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THE DEFORMATION RETRACT OF BOTH COVERING SPACE OF THE COMPLEX PROJECTIVE SPACE AND ITS TOPOLOGICAL FOLDING

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ABSTRACT

In this paper the relation between the deformation retract of a manifold homeomorphic to the complex projective n-space cp^n and the deformations retract of its covering space is investigated. Furthermore, this relation discussed after and before the isometric and topological folding of the covering space into itself. Theorems governing the considered relation are given.

1. INTRODUCTION

Aiming to our study we consider the covering spaces of the complex projective n-space cp^n and cp^{n-1}, also cp^1. The present article is considered as a continuation of [1]. A continuous map $p : \bar{M} \rightarrow M$ is called a covering projection if each point $x \in M$ has an open neighbourhood evenly covered by p, $\bar{M}$ is called a covering space of a manifold M. In this work we prove that among all the deformation retracts of the considered covering spaces there is a deformation retract which is a covering space of the deformation retract of a manifold M homeomorphic to the complex projective space. The relation between these deformation retracts after and before the isometric and topological folding of the covering space into itself are studied.

2. DEFINITIONS AND PRELIMINARIES

In complex (n + 1)-space c^{n+1}, consider the subspace defined by $|z| = 1$, where if $z = (z_0, z_1, \ldots, z_n)$, we define $|z|^2 = |z_0|^2 + \ldots + |z_n|^2$; clearly this space is the hypersphere s^{2n+1}. We identify $z \backsim z'$ on s^{2n+1} if $z' = cz$, where c is a complex number of absolute value 1; the resulting quotient space is called cp^n, cp^n is a compact connected 2n-dimensional manifold.

A map $F : M \to N$, where M, N are C^∞ Riemannian manifolds of dimensions m, n respectively, is said to be an isometric folding of M into N, if and only if for any piecewise geodesic path $\gamma : J \to M$, the induced path $F \circ \gamma : J \to N$ is a piecewise geodesic and of the same length as γ, $J = [0, 1]$. If F does not preserve length, then F is a topological folding [2], [3]. Any manifold M homeomorphic to the complex projective space cp^n, the deformation retract of M is homeomorphic to the deformation retract of cp^n.

A subset A of a topological sapce M is a deformation retract of M if there exists a retraction $R : M \to A$ and a homotopy $\varnothing : M \times I \to M$ such that

$$\left.\begin{aligned} \varnothing(x,0) &= x \\ \varnothing(x,1) &= R(x) \end{aligned}\right\} \quad x \in M$$

$$\varnothing(a,t) = a, \ a \in A \text{ and } t \in I = [0,1] \ [4].$$

By using the Lagrangian equations:

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \psi_i'}\right) - \frac{\partial T}{\partial \psi_i} = 0, \quad i = 1,2,\ldots,n$$

we determined a geodesic $s^{2n-1} \subset s^{2n+1}$.

Let A be a deformation retract of topological space M $i : A \subset M$ be the inclusion, $r : M \to A$ be the retraction and $H : M \times I \to M$ be the deformation such that $H_0 = 1_X$ and $H_1 = i \circ r$. Let η be a fibre-bundle over M with total space E, fibre F, and projection $P : E \to M$. Since homotopic maps induce isomorphic bundles and since $1_X \simeq i \circ r$, $\eta = (i \circ r)' \eta = r' \circ i' (\eta) = r' (\eta /A)$ where η /A denotes the restriction of η to A and $\imath' (\eta /A)$ its pull-back to M by the retraction r.

Define a retraction $\bar{r} : \eta = r' (\eta /A) \to \eta /A$ by $\bar{r} (x,e_A) = e_A$ for $e_A \in p^{-1} (A)$ and $x \in M$ such that $r (x)=p (e_A)$ and a deformation $\bar{H} : \eta \times I = r' (\eta /A) \times I \to r' (\eta /A) = \eta$ by $\bar{H} (x, e_A, t) = (H (x, t), e_A)$. Then $\bar{H}_0 = 1$ and $\bar{H}_1 = j \circ \bar{r}$ where $j : \eta /A \subset \eta$ is the inclusion. Hence $\bar{r}$ defines a deformation retraction of η to its restriction η /A to A This proves the following.

Proposition: Let A be a deformation retract of a topological space M and η be a fibre bundle over M. Then the restriction η /A of η to A is a deformation retract of η.

