Commun. Fac. Sci. Univ. Ank. Series A_1
V. 43. pp. 41-47 (1994)

ON HOLDITCH'S THEOREM AND POLAR INERTIA MOMENTUM

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(Received Dec. 8, 1992: Revised April. 22, 1994; Aceepted June 17, 1994)

ABSTRACT

A. Broman generalised Holditch's Theorem to the closed rectifiable curves [2]. In this paper we obtained an interesting expression, which is similar to Holditch's Formula, for the polar inertia momentums of the closed rectifiable curves and we show that the ratio between the polar inertia momentums of the curves and the areas bounded by them is 1 / 2.

I. INTRODUCTION

Hamnet Holditch, president of Casuj College in Cambridge during the middle part of the last century, showed that the ring area between a closed convex curve α and the curve β traced out by a point on a chord of fixed length that slides around with both endpoints on α is equal to πab, that is $I_\alpha - I_\beta = \pi ab$ [1]. Where a and b denote the distances of endpoints to dividing point and I_α and I_β denote the areas bounded by the curves α and β, respectively.

This study is discussed by A. Broman and pointing out some limitations he gave a modern explanation of Holditch's Theorem [2] as figure 1. After he generalised this theorem as follows:

Theorem 1. Let a be a closed rectifiable curve with parametric representation $x = x(t)$, $y = (t)$, $0 \leq t \leq 1$. Let $\theta = \theta(t)$, $0 \leq t \leq 1$, be a continuous function of bounded variation with $\theta(1) = \theta(0) + 2n\pi$, n denoting an integer. Let a and b be positive numbers. Let $A = A(t)$, $0 \leq t \leq 1$, be the point traverse α, and for each t, let $B = B(t)$ be the point such that AB is a line segment with length $a + b$ and direction angle $\theta = \theta(t)$. Let $D = D(t)$ be the point of AB at the distance a from A. Denote by β and δ the curves traced out by B and D, respectively, as A traverses α. Set

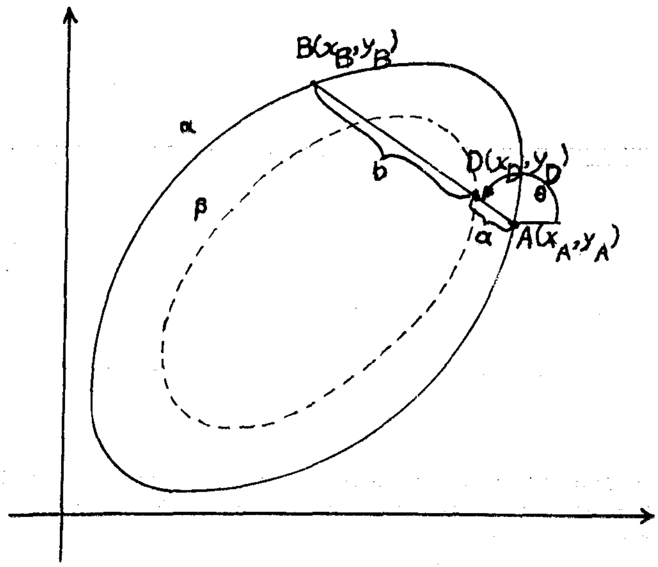

FIGURE 1

$$I_\alpha = \int_\alpha x dy, \; I_\beta = \int_\beta x dy, \; I_\delta = \int_\delta x dy.$$

Then

$$I_\delta = \frac{1}{a+b} \{bI_\alpha + aI_\beta\} - n\pi ab.$$

II. HOLDITCH'S FORMULA AND INERTIA MOMENTUM

Let r be the distance of a point on the curve α to the origin point O and θ denote the direction angle. For the distribution of mass with the density $d\theta$, the polar inertia momentum of α according to O is given by

$$T_\alpha = \int_\alpha r^2 d\theta$$

where the integration is taken along the curve α [3].

Now we will begin with an elementary problem. In a circle α of radius r and center origin O, a chord is divided into parts of lengths a and b by a point D. As the chord makes a complete sweep around the circle, an inner circle, the locus of D, is traced out (see figure 2, where the dotted curve is the locus). Let β be the locus of the point D and z its radius.

