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AUTHORS: Nuri KURUOGLU

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## ON THE CHARACTERISTIC PROPERTIES OF THE $a$ -PEDAL SURFACES IN THE EUCLIDEAN SPACE $E^3$

NURİ KURUOĞLU, AYHAN SARIOĞLUGİL

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### ABSTRACT

The purpose of this paper is, first, to present the definition of pedal surface and to give some new characteristic properties of the pedal surfaces in the Euclidean 3-space  $E^3$ , ([1] Chapter 1); second, after the definition of  $a$ -pedal surface to give some new characteristic properties of the  $a$ -pedal surfaces related to the support function, Gauss curvature, mean curvature, the area element, the first and second fundamental forms and their coefficients (Chapter 2).

Using the classical methods of surface theory in differential geometry we have established that the support function of the  $a$ -pedal surface ( $\bar{h}_a$ ) is equal to  $h^{a+1}/P_a$ ,  $h$  being the support function

of the original surface and  $P_a^2 = h^2 + a^2 \nabla^{\text{III}}(h, h)$ , where  $\nabla^{\text{III}}(h, h)$  is the second Beltrami's operator with respect to third fundamental form.

Moreover, for the special case  $a=1$ , we get the results of the paper [2].

### I. INTRODUCTION

Let  $M$  be a smooth, closed surface in the Euclidean 3-space  $E^3$  and  $O$  be a point in the interior of  $M$ . If  $X$  is the position vector at a point  $p \in M$ , with respect to  $O$  as origin and  $N$  is the inner unit normal of  $M$ , then the surface with the position vector  $\bar{X} = -hN$ , with respect to  $O$  as origin, is called the pedal surface of  $M$ , with respect to  $O$ , [2].

Geometrically we can construct the pedal surface  $M$  as follows:

We draw the tangent plane  $\Sigma$  at  $p$  on  $M$  and from  $O$  we get the normal to the plane  $\Sigma$ . The normal meets  $\Sigma$  at a point  $\bar{p}$ . The locus of all points  $\bar{p}$  which correspond to all the points  $p$  on  $M$  will give the pedal surface. Therefore, the position vector of the point  $\bar{p} \in \bar{M}$  can be given by

$$(1.1) \quad \bar{X} = -hN$$

where  $h$  is the support function of  $M$  at  $p$  and  $N$  is the unit normal of  $M$ . Then, for the support function  $h$ , we can give the following formula.

$$(1.2) \quad h = - \langle X, N \rangle$$

$X$  being the position vector of  $p \in M$  and  $\langle, \rangle$  is the usual inner product of  $E^3$ , [3].

In this paper the length of the position vector  $X$  of the point  $p \in M$  will be denoted by  $\rho$ . If we consider a local parameter system on  $M$  with parameters  $u, v$ , then the position vector  $X$  has expression  $X(u, v)$  and the normal to the pedal surface  $\bar{M}$  is given by

$$(1.3) \quad \bar{N} = \frac{\bar{X}_i \times \bar{X}_j}{\|\bar{X}_i \times \bar{X}_j\|},$$

where  $\bar{X}_i, \bar{X}_j$  are partial derivatives, with respect to  $u$  and  $v$ , respectively. We also have

$$(1.4) \quad \bar{h} = - \langle \bar{X}, \bar{N} \rangle$$

for the support function  $\bar{h}$  of the pedal surface  $\bar{M}$  at  $\bar{p}$ . If we substitute the equation (1.1) in the equation (1.4) we get

$$(1.5) \quad \bar{h} = h \langle N, \bar{N} \rangle$$

Moreover by differentiating the position vector  $\bar{X}$  with respect to  $u$  and  $v$  we get

$$(1.6) \quad \bar{X}_i = - h_i N - h N_i, \quad i=u, v.$$

We can express  $X$  locally in terms of  $h, N$  and the inverse tensor ( $n^{ik}$ ) of the third fundamental form  $III = (n_{ik})$  of  $M$ , with respect to an arbitrary parameter system, namely,

$$(1.7) \quad X = -hN - \sum_{i,k} n^{ik} h_i N_k, \quad i, k=u, v,$$

where  $h_i, N_k$  are the partial derivatives, with respect to the local parameters  $u$  and  $v$ , [2].