3. THE MAIN RESULTS

Theorem 1: Let $P_1 : s^7 \to cp^3$ be a covering projection of the hypersphere s^7 onto the complex projective space cp^3 in c^4, assume $\varnothing_i(s^7)$ the set of all deformation retracts of s^7, $\overline{\varnothing}_i(cp^3)$ the corresponding set of cp^3, then there exist one deformation retract $\varnothing_1(s^7)$, also there is an induced one deformation retract $\overline{\varnothing}_1(cp^3)$ and $\varnothing_1(s^7)$ is a covering space of $\overline{\varnothing}_1(cp^3)$ with covering projection p_2, $p_2 \circ \varnothing_1 = \overline{\varnothing}_1 \circ p_1$.

Proof: Let $\varnothing : (s^7 - \{a_i\}) \times I \to (s^7 - \{a_i\})$ be the deformation retract of s^7 with retraction $r : (s^7 - \{a_i\}) \to s^5$, where $s^n = \{(x_0, x_1, \ldots, x_{n-1}) : x^2_0 + \ldots + x^2_n = 1\}$, $\{a_i\}$ any two antipodal points.) There are many types of $\varnothing$ say : $\varnothing_i(s^7)$, take $\varnothing = \varnothing_1((z_0,z_1,z_2,z_3,); t) = (z_0, z_1,z_2,z_3)\ [1 - t] + (z_0,z_1,z_2,0)\ t$, $t \in [0,1]$, where

$$\left.\begin{array}{lll} \varnothing_1(x,0) & = x & , \\ \varnothing_1(x,1) & = r(x) & \\ r(x) & = s^5 & \end{array}\right\} \quad x \in (s^7 - \{a_i\})$$

Assume $\overline{\varnothing} : \{cp^3 - \beta^i\} \times I \to \{cp^3 - \beta^i\}$ is the deformation retract of the complex projective space cp^3, β^i is any point in cp^3. The point in cp^3 is $(z; cz)$, $|c| = 1$. We identify $z \backsim z'$ on s^7 if $z' = cz$, take $c = e^{i\pi}$ the resulting quotient space is called cp^3. The parametric equation for cp^3 is given by

$$\begin{aligned} \underline{r} = [&r \cos\psi_1 \prod_{k=1}^{6} \sin\psi_{k+1},\quad r \prod_{k=1}^{7} \sin\psi_k, \ldots, r\cos\psi_{i-1} \prod_{k=i}^{7} \sin\psi_k, \\ &\ldots, r\cos\psi_7, - r\cos\psi_1 \prod_{k=1}^{6} \sin\psi_{k+1}, - r \prod_{k=1}^{7} \sin\psi_k, \ldots, \\ &- r\cos\psi_{i-1} \prod_{k=1}^{7} \sin\psi_k, \ldots, - r\cos\psi_7] =: [\alpha; -\alpha], \\ &i = 3,4,\ldots,7. \end{aligned} \tag{I}$$

We discuss the deformation retract of $\{cp^3 - \beta^i\}$ by using Lagrangian equation:

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \psi'}\right) - \frac{\partial T}{\partial \psi} = 0, \text{ where } \psi = \psi_1, \psi_2, \ldots, \psi_7 \tag{1}$$

put $ds^2 = \left(\sum_{i=0}^{7} \overline{dx}_i^2; - \sum_{i=0}^{7} \overline{dx}_i^2\right)$ and $T = \frac{1}{2}\,\overline{ds}^2$, by using (1), we have

$$\frac{d}{ds}\left(r^2 \prod_{i=2}^{7} \sin^2 \psi_i \, \psi_1'\right) = 0, \tag{2}$$

$$\frac{d}{ds}\left(r^2 \prod_{i=3}^{7} \sin^2 \psi_i \, \psi_2'\right) - r^2 \sin \psi_2 \cos \psi_2 \prod_{i=3}^{7} \sin^2 \psi_i \, \psi_1'^2 = 0 \tag{3}$$

