-\vec{\xi}_v) = \vec{\xi}_{\underset{1}{=}} \cdot \vec{\xi}_{\underset{2}{=}} \sqrt{\bar{E}\bar{G}} = \vec{\xi}_1 \cdot \vec{\xi}_2 \sqrt{EG} = 0$$

$$\bar{\mathcal{G}}^{**} = \vec{g}_v^{**2} = (-\vec{\xi}_v) = \vec{\xi}^2_{\underset{2}{=}} \bar{\bar{G}} = \vec{\xi}^2_{\underset{2}{=}} G = \bar{r}^2 G, \tag{4.5}$$

$$d\bar{\sigma}^{**2} = r^2 Edu^2 + \bar{r}^2 Gdv^2 = [\bar{I}^{**}] \tag{4.6}$$

and

$$\bar{e}^{**} = \vec{r}_u \cdot \vec{g}_u^{**} = \vec{r}_u \cdot (-\vec{\xi}_u) = -\vec{r}_{\underset{1}{=}} \cdot \vec{\xi}_{\underset{1}{=}} \bar{\bar{E}} = -\vec{r}_1 \cdot \vec{\xi}_1 E = -\left(\frac{1}{\bar{r}}\right) \frac{K}{\bar{q}} E$$

$$\bar{f}^{**} = \vec{r}_v \cdot \vec{g}_u^{**} = \vec{r}_v \cdot (-\vec{\xi}_u) = -\vec{r}_{\underset{2}{=}} \cdot \vec{\xi}_{\underset{1}{=}} \sqrt{\bar{\bar{E}}\bar{\bar{G}}} = -\vec{r}_2 \cdot \vec{\xi}_1 \sqrt{EG} = 0$$

$$\bar{f}^{**\prime} = \vec{r}_u \cdot \vec{g}_v^{**} = \vec{r}_u \cdot (-\vec{\xi}_v) = -\vec{r}_{\underset{1}{=}} \cdot \vec{\xi}_{\underset{2}{=}} \sqrt{\bar{\bar{E}}\bar{\bar{G}}} = -\vec{r}_1 \cdot \vec{\xi}_2 \sqrt{EG} = 0 \tag{4.7}$$

$$\bar{g}^{**} = \vec{r}_v \cdot \vec{g}_v^{**} = \vec{r}_v \cdot (-\vec{\xi}_v) = -\vec{r}_{\underset{2}{=}} \cdot \vec{\xi}_{\underset{2}{=}} \bar{\bar{G}} = -\vec{r}_2 \cdot \vec{\xi}_2 G = 0,$$

$$d\vec{r} \cdot d\vec{g}^{**} = -d\vec{r} \cdot d\vec{\xi} = \left(\frac{1}{\bar{r}}\right)_1 \frac{K}{\bar{q}} Edu^2 = ([\bar{I}\bar{I}^{**}] \tag{4.8}$$

are written.

Also from (4.3) and (4.7), we see that the congruences $\vec{y}^*$ and $\vec{y}^{**}$ are normal congruences.

Since (4.2) and (4.6) are same, we may state the following theorem:

4.2. Theorem

The first fundamental forms or the linear elements of their spherical representations of the congruences $\vec{y}^*$ and $\vec{y}^{**}$ are equal.

From (4.1) which is identical of (4.5), for $\bar{\mathcal{H}}^*$ and $\bar{\mathcal{H}}^{**}$, we find,

$$\bar{\mathcal{H}}^{*2} = \bar{\mathcal{H}}^{*2} = K^2EG \tag{4.9}$$

Since EG is always different from zero, we may state the following theorem:

4.3. Theorem

Since the surface $\vec{x}$ cannot be developable, the congruences $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ cannot be cylindrical congruences.

Since the congruences $\vec{\bar{y}}^*$, $\vec{\bar{y}}^{**}$ are normal congruences, for these we may write $(a_{\underset{2}{=}} + a^2 = 0,\ a \neq 0)$ and $(\bar{\alpha}_{\underset{2}{=}} + \bar{\alpha}^2 = 0,\ \bar{\alpha} \neq 0)$ consequtively, which are similar to the conditions $(\bar{q}_1 + q^2 = 0,\ \bar{q} \neq 0)$ and $(q_2 + q^2 = 0,\ q \neq 0)$ of the congruences $\vec{y}$ and $\vec{\bar{y}}$. Since the limit points of the normal congruences $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ coincide with their focal points,