In addition, for the second Beltrami's operator  $\nabla^{\text{III}}(h, h)$ , with respect to third fundamental form, the following formula can be given.

$$(1.8) \quad \nabla^{\text{III}}(h, h) = \frac{h_i^2 n_{22} + h_j^2 n_{11} - 2h_i h_j n_{12}}{n_{11} n_{22} - n_{12}^2}, \quad i=u, j=v,$$

where  $n_{ij}$ ,  $1 \leq i, j \leq 2$ , are the coefficients of the third fundamental form of  $M$ , [1]. The following characteristic properties about the pedal surface can be given without proof from the paper [2].

**Corollary 1.1.**

$$(1.9) \quad \rho^2 = h^2 + \nabla(h,h),$$

where  $\rho^2 = \langle X, X \rangle$ , [2].

**Theorem 1.2.** For the coefficient  $g_{ij}$  of the first fundamental form of  $\bar{M}$ ,  $1 \leq i, j \leq 2$ , we have, [2].

$$(1.10) \quad \begin{aligned} \bar{g}_{11} &= h_1^2 + h^2 n_{11} \\ \bar{g}_{22} &= h_2^2 + h^2 n_{22} \\ \bar{g}_{12} &= h_1 h_2 + h^2 n_{12}, \quad i=u, j=v. \end{aligned}$$

**Corollary 1.3.**

$$(1.11) \quad \begin{aligned} \bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2 &= \rho^2 h^2 (n_{11} n_{22} - n_{12}^2), \\ \text{or} \\ \bar{g}_{11} \bar{g}_{22} - \bar{g}_{12}^2 &= \rho^2 h^2 K^2 (g_{11} g_{22} - g_{12}^2), \end{aligned}$$

where  $K$  denotes the Gauss curvature of  $M$  and  $g_{ij}$ ,  $1 \leq i, j \leq 2$ , are the coefficients of the first fundamental form of  $M$ , [2].

**Theorem 1.4.** The support function  $h$  of the pedal surface  $\bar{M}$  is equal  $h^2/\rho$ , [2].

**Theorem 1.5.**

$$(1.12) \quad \begin{aligned} \bar{b}_{11} &= \frac{1}{\rho h} (-h^2 b_{11} + 2h g_{11}), \\ \bar{b}_{22} &= \frac{1}{\rho h} (-h^2 b_{22} + 2h g_{22}), \\ \bar{b}_{12} &= \bar{b}_{21} = \frac{1}{\rho h} (-h^2 b_{12} + 2h g_{12}), \end{aligned}$$

where  $\bar{b}_{ij}$ ,  $b_{ij}$ ,  $1 \leq i, j \leq 2$ , are the coefficients of the second fundamental forms of  $\bar{M}$  and  $M$ , respectively, [2].

**Corollary 1.6.**

$$(1.13) \quad \bar{II} = \frac{1}{\rho h} (-h^2 II + 2h \bar{I}),$$

where  $\bar{\Pi}$ ,  $\bar{I}$  are the second and first fundamental forms of the pedal surface  $\bar{M}$  and  $\Pi$  is the second fundamental form of  $M$ , [2].

**Corollary 1.7.** The mean curvature  $\bar{H}$  and The Gauss curvature  $\bar{K}$  of the pedal surface  $\bar{M}$  satisfy the following equations, [2].

$$(1.14) \quad \bar{H} = \frac{1}{2\rho^3 h K} (4\rho^2 h K - \overset{\Pi}{\nabla}(h, h) - 2h^2 H),$$

$$(1.15) \quad \bar{K} = \frac{1}{\rho^4 h K} (4\rho^2 h K + h - 2 \overset{\Pi}{\nabla}(h, h) - 4h^2 H),$$

where  $H$ ,  $K$  are the mean and Gauss curvatures of  $M$ , respectively, and

$\overset{\Pi}{\nabla}(h, h)$  is the first Beltrami's operator, with respect to the second fundamental form of  $M$ .

## II. THE CHARACTERISTIC PROPERTIES OF THE $a$ -PEDAL SURFACES IN THE EUCLIDEAN SPACE $E^3$

In this section, after the definition of  $a$ -pedal surface, we would like to give some new characteristic properties of the  $a$ -pedal surfaces defined in  $E^3$ .