If we take $\psi_2 = 0$, whence $\sin \psi_2 = 0$, substitute in (I) we obtain

$$x_0 = 0,$$
$$x_1 = 0,$$
$$x_2 = r \prod_{k=3}^{7} \sin \psi_k$$
$$x_3 = r \cos \psi^3 \prod_{k=4}^{7} \sin \psi_k,$$
$$\vdots$$
$$x_7 = r \cos \psi_7;$$
$$-x_0 = 0,$$
$$-x_1 = 0,$$
$$-x_2 = -r \prod_{k=3}^{7} \sin \psi_k,$$
$$-x_3 = -r \cos \psi_3 \prod_{k=4}^{7} \sin \psi_k$$
$$\vdots$$
$$-x_7 = -r \cos \psi_7.$$

Now, we define the deformation retract of cp^3 (I) onto a geodesic $cp^2 \subset cp^3$ as follows:

$$\overline{\varnothing}_1(m,t) = [(r \cos \psi_1 \prod_{k=1}^{6} \sin \psi_{k+1}, \; r \prod_{k=1}^{7} \sin \psi_k, \ldots,$$

$$r \cos \gamma_{i-1} \prod_{k=i}^{7} \sin \psi_k, \ldots, r \cos \psi_7;$$

$$- r \cos \psi_1 \prod_{k=1}^{6} \sin \psi_{k+1}, \; - r \prod_{k=1}^{7} \sin \psi_k, \; \ldots,$$

$$- r \cos \psi_7) - \{\beta^i\}] \, [1 - t] + t [0,0,r \prod_{k=3}^{7} \sin \psi_k,$$

$$r \cos \psi_3 \prod_{k=4}^{7} \sin \psi_k, \; \ldots, \; r \cos \psi_7; -0, -0,$$

$$- r \prod_{k=3}^{7} \sin \psi_k, \; \ldots, \; - r \cos \psi_7],$$

$$\overline{\varnothing}_1(m,1) = [0,0,r \prod_{k=3}^{7} \sin \psi_k, \; \ldots, \; r \cos \psi_7; -0, -0,$$

$$- r \prod_{k=3}^{7} \sin \psi_k, \; \ldots, \; - r \cos \psi_7] = cp^2, \qquad (4)$$

then the diagram

$$\begin{array}{ccc} s^7 & \xrightarrow{\varnothing_1} & s^5 \\ p_1 \downarrow & & \downarrow p_2 \\ cp^3 & \xrightarrow[\overline{\varnothing}_1]{} & cp^2 \end{array}$$

is commutative, so $\overline{\varnothing}_1 \circ p_1 = p_2 \circ \varnothing_1$.

Theorem 2: Let $F_1 : s^7 \to s^7$ such that $F_1(x_0,x_1,x_2,\ldots,x_7) = (|x_0|, |x_1|, x_2, \ldots, x_7)$ be an isometric folding, there is an induced isometric folding $\bar{F}_1 : cp^3 \to cp^3$ such that $\varnothing_1 \circ F_1 = \varnothing_1 \Rightarrow \overline{\varnothing}_1 \circ \bar{F}_1 = \overline{\varnothing}_1$, for any foldings homeomorphic to F_1, $\bar{F}_1$, $\varnothing_1$, $\overline{\varnothing}_1$ defined in Theorem 1.

Proof: Consider the following diagram

$$\begin{array}{ccccc} s^7 & \xrightarrow{F_1} & s^7 & \xrightarrow{\varnothing_1} & s^5 \\ p_1 \downarrow & & & & \downarrow p_2 \\ cp^3 & \xrightarrow[\bar{F}_1]{} & cp^3 & \xrightarrow[\overline{\varnothing}_1]{} & cp^2 \end{array}$$

from the diagram we have $\overline{\varnothing}_1 \circ \bar{F}_1 \circ p_1 = p_2 \circ \varnothing_1 \circ F_1$, $\overline{\varnothing}_1$ since $\circ\, p_1 = p_2 \circ \varnothing_1$, then $\overline{\varnothing}_1 \circ \bar{F}_1 \circ p_1 = p_2 \circ \varnothing_1 = \overline{\varnothing}_1 \circ p_1 \Rightarrow \overline{\varnothing}_1 \circ \bar{F}_1 = \overline{\varnothing}_1$. If $F_2(x_0, x_1, x_2, \ldots, x_7) = (x_0, x_1, |x_2|, |x_3|, x_4, \ldots, x_7)$ from the proof of Theorem 1 we obtain

$$\varnothing_1 \circ F_2 = \varnothing_1 \circ F_2(x_0, x_1, \ldots, x_7) = \varnothing_1((x_0,x_1, |x_2|, |x_3|, \ldots, x_7), 1)$$

$= ((0,0, |x_2|, |x_3|, \ldots, x_7), 1) \neq \varnothing_1((x_0,x_1,\ldots, x_4), 1)$ according to this $\overline{\varnothing}_1 \circ \bar{F}_2 \neq \overline{\varnothing}_1$.