1) for $\vec{\bar{y}}^*$, from (4.1) and (4.3) we find

$$\left.\begin{aligned} \bar{l}^*_{\mathrm{I}} &= \bar{\rho}^*_{\mathrm{I}} = -\frac{r_2}{Kq} = \frac{1}{r} - \frac{1}{r} \\ \bar{l}^*_{\mathrm{II}} &= \bar{\rho}^*_{\mathrm{II}} = 0 \end{aligned}\right. \tag{4.10}$$

2) for $\vec{\bar{y}}^{**}$, from (4.5) and (4.7), we find

$$\left.\begin{aligned} \bar{l}^{**} &= \rho^{**}_{\mathrm{I}} = -\frac{\bar{r}_1}{K\bar{q}} = \frac{1}{r} - \frac{1}{\bar{r}} = -\left(\frac{1}{\bar{r}} - \frac{1}{r}\right) \\ \bar{l}^{**}_{\mathrm{II}} &= \rho^{**}_{\mathrm{II}} = 0. \end{aligned}\right. \tag{4.11}$$

Their middle points become

1) For $\vec{\bar{y}}^*$,

$$\bar{m}^* = \frac{1}{2}\left(\frac{1}{\bar{r}} - \frac{1}{r}\right), \tag{4.12}$$

2) For $\vec{\vec{y}}^{**}$,

$$\bar{m}^{**} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{\bar{r}}\right) = -\bar{m}^{*}. \tag{4.13}$$

If we take

$$\bar{\rho}^{*}{}_{I} - \rho^{*}{}_{II} = \frac{1}{\bar{r}} - \frac{1}{r} = \bar{\rho}^{*}$$

and

$$\bar{\rho}^{**}{}_{II} - \bar{\rho}^{*}{}_{II} = \frac{1}{\bar{r}} - \frac{1}{r} = \bar{\rho}^{**} = \bar{\rho}^{*},$$

1) Taking the middle surface

$$\vec{\vec{m}}^{*} = \vec{r} + \frac{\bar{\rho}^{*}}{2}\vec{\vec{g}}^{*} = \vec{r} + \frac{\bar{\rho}^{*}}{2}\vec{\xi} \tag{4.14}$$

of $\vec{\vec{y}}^{*}$ as the reference surface, the first focal surface generated by the focal point $\bar{\rho}^{*}$ becomes

$$\vec{\vec{k}}^{*} = \vec{\vec{m}}^{*} + \frac{\bar{\rho}^{*}}{2}\vec{\vec{g}}^{*} = \vec{x} + \frac{1}{\bar{r}}\vec{\xi} = \vec{\vec{r}}, \tag{4.15}$$

the second focal surface generated by the focal point $\rho^{*}{}_{II}$ becomes

$$\vec{\vec{p}}^{*} = \vec{\vec{m}}^{*} - \frac{\bar{\rho}^{*}}{2}\vec{\vec{g}}^{*} = \vec{r} = \vec{x} + \frac{1}{r}\xi = \vec{z} + \frac{1}{\bar{b}}\vec{n}. \tag{4.16}$$

2) Taking the middle surface

$$\vec{\vec{m}}^{**} = \vec{r} + \frac{\bar{\rho}^{**}}{2}\vec{\vec{g}}^{**} = \vec{\vec{r}} - \frac{\bar{\rho}^{**}}{2}\vec{\xi} \tag{4.17}$$

of $\vec{\vec{y}}^{**}$ as the reference surface, the focal surfaces generated by the focal points $\bar{\rho}^{**}{}_{I}$ and $\bar{\rho}^{***}{}_{II}$ become consequtively,

$$\vec{\vec{k}}^{**} = \vec{m}^{**} + \frac{\rho^{**}}{2}\vec{\vec{g}}^{**} = \vec{x} + \frac{1}{r}\xi = \vec{r} \tag{4.18}$$

$$\vec{\bar{p}}^{**} = \vec{m}^{**} - \frac{\bar{\rho}^{**}}{2}\,\vec{\bar{g}}^{**} = \vec{\bar{r}} = \vec{x} + \frac{1}{\bar{r}}\,\vec{\xi} = \vec{\bar{z}} + \frac{1}{\beta}\,\vec{\bar{n}}. \quad (4.19)$$

From these the following theorem may be stated:

4.4. Theorem

1°) The distance between the focal points of the congruence $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ is equal to the distance between the central points of the base surface $\vec{x}$. i.e. the focal surfaces of $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ coincide with the central surface of $\vec{x}$ $(\vec{\bar{p}}^* = \vec{\bar{k}}^{**} = \vec{r},\ \vec{\bar{k}}^* = \vec{\bar{p}}^{**} = \vec{\bar{r}})$.