**Definition 2.1.** Let  $M$  be a smooth, convex, closed surface in  $E^3$  and  $O$  be a point in the interior of  $M$ . If  $X$  is the position vector of a point  $p \in M$ , with respect to  $O$  as origin and  $N$  is the inner unit normal vector field of  $M$  at  $p$ , then, for a given real number  $a$ , the surface with the position vector  $\bar{X}_a = -h^a N$ , with respect to  $O$  as origin, is called the  $a$ -pedal surface of  $M$ , with respect to  $O$ , and denoted by  $\bar{M}_a$ . Where  $h$  is the support function of  $M$  at  $p \in M$ .

The coefficients of first fundamental form of  $\bar{M}_a$  can be given in terms of the support function  $h$  and the third fundamental form of  $M$ , by the following theorem.

**Theorem 2.1.** For the coefficients  $(\bar{g}_a)_{ij}$  of first fundamental form of  $\bar{M}_a$ ,  $1 \leq i, j \leq 2$ , we have

$$(2.1) \quad \begin{aligned} (\bar{g}_a)_{11} &= a^2 h^{2(a-1)} h_1^2 + h^{2a} n_{11}, \\ (\bar{g}_a)_{22} &= a^2 h^{2(a-1)} h_2^2 + h^{2a} n_{22}, \\ (\bar{g}_a)_{12} &= a^2 h^{2(a-1)} h_1 h_2 + h^{2a} n_{12}, \quad i = u, j = v. \end{aligned}$$

**Proof:** By differentiating the position vector  $\bar{X}_a$  with respect to  $u$  and  $v$ , we get

$$(2.2) \quad (\bar{X}_a)_i = ah^{a-1}h_i N - h^a N_i, \quad i = u, v.$$

Because of (2.2) and the definition, of the coefficients of first fundamental form, we can easily get the results of the theorem.

For the special case  $a = 1$ , we have the results of the Theorem 1.2.

**Corollary 2.2.**

$$(2.3) \quad (\bar{g}_a)_{11}(\bar{g}_a)_{22} - (\bar{g}_a)_{12}^2 = h^{4a-2} P_a^2(n_{11}n_{22} - n_{12}^2)$$

or

$$(2.4) \quad (\bar{g}_a)_{11}(\bar{g}_a)_{22} - (\bar{g}_a)_{12}^2 = K^2 h^{4a-2} P_a^2(g_{11}g_{22} - g_{12}^2)$$

where  $K$  is Gauss curvature of  $M$  and

$$(2.5) \quad P_a^2 = h^2 + a^2 + \nabla^{\text{III}}(h, h).$$

For  $a = 1$ , we have Corollary 1.1 and the Corollary 1.3.

From the Corollary 2.2. and the definition of the area element of  $a$ -pedal surface  $\bar{M}_a$ , we have the following corollary.

**Corollary 2.3.**

$$d\bar{A}_a = Kh^{2a-1}P_a dA.$$

Let  $\bar{N}_a$  be the unit inner normal of the  $a$ -pedal  $\bar{M}_a$ . If we rewrite the equation (1.3) for  $\bar{N}_a$ , we get

$$(2.6) \quad \bar{N}_a = \frac{(\bar{X}_a)_i \times (\bar{X}_a)_j}{\|(\bar{X}_a)_i \times (\bar{X}_a)_j\|}, \quad i=u, j=v.$$

Moreover, if we substitute the (2.2) in this last equation, we observe that

$$(2.7) \quad \bar{N}_a = \frac{ah_i N \times N_j - ah_j N \times N_i + h N_i \times N_j}{KP_a \sqrt{g_{11}g_{22} - g_{12}^2}}.$$

**Theorem 2.4.** For the support function  $\bar{h}_a$  of the  $a$ -pedal surface  $\bar{M}_a$ , we have

$$\bar{h}_a = h^{a+1} \cdot P_a^{-1}.$$

**Proof:** If we consider (2.7) and Corollary 2.2 together with (1.5), we get the result, namely,

$$(2.8) \quad \bar{h}_a = h^{a+1} \cdot P_a^{-1},$$

where  $P_a^2 = h^2 + a^2 \overset{\text{III}}{\nabla}(h, h)$ .

If  $a = 1$ , we get the Theorem 1.4.

