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QUASI - HADAMARD PRODUCT OF p - VALENT FUNCTIONS

M.K. AOUF., A. SHAMANDY and M.F. YASSEN

Department of Mathematics Faculty of Science University of Mansoura Mansoura, EGYPT.

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ABSTRACT

The authors establish certain results concerning the quasi-Hadamard product of analytic and p-valent functions with negative coefficients analogous to the results due to Vinod Kumar.

1. INTRODUCTION

Let $S_p(\alpha, \beta, \lambda)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z: |z| < 1\}$ and satisfy the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\alpha \dfrac{zf'(z)}{f(z)} + p - \lambda(\alpha+1)} \right| < \beta \tag{1.2}$$

for some α $(0 \leq \alpha \leq 1)$, β $(0 < \beta \leq 1)$, λ $(0 \leq \lambda < p)$ and for all $z \in U$. The class $S_p(\alpha, \beta, \lambda)$ was studied by Owa and Aouf [4].

Throughout the paper, let the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0,\ a_{p+n} \geq 0,\ p \in N), \tag{1.3}$$

$$f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i}\, z^{p+n} \quad (a_{p,i} > 0,\ a_{p+n,i} \geq 0,\ p \in N), \tag{1.4}$$

$$g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \quad (b_p > 0,\ b_{p+n} \geq 0,\ p \in N), \tag{1.5}$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{p+n,j} z^{p+n} \quad (b_{p,j} > 0,\ b_{p+n,j} \geq 0,\ p \in N), \tag{1.6}$$

be analytic and p–valent in U.

Let $S^*_p(\alpha, \beta, \lambda)$ denote the class of functions f(z) of the form (1.3) and satisfying (1.2) for some α, β, λ and for all $z \in U$. Also let $C^*_p(\alpha, \beta\ \lambda)$ denote the class of functions of the form (1.3) such that

$$\frac{zf'(z)}{p} \in S^*_p(\alpha, \beta, \lambda).$$

We note that when $a_p = \alpha = \beta = 1$, the classes $S^*_p(1, 1, \lambda) = T^*(p, \lambda)$ and $C^*_p(1, 1, \lambda) = C(p, \lambda)$ were studied by Owa [3].

Using similar arguments as given by Owa [3] we can easily prove the following analogous results for functions in the classes $S^*_p(\alpha, \beta, \lambda)$ and $C^*_p(\alpha, \beta, \lambda)$.

A function f(z) defined by (1.3) belongs to the class $S^*_p(\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} [\{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}\, a_{p+n}] \leq \beta(1+\alpha)(p-\lambda)\, a_p \tag{1.7}$$

and f(z) defined by (1.3) belongs to the class $C^*_p(\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)\{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}\, a_{p+n}\right] \leq \beta(1+\alpha)(p-\lambda)\, a_p. \tag{1.8}$$

We now introduce the following class of analytic and p–valent functions which plays an important role in the discussion that follows:

A function f(z), defined by (1.3), belongs to the class $S^*_{p,k}(\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=1}^{\infty}\left[\left(\frac{p+n}{p}\right)^{k}\{n(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\}\,a_{p+n}\right]\leq\beta(1+\alpha)(p-\lambda)\,a_{p}, \tag{1.9}$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $0 \leq \lambda < p$ and k is any fixed nonnegative real number.

We note that, for every nonnegative real number k, the class $S^*_{p,k}(\alpha, \beta, \lambda)$ is nonempty as the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{\beta(1+\alpha)(p-\lambda)\,a_p}{\left(\frac{p+n}{p}\right)^{k}\{n(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\}}\,\lambda_{p+n} z^{p+n}, \tag{1.10}$$

where $a_p > 0$, $\lambda_{p+n} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n} \leq 1$, satisfy the inequality (1.9).

It is evident that $S^*_{p,1}(\alpha, \beta, \lambda) = C_p^*(\alpha, \beta, \lambda)$ and, for $k=0$, $S^*_{p,k}(\alpha, \beta, \lambda)$ is identical to $S^*_p(\alpha, \beta, \lambda)$. Further, $S^*_{p,k}(\alpha, \beta, \lambda) \subset S^*_{p,k}(\alpha, \beta, \lambda)$ if $k > h \geq 0$, the containment being proper. Whence, for any positive integer k, we have the inclusion relation

$S^*_{p,k}(\alpha, \beta, \lambda) \subset S^*_{p,k-1}(\alpha, \beta, \lambda) \subset \ldots \subset S^*_{p,2}(\alpha, \beta, \lambda) \subset C^*_p(\alpha, \beta, \lambda) \subset S^*_p(\alpha, \beta, \lambda)$.

Let us define the quasi–Hadamard product of the functions f(z) and g(z) by

$$f*g(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}. \tag{1.11}$$

Similarly, we can define the quasi–Hadamard product of more then two functions.

In this paper we establish certain results concerning the quasi–Hadamard product of functions in the classes $S^*_{p,k}(\alpha, \beta, \lambda)$, $S^*_p(\alpha, \beta, \lambda)$ and $C^*_p(\alpha, \beta, \lambda)$ analogous to the results due to Vinod Kumar [1, 2].