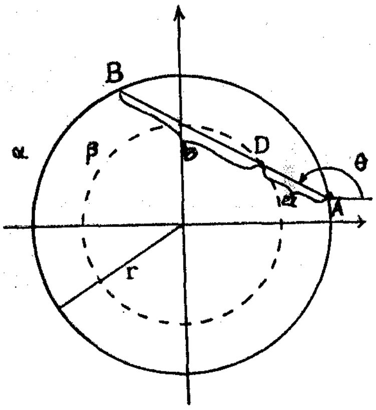

FIGURE 2

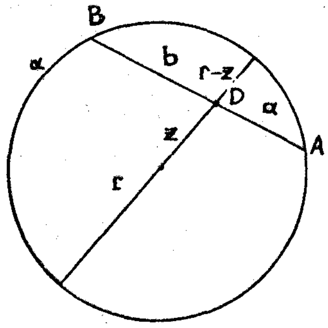

FIGURE 3

The polar inertia momentums of α and β, according to the origin point O, are

$$T_\alpha = \int_0^{2\pi} r^2 d\theta = 2\pi r^2 \tag{2}$$

and

$$T_\beta = \int_0^{2\pi} z^2 d\theta = 2\pi z^2 \tag{3}$$

respectively, where r and z are independent of θ. From (2) and (3)

$$T_\alpha - T_\beta = 2\pi (r^2 - z^2). \tag{4}$$

If it is considered

$$ab = (r + z)(r - z)$$

(see Figure 3) and the equation (4) then we obtain

$$T_\alpha - T_\beta = 2\pi ab. \tag{5}$$

Now, we can give the following theorem which shows that the relation (5) is valid when the outer circle is replaced by a more general curve.

Theorem 2. Let α be a convex closed curve, and suppose that $A = A(t)$ and $B = B(t)$, $0 \leq t \leq 1$, are points that traverse α counter clockwise one revolution as t increases from 0 to 1, so that AB is a chord of α with a constant length $a + b$. Let $D = D(t)$ be the dividing point of AB so that $|AD| = a$. Suppose that the direction angle $\theta = \theta(t)$ of the chord AB (see figure 1) is an increasing continuous function of t with $\theta(1) = \theta(0) + 2\pi$. Let β be the locus of D, and assume that β is a simple closed curve. Then the difference between the polar inertia momentums of α and β according to the origin point O is $2\pi ab$.

Proof: The polar inertia momentum of α according to the origin point O is

$$T_\alpha = \int_\alpha (x^2_A + y^2_A)\, d\theta \tag{6}$$

or

$$T_\alpha = \int_\alpha (x^2_B + y^2_B)\, d\theta.$$

Suppose that $s = a + b$ then

$$T_\alpha = \int_\alpha (x^2_B + y^2_B)\, d\theta = \int_\alpha ((x_A + s\cos\theta)^2 + (y_A + s\sin\theta)^2)\, d\theta$$

or

$$T_\alpha = \int_\alpha (x^2_A + y^2_A)\, d\theta + 2s \int_\alpha (x_A \cos\theta + y_A \sin\theta) d\theta + s^2 \int_\alpha d\theta.$$

in the last equality, when equation (6) is considered we observe that

$$\int_\alpha (x_A \cos\theta + y_A \sin\theta)\, d\theta = - s\pi. \tag{7}$$

The polar inertia momentum of β according to the origin point O is

$$T_\beta = \int_\beta (x^2_D + y^2_D)\, d\theta = \int_\alpha ((x_A + a\cos\theta)^2 + (y_A + a\sin\theta)^2) d\theta$$

or

$$T_\beta = T_\alpha + 2a \int_\alpha (x_A \cos\theta + y_A \sin\theta)\, d\theta + a^2 \int_\alpha d\theta. \tag{8}$$

On the other hand we have

$$\int_\alpha d\theta = 2\pi. \tag{9}$$

Considering (7) and (9) in (8) we obtain

$$T_\beta = T_\alpha - \pi ab. \tag{10}$$

This is the desired result.

Although the assumptions that α is convex and β is a simple closed curve seem essential to Theorem 2, we can relax these requirements and we give the following theorem which is a generalisation of Theorem 2.