The coefficients of the second fundamental form of  $\bar{M}_a$  can be given in terms of the support function  $h$ , the coefficients of the first fundamental form of  $\bar{M}_a$  and the coefficients of the second fundamental form of  $M$ , by the following theorem.

**Theorem 2.5.**

$$(2.9) \quad \begin{aligned} (b_a)_{11} &= h^{-a} P_a^{-1} \{-ah^{2a} b_{11} + (a+1)h (\bar{g}_a)_{11}\} \\ (b_a)_{22} &= h^{-a} P_a^{-1} \{-ah^{2a} b_{22} + (a+1)h (\bar{g}_a)_{22}\} \\ (b_a)_{12} &= h^{-a} P_a^{-1} \{-ah^{2a} b_{12} + (a+1)h (\bar{g}_a)_{12}\} \end{aligned}$$

where  $(\bar{b}_a)_{ij}$ ,  $b_{ij}$ ,  $1 \leq i, j \leq 2$ , are the coefficients of the second fundamental forms  $\bar{M}_a$ ,  $M$ , respectively.

**Proof:** If we differentiate the vector  $(\bar{X}_a)_i$  and the support function  $h$  of  $M$  with respect to the parameters  $u$  and  $v$ , and if we use the formulas of  $(\bar{b}_a)_{ij}$ ,  $1 \leq i, j \leq 2$ , we obtain the results of the theorem.

For  $a = 1$ , we have the results of the Theorem 1.5.

As a consequence of the Theorem 2.1, from the definition of first and second fundamental forms of a surface, we have the following corollary.

**Corollary 2.6.**

$$(2.10) \quad \Pi_a = h^{-a} P_a^{-1} \{-ah^{2a}\Pi + (a+1)h \bar{I}_a\},$$

where  $\Pi_a$ ,  $\Pi$  and  $\bar{I}_a$  are the second and first fundamental forms of the  $a$ -pedal surface  $\bar{M}_a$  or, of the original surface  $M$ .

For  $a = 1$ , we get the Corollary 1.6.

**Theorem 2.7.** The mean curvature  $\bar{H}_a$  of the  $a$ -pedal surface  $\bar{M}_a$  satisfies

$$(2.11) \quad \bar{H}_a = \frac{1}{2h^a K P_a^3} \{2(a+1)h K P_a^2 - a^3 (a+1) \overset{\text{II}}{\nabla}(h, h) - 2ah^2 H\},$$

where  $\overset{\text{II}}{\nabla}(h, h)$  is the first Beltrami's operator, with respect to second fundamental form,  $K$  and  $H$  are the Gauss and the mean curvatures of  $M$ .

**Proof.** If we consider (2.1) and (2.9) together with the equation

$$\bar{H}_a = \frac{(\bar{b}_a)_{11}(\bar{g}_a)_{22} - 2(\bar{b}_a)_{12}(\bar{g}_a)_{12} + (\bar{b}_a)_{22}(\bar{g}_a)_{11}}{2 [(\bar{g}_a)_{11}(\bar{g}_a)_{22} - (\bar{g}_a)_{12}^2]}$$

and if we use the first Beltrami's operator, with respect to second fundamental form of  $M$ , we get the result.

For  $a = 1$ , we have the Corollary 1.7.

**Theorem 2.7.** The Gauss curvature  $\bar{K}_a$  of the  $a$ -pedal surface  $\bar{M}_a$  satisfies

$$(2.13) \quad \bar{K}_a = \frac{1}{h^{2a} K^2 P_a^4} \{ (a+1)^2 h^2 K^2 P_a^2 + a^2 h^2 K - a^3 (a+1) h^2 K \nabla^{\text{II}}(h, h) - 2a (a+1) h^3 H K \}.$$

**Proof:** The Gauss Curvature of  $\bar{M}_a$ , in terms of first and second fundamental forms of  $\bar{M}_a$ , can be given by

$$(2.14) \quad \bar{K}_a = \frac{(\bar{b}_a)_{11}(\bar{b}_a)_{22} - (\bar{b}_a)_{12}^2}{(\bar{g}_a)_{11}(\bar{g}_a)_{22} - (\bar{g}_a)_{12}^2}$$

Using Theorem 2.1, Theorem 2.4 and Theorem 2.5, we obtain result of the theorem. For the special case  $a=1$ , we have the Corollary 1.7.

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