Theorem 3: Let $p_2 : s^5 \rightarrow cp^2$ be a covering projection, $\eta_i(s^5)$ the set of all deformation retracts of s^5, $\tilde{\eta}_i(cp^2)$ the corresponding set of cp^2, then there is an induced one deformation retract $\eta_1(s^5)$, also there is an induced one deformation retract $\tilde{\eta}_1(cp^2)$ and $\eta_1(s^5)$ is a covering space of $\tilde{n}_1(cp^2)$ with covering projection p_3 and $p_3 \circ \eta_1 = \tilde{n}_1 \circ p_2$.

Proof: Consider $\eta_1 : (s^5 - \{a_i\}) \times I \rightarrow (s^5 - \{a_i\})$ with $\eta_1(m,t) = (x_2, x_3, x_4, x_5, x_6, x_7)(1 - t) + (0,0,0,x_5,x_6,x_7).t$ (5)

with retraction $r' : (s^5 - \{a_i\}) \rightarrow s^2$, $r'(s^5 - \{a_i\}) = (0,0,0,x_5,x_6,x_7)$, also for cp^2 we define the deformation retract as follows: The parametric equation of the complex projective space cp^2 is given from (4) by

$$\underline{r} = [r \prod_{k=3}^{7} \sin\psi_k,\ r\cos\psi_3 \prod_{k=6}^{7} \sin\psi_k, \ldots, r\cos\psi_7;$$

$$- r \prod_{k=3}^{7} \sin\psi_k,\ - r\cos\psi_3 \prod_{k=4}^{7} \sin\psi_k, \ldots, - r\cos\psi_7].$$

Since $ds^2 = \left(\sum_{i=3}^{7} dx_i^2; - \sum_{i=3}^{7} dx_i^2 \right)$, $T = \frac{1}{2} ds^2$ we obtain the deformation retract of cp^2 by using Lagrangian equations $\frac{d}{ds}\left(\frac{\partial T}{\partial \psi'_i}\right) - \frac{\partial T}{\partial \psi_i} = 0$, $\psi_i = 3,4,\ldots, 7$, choose $\psi_5 = 0$, as in proof of Theorem 1 we obtain the $\tilde{n}_i(cp^2)$, take

$$\overline{\eta}_1 : \{cp^2 - \beta^i\} \times I \rightarrow \{cp^2 - \beta^i\}$$

such that

$$\overline{\eta}_1(m,t) = \{[r \prod_{k=3}^{7} \sin\psi_k,\ r\cos\psi_3 \prod_{k=4}^{7} \sin\psi_k, \ldots, r\cos\psi_7;$$

$$- r \prod_{k=3}^{7} \sin \psi_k, - r \cos \psi_3 \prod_{k=4}^{7} \sin \psi_k, \ldots, - r \cos \psi_7]$$

$$- \beta^i\} (1 - t) + t \{0,0,0,0,0,r \prod_{k=6}^{7} \sin \psi_k, r \cos \psi_6 \sin \psi_7,$$

$$r \cos \psi_7; - 0, - 0, - 0, - 0, - 0, - r \prod_{k=6}^{7} \sin \psi_k,$$

$$- r \cos \psi_6 \sin \psi_7, - r \cos \psi_7\}$$

with retraction

$$\bar{r}'(cp^2 - \beta^i) = \tilde{\eta}_1(m,1) = \{0,0,0,0,0,r \prod_{k=6}^{7} \sin \psi_k, r \cos \psi_6 \sin \psi_7,$$

$$k \cos \psi_7; -0, -0, -0, -0, -0, - r \prod_{k=6}^{7} \sin \psi_k,$$

$$- r \cos \psi_6 \sin \psi_7, - r \cos \psi_7\},$$

and the diagram

$$\begin{array}{ccc} s^5 & \xrightarrow{\eta_1} & s^2 \\ p_2 \downarrow & & \downarrow p_3 \\ cp^2 & \xrightarrow[\tilde{\eta}_1]{} & cp^1 \end{array}$$

is commutative i.e. $\tilde{\eta}_1 \circ p_1 = p_1 \circ \eta_1$.