Theorem 3. Let α be a closed rectifiable curve with parametric representation $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$. Let $\theta = \theta(t)$, $0 \leq t \leq 1$, be a continuous function of bounded variation with $\theta(1) = \theta(0) + 2n\pi$, n denoting an integer. Let a, b be positive numbers. Let point

$A = A(t)$, $0 \leq t \leq 1$, traverse α, and for each t, let $B = B(t)$ be the point such that AB is line segment with length $a + b$ and direction angle $\theta = \theta(t)$. Let $D = D(t)$ be the point of AB at the distance a from A. Denote the curves traced out by B and D with β and δ, respectively, as A traverses α. Then

$$T_\delta = \frac{1}{a+b}\{bT_\alpha + aT_\beta\} - 2n\pi ab. \tag{11}$$

Proof: The assumptions that α is rectifiable and θ is of bounded variation imply that β and δ are rectifiable. The condition $\theta(1)=\theta(0)+2n\pi$ obviously has the effect that the line segment AB comes back to starting position when the point A has gone around the curve α (figure 4). The integer n is known as winding number of AB [4]. The polar inertia momentum of curve β drawn by the point B according to the origin point O, is

$$T_\beta = \int_\beta (x^2_B + y^2_B)\, d\theta = \int_\delta ((x_D + b \cos\theta)^2 + (y_D + b \sin\theta)^2) d\theta$$

or

$$T_\beta = \int_\delta (x^2_D + y^2_D)\, d\theta + 2b \int_\delta (x_D \cos\theta + Y_D \sin\theta)\, d\theta + b^2 \int_\delta d\theta.$$

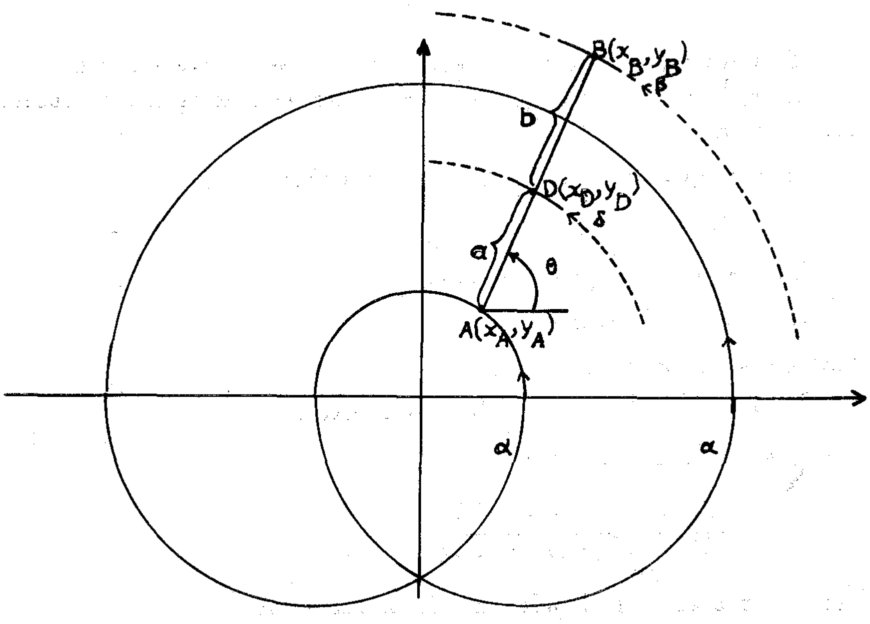

FIGURE 4

Denote

$$T = \int_\delta (x_D \cos\theta + y_D \sin\theta)\, d\theta, \; T_\theta = \int_\delta d\theta.$$

Then

$$T_\beta = T_\delta + 2bT + b^2 T_\theta.$$

Analogously it can be written

$$T_\alpha = T_\delta - 2aT + a^2 T_\theta.$$

From the last equations, we get

$$bT_a + aT_\beta = (a + b)\, T_\delta + ab\,(a + b)\, T_\theta.$$

If it is considered that

$$T_\theta = \int_\delta d\theta = 2n\pi$$

we obtain (11). This is desired result.

From (11), if the polar inertia momentums of α and β according to the origin point O is known, the polar inertia momentum of δ according to the origin point O can be easily found and for the polar inertia momentums of them, we have the following difference equation:

$$T_\delta - \frac{1}{a+b}\,\{bT_a + aT_\beta\} = z2n\pi ab.$$

This difference does not depend on the chosen curve α and the motion of the chord, but it depends on the distances of endpoints to chosen point on the chord AB.

In the special case $\alpha = \beta$, it follows that

$$T_\delta = T_\alpha - 2n\pi ab.$$

In the case of $n = 1$ we have

$$T_\delta = T_\alpha - 2\pi ab$$

that gives Theorem 2.

Finally, we can give the following result.

Result: There exist the ratio

$$\frac{bI_\alpha + aI_\beta - (a + b)\, I_\delta}{bT_\alpha + aT_\beta - (a + b)\, T_\delta} = \frac{1}{2}$$

between the areas and polar inertia momentums.

Proof: It can be easily proved by using Theorem 1 and Theorem 3.

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