2. THE MAIN THEOREMS

Theorem 1. Let functions $f_i(z)$ defined by (1.4) be in the class $C^*_p(\alpha, \beta, \lambda)$ for every $i = 1, 2, \ldots, m$; and let the functions $g_j(z)$

defined by (1.6) be in the class $S^*_p(\alpha, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi-Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $S^*_{p,\, 2m+q-1}(\alpha, \beta, \lambda)$.

Proof: We denote the quasi-Hadamard product $f_1 * f_2 * \ldots * f_m * g_1 * g_2 \ldots * g_q(z)$ by the function $g(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{ \prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q} b_{p,\, j} \right\} z^p - \sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{m} a_{p+n,\, i} \prod_{j=1}^{q} b_{p+n,\, j} \right\} z^{p+n}. \quad (2.1)$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right)^{2m+q-1} \{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\} \left\{ \prod_{i=1}^{m} a_{p+n,\, i} \prod_{j=1}^{q} b_{p+n,j} \right\} \right]$$

$$\leq \beta(1+\alpha)(p-\lambda) \left(\prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q} b_{p,\, j} \right). \quad (2.2)$$

Since $f_i(z) \in C^*_p(\alpha, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p} \right) \{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\} a_{p+n,\, i} \right] \leq \beta(1+\alpha)(p-\lambda) a_{p,\, i}, \quad (2.3)$$

for every $i = 1, 2, \ldots, m$. Therefore

$$\left(\frac{p+n}{p} \right) \{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda) \} a_{p+n,\, i} \leq \beta(1+\alpha)(p-\lambda) a_{p,\, i}$$

or

$$a_{p+n,\, i} \leq \left[\frac{\beta(1+\alpha)(p-\lambda)}{\left(\frac{p+n}{p} \right) \{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}} \right] a_{p,\, i},$$

for every $i = 1, 2, \ldots, m$. The right-hand expression of this last inequality is not greater than $\left(\frac{p+n}{p} \right)^{-2} a_{p,\, i}$. Hence

$$a_{p+n,\, i} \leq \left(\frac{p+n}{p}\right)^{-2} a_{p,\, i}, \tag{2.4}$$

for every $i = 1, 2, \ldots, m$. Similarly, for $g_j(z) \in S^*_p(\alpha, \beta, \lambda)$, we have

$$\sum_{n=1}^{\infty} \left[\{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}\, b_{p+n,\, j} \right] \leq \beta(1+\alpha)(p-\lambda)\, b_{p,\, j} \tag{2.5}$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$b_{p+n,\, j} \leq \left(\frac{p+n}{p}\right)^{-1} b_{p,\, j}, \tag{2.6}$$

for every $j = 1, 2, \ldots, q$.

Using (2.4) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q-1$, and (2.5) for $j = q$, we get

$$\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)^{2m+q-1} \{n(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\} \left\{ \prod_{i=1}^{m} a_{p+n,\, i} \prod_{j=1}^{q} b_{p+n,\, j} \right\} \right]$$

$$\leq \sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)^{2m+q-1} \{n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}\, b_{p+n,\, q} \right.$$

$$\left. \cdot \left\{ \left(\frac{p+n}{p}\right)^{-2m} \left(\frac{p+n}{p}\right)^{-(q-1)} \prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q-1} b_{p,\, j} \right\} \right]$$

$$= \sum_{n=1}^{\infty} \left[\{ n(1+\alpha\beta) + \beta(1+\alpha)(p-\lambda)\}\, b_{p+n,\, q} \right] \left(\prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q-1} b_{p,\, j} \right)$$

$$\leq \beta(1+\alpha)(p-\lambda) \left(\prod_{i=1}^{m} a_{p,\, i} \prod_{j=1}^{q} b_{p,\, j} \right).$$

Hence $h(z) \in S^*_{p,\, 2m+q-1}(\alpha, \beta, \lambda)$. This completes the proof.

We note that the required estimate can also be obtained by using (2.4) for $i = 1, 2, \ldots, m-1$, (2.6) for $j = 1, 2, \ldots, q$, and (2.3) for $i = m$.

Now we discuss the applications of Theorem 1. Taking into account the quasi-Hadamard product of the functions $f_1(z), f_2(z), \ldots, f_m(z)$ only, in the proof of Theorem 1, and using (2.4) for $i = 1, 2, \ldots, m-1$ and (2.3) for $i = m$, we are led to

Corollary 1. Let the fuctions $f_i(z)$ defined by (1.4) belong to the class $C^*_p(\alpha, \beta, \lambda)$ for every $i = 1, 2, \ldots, m$. Then the quasi–Hadamard product $f_1 * f_2 * \ldots * f_m(z)$ belongs to the class $S^*_{p, 2m-1}(\alpha, \beta, \lambda)$.

Next, taking into account the quasi–Hadamard product of the functions $g_1(z), g_2(z), \ldots, g_q(z)$ only, in the proof of Theorem 1, and using (2.6) for $j = 1, 2, \ldots, q-1$ and (2.5) for $j = q$, we are led to.

Corollary 2. Let the functions $g_j(z)$ defined by (1.6) belong to the class $S^*_p(\alpha, \beta, \lambda)$ for every $j = 1, 2, \ldots, q$. Then, the quasi–Hadamard product $g_1 * g_2 * \ldots * g_q(z)$ belongs to the class $S^*_{p, q-1}(\alpha, \beta, \lambda)$.

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