Theorem 4: Let $F_1 : s^5 \to s^5$ such that $F_1(x_2,x_3,x_4,x_5,x_6,x_7,) = (|x_2|, |x_3|, x_4, x_5, x_6, x_7)$ or $F_1 = (x_2, |x_3|, |x_4|, x_5, x_6, x_7)$ be an isometric folding, there is an induced isometric folding $\bar{F}_1 : cp^2 \to cp^2$ such that $\eta_1 \circ F_1 = \psi_1 \Rightarrow \bar{\psi}_1 \circ \bar{F}_1 = \bar{\psi}_1$ and this is true for any foldings homeomorphic to F_1, $\bar{F}_1$.

Proof: We define the following diagram

$$\begin{array}{ccccc} s^5 & \xrightarrow{F_1} & s^5 & \xrightarrow{\eta_1} & s^2 \\ p_2 \downarrow & & & & \downarrow p_3 \\ cp^2 & \xrightarrow[\bar{F}_1]{} & cp^2 & \xrightarrow[\tilde{\eta}_1]{} & cp^1 \end{array}$$

we have $\bar{\eta}_1$ o $\bar{F}_1$ o $p_2 = p_3$ o η_1 o F_1 put η_1 o $F_1 = \eta_1$ we obtain $\bar{\eta}_1$ o $\bar{F}_1$ o $p_2 = p_3$ o η_1, from Theorem 3, we get

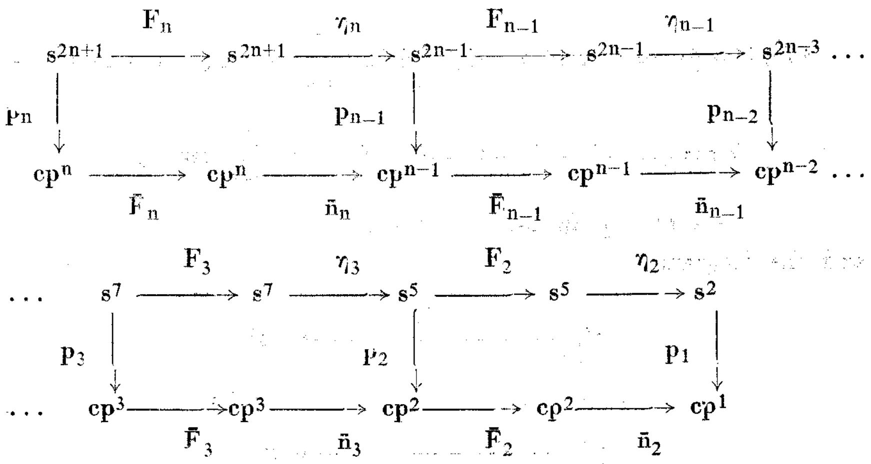
$$\bar{\eta}_1 \text{ o } \bar{F}_1 \text{ o } p_2 = p_3 \text{ o } \eta_1 = \bar{n}_1 \text{ o } p_2 \Rightarrow \bar{n}_1 \text{ o } \bar{F}_1 = \bar{n}_1.$$

If we take $F_2(x_2,x_3,x_4,x_5,x_6,x_7) = (x_2,x_3,x_4,x_5,|x_6|,|x_7|)$ we obtain

$$\eta_1 \text{ o } F_2 = \eta_1((x_2,x_3,x_4,x_5,|x_6|,|x_7|), 1) = (0,0,0,x_5,|x_6|,|x_7|)$$

from (5) we have η_1 o $F_1 \neq \eta_1$ (m,1) $\Rightarrow \bar{\eta}_1$ o $\bar{F}_2 \neq \bar{\eta}_2$.

The above results of Theorems (1), (2), (3) and (4) will be generalized for cp^n, cp^{n-1}, ..., cp^3, cp^2, cp^1 and we arrive to the following diagram:



such that $\bar{\eta}_i$ o $\bar{F}_i = \bar{\eta}_i$, η_i o $F_i = \eta_i$. $i=2,3,\ldots,n$, p_{i-1} o η_i o $F_i = \bar{\eta}_i$ o $\bar{F}_i$ o p_i or p_{i-1} o $\eta_i = \bar{\eta}_i$ o p_i $i=2,3,4,\ldots,n$.

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