2°) One focal surface of $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ each, coincides with the focal surface of $\vec{y}$ and $\vec{\bar{y}}$ each and become the reference surface $(\vec{p} = \vec{\bar{p}}^* = \vec{\bar{k}}^{**} = \vec{r},\ \vec{\bar{p}} = \vec{\bar{k}}^* = \vec{\bar{p}}^{**} = \vec{\bar{r}})$.

On the other hand, from the definition of b and (2.3) we find

$$\frac{1}{\bar{b}} - \frac{1}{b} = \rho \quad (4.20)$$

and from the definition of $\bar{\beta}$ and (2.7) we find

$$\frac{1}{\beta} - \frac{1}{\bar{\beta}} = \bar{\rho}. \quad (4.21)$$

From these, we find

$$\vec{k} = \vec{z} + \frac{1}{b}\,\vec{n}$$

and

$$\vec{\bar{k}} = \vec{\bar{z}} + \frac{1}{\bar{\beta}}\,\vec{\bar{n}}. \quad (4.23)$$

From 4.4. Theorem 2°, (4.16) and (4.18) we get

$$\vec{p} = \vec{z} + \frac{1}{\bar{b}} \vec{n}$$

$$\vec{\bar{p}} = \vec{\bar{z}} + \frac{1}{\beta} \vec{\bar{n}}.$$

Therefore the following theorem may be stated:

4.5. Theorem

The distance between the focal points of the congruences $\vec{y}$ and $\vec{\bar{y}}$ is equal to the distance between the central points of the surfaces $\vec{z}$ and $\vec{\bar{z}}$. i.e. the focal surfaces of $\vec{y}$ coincide with the central surfaces of $\vec{z}$, the focal surfaces of $\vec{\bar{y}}$ coincide with the central surfaces of z.

Since the congruences $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ are the normal congruences, their principal surfaces which are developable,

1) for $\vec{\bar{y}}^*$

$$EG \frac{r_2 K}{q} dudv = 0, \tag{4.24}$$

2) for $\vec{\bar{y}}^{**}$

$$-EG \frac{\bar{r}_1 K}{\bar{q}} dudv = 0 \tag{4.25}$$

are found. Since $EG \neq 0$, $r_2 \neq 0$ $r_1 \neq 0$ and $K \neq 0$ in (4.24) and (4.25), the following theorem may be stated:

4.6. Theorem

The principal surfaces which are developable of the normal congruences $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$ are parametric surfaces.

The mean ruled surfaces of the congruences $\vec{\bar{y}}^*$ and $\vec{\bar{y}}^{**}$, we find

1) for $\vec{\bar{y}}^*$

$$\bar{\mathcal{E}}^* \, du^2 - \bar{\mathcal{G}}^* \, dv^2 = 0 \tag{4.26}$$

and

2) for $\vec{y}^{**}$

$$\bar{\mathcal{G}}^{**} \, dv^2 - \bar{\mathcal{E}}^{**} \, du^2 = 0. \tag{4.27}$$

Since $\bar{\mathcal{E}}^* = \bar{\mathcal{E}}^{**}$ and $\bar{\mathcal{G}}^* = \bar{\mathcal{G}}^{**}$, the folloving theorem may be stated:

4.7. Theorem

The mean ruled surfaces of the normal congruences $\vec{y}^*$ and $\vec{y}^{**}$ coincide. As the condition of the congruences $\vec{y}^*$ and $\vec{y}^{**}$ to become isotropic congruences we find

1) for $\vec{y}^*$

$$-\left(\frac{1}{r}\right)_2 \frac{K}{q} G = 0, \tag{4.28}$$

2) for $\vec{y}^{**}$

$$-\left(\frac{1}{r}\right)_1 \frac{K}{\bar{q}} E = 0. \tag{4.29}$$

Since $E \neq 0$, $G \neq 0$, $r_2 \neq 0$, $r_1 \neq 0$, we may state the followiog theorem:

4.8. Theorem

Since the base surface $\vec{x}$ cannot be developable and canal surface, the congruences $\vec{y}^*$ and $\vec{y}^{**}$ cannot become isotropic congruences.

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