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ON p - VALENT FUNCTIONS WITH NEGATIVE AND MISSING COEFFICIENTS II

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ABSTRACT

Let $P_k(p, A, B, \alpha)$ be the class of functions $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$, $k \geq 2$, analytic and p-valent in the unit disc $U = \{z: |z| < 1\}$ and satisfying the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{[pB + (A-B)(p-\alpha)] - B\dfrac{zf'(z)}{f(z)}} \right| < 1, \quad z \in U,$$

where $-1 \leq B < A \leq 1$, $-1 \leq B < 0$, and $0 \leq \alpha < p$.

In this paper we obtain coefficient estimate, distortion and closure theorems and the radius of convexity for the class $P_k(p, A, B, \alpha)$. We also obtain class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt, \quad c > -p$$

for the class $P_k(p, A, B, \alpha)$. Conversely when $F(z) \in P_k(p, A, B, \alpha)$, radius of p-valence of $f(z)$ has been determined.

1. INTRODUCTION

Let S_p $(p \geq 1)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z: |z| < 1\}$. For $-1 \leq B < A \leq 1$, $-1 \leq B < 0$, and $0 \leq \alpha < p$, let $P^*(p, A, B, \alpha)$ be the class of those functions $f(z)$ of S_p for which

$\frac{zf'(z)}{f(z)}$ is subordinate to $\frac{p + [pB + (A-B)(p-\alpha)]z}{1 + Bz}$. In other words

$f(z) \in P^*(p, A, B, \alpha)$ if and only if there exists a function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{zf'(z)}{f(z)} = \frac{p + [pB + (A-B)(p-\alpha)]\, w(z)}{1 + B\, w(z)}, \quad z \in U. \tag{1.2}$$

The condition (1.2) is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{[pB + (A-B)(p-\alpha)] - B\frac{zf'(z)}{f(z)}} \right| < 1, \quad z \in U. \tag{1.3}$$

Let T_p denote the subclass of S_p consisting of functions analytic and p–valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}|\, z^{p+n} \quad (k \geq 2). \tag{1.4}$$

Let us define

$$P_k(p, A, B, \alpha) = P^*(p, A, B, \alpha) \cap T_p.$$

Aouf [1, 2], Gupta and Jain [5, 6], Goel and Sohi [4], Sarangi and Uralegaddi [29], Shukla and Dashrath [10], and Silverman [11] have studied certain subclasses of analytic functions with negative coefficients and Kumar [7] and Aouf [3] have recently studied certain subclasses of analytic functions with negative and missing coefficients.

In this paper, under the assumption $-1 \leq B < 0$, and $k \geq 2$, we obtain coefficient estimate, distortion theorem, covering theorem and radius of convexity for the class $P_k(p, A, B, \alpha)$. We also obtain the class preserving integral operators of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt, \quad c > -p \tag{1.5}$$

for the class $P_k(p, A, B, \alpha)$. Conversely, when $F(z) \in P_k(p, A, B, \alpha)$, we determine the radius of p–valence of $f(z)$ defined by (1.5). Lastly we show that the class $P_k(p, A, B, \alpha)$ is closed under "arithmetic mean" and "convex linear combinations".

Remark: We observe that our distortion theorem (Theorem 2) improves the results of Sarangi and Patil [8, Theorem 3].

2. COEFFICIENT ESTIMATE

Theorem 1. Let the function f(z) be defined by (1.4). Then f(z) $\in P_k(p, A, B, \alpha)$ if and only if

$$\sum_{n=k}^{\infty} [(1-B)n + (A-B)(p-\alpha)] \, |a_{p+n}| \leq (A-B)(p-\alpha). \tag{2.1}$$

The result is sharp.

Proof: Let $|z| = 1$, then

$$|zf'(z) - pf(z)| - |[pB + (A-B)(p-\alpha)] f(z) - Bzf'(z)|$$

$$= \Big| -\sum_{n=k}^{\infty} n \, |a_{p+n}| \, z^{p+n} \Big| - \Big| (A-B)(p-\alpha) z^p + \sum_{n=k}^{\infty} [nB + (B-A)(p-\alpha)] \, |a_{p+n}| \, z^{p+n} \Big|$$

$$\leq \sum_{n=k}^{\infty} [(1-B)n + (A-B)(p-\alpha)] \, |a_{p+n}| - (A-B)(p-\alpha) \leq 0$$

(since $-1 \leq B < 0$).

Hence by the principle of maximum modulus $f(z) \in P_k(p, A, B, \alpha)$.

Conversely, suppose that

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{[pB + (A-B)(p-\alpha)] - B\dfrac{zf'(z)}{f(z)}} \right|$$

$$= \left| \frac{-\sum_{n=k}^{\infty} n \, |a_{p+n}| \, z^{p+n}}{(A-B)(p-\alpha) z^p + \sum_{n=k}^{\infty} [nB + (B-A)(p-\alpha)] \, |a_{p+n}| \, z^{p+n}} \right| < 1, \; z \in U.$$

Since $|\operatorname{Re}(z)| \leq |z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{n=k}^{\infty} n\,|a_{p+n}|\,z^{p+n}}{(A-B)(p-\alpha)z^p + \sum_{n=k}^{\infty}[nB+(B-A)(p-\alpha)]\,|a_{p+n}|\,z^{p+n}}\right\} < 1. \quad (2.2)$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=k}^{\infty}[(1-B)n+(A-B)(p-\alpha)]\,|a_{p+n}| \leq (A-B)(p-\alpha).$$

This completes the proof of the theorem.

Sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)(p-\alpha)}{(1-B)n+(A-B)(p-\alpha)}\,z^{p+n} \quad (n \geq k,\ k \geq 2). \qquad (2.3)$$

3. DISTORTION PROPERTIES

Theorem 2. If a function f(z) defined by (1.4) is in the class $P_k(p, A, B, \alpha)$, then for $|z| = r$

$$r^p - \frac{(A-B)(p-\alpha)}{(1-B)k+(A-B)(p-\alpha)}\,r^{p+k} \leq |f(z)|$$

$$\leq r^p + \frac{(A-B)(p-\alpha)}{(1-B)k+(A-B)(p-\alpha)}\,r^{p+k} \qquad (3.1)$$

and

$$p\,r^{p-1} - \frac{(A-B)(p-\alpha)(p+k)}{(1-B)k+(A-B)(p-\alpha)}\,r^{p+k-1} \leq |f'(z)|$$

$$\leq p\,r^{p-1} + \frac{(A-B)(p-\alpha)(p+k)}{(1-B)k+(A-B)(p-\alpha)}\,r^{p+k-1} \qquad (3.2)$$

All the inequalities are sharp.

Proof: From Theorem 1, we have

$$[(1-B)\,k + (A-B)\,(p-\alpha)] \sum_{n=k}^{\infty} |a_{p+n}|$$

$$\leq \sum_{n=k}^{\infty} [(1-B)\,n + (A-B)\,(p-\alpha)]\,|a_{p+n}| \leq (A-B)\,(p-\alpha).$$

This implies that

$$\sum_{n=k}^{\infty} |a_{p+n}| \leq \frac{(A-B)\,(p-\alpha)}{(1-B)\,k + (A-B)\,(p-\alpha)} \tag{3.3}$$

Thus

$$|f(z)| \leq |z|^p + \sum_{n=k}^{\infty} |a_{p+n}|\,|z|^{p+n}$$

$$\leq r^p + r^{p+k} \sum_{n=k}^{\infty} |a_{p+n}|$$

$$\leq r^p + \frac{(A-B)\,(p-\alpha)}{(1-B)\,k + (A-B)\,(p-\alpha)}\, r^{p+k}.$$

Also

$$|f(z)| \geq |z|^p - \sum_{n=k}^{\infty} |a_{p+n}|\,|z|^{p+n}$$

$$\geq |z|^p - r^{p+k} \sum_{n=k}^{\infty} |a_{p+n}|$$

$$\geq r^p - \frac{(A-B)\,(p-\alpha)}{(1-B)\,k + (A-B)\,(p-\alpha)}\, r^{p+k}.$$

Further

$$|f'(z)| \leq p\,r^{p-1} + \sum_{n=k}^{\infty} (p+n)\,|a_{p+n}|\,|z|^{p+n-1}$$

$$\leq p\,r^{p-1} + r^{p+k-1} \sum_{n=k}^{\infty} (p+n)\,|a_{p+n}|. \tag{3.4}$$

In view of Theorem 1.

$$\sum_{n=k}^{\infty} (1-B)\left[p + n - \frac{p\,(1-B) + (B-A)\,(p-\alpha)}{(1-B)}\right] |a_{p+n}| \leq (A-B)(p-\alpha)$$

or

$$\sum_{n=k}^{\infty} (1-B)\,(p + n)\,|a_{p+n}| \leq (A-B)\,(p-\alpha)$$

$$+ \Big[p\,(1-B) + (B-A)\,(P-\alpha)\Big] \sum_{n=k}^{\infty} |a_{p+n}| \tag{3.5}$$

(3.5), with the help of (3.3), implies that

$$\sum_{n=k}^{\infty} (p+n)\,|a_{p+n}| \leq \frac{(A-B)\,(p-\alpha)\,(p+k)}{(1-B)\,k + (A-B)\,(p-\alpha)}\,. \tag{3.6}$$

A substitution of (3.6) into (3.4) yields the right-hand inequality of (3.2).

On the other-hand,

$$|f'(z)| \geq p\,r^{p-1} - r^{p+k-1} \sum_{n=k}^{\infty} (p+n)\,|a_{p+n}|$$

$$\geq p\,r^{p-1} - \frac{(A-B)\,(p-\alpha)\,(p+k)}{(1-B)\,k + (A-B)\,(p-\alpha)}\, r^{p+k-1}$$

This completes the proof of Theorem 2.

Equality in (3.1) and (3.2) is obtained if we take

$$f(z) = z^p - \frac{(A-B)\,(p-\alpha)}{(1-B)\,k + (A-B)\,(p-\alpha)}\, z^{p+k}\ (z = \pm r).$$

Corollary 1. If $f(z) \in P_k(p, A, B, \alpha)$, then the disc U is mapped by f(z) onto a domain that contains the disc

$$|w| < \frac{(1-B)\,k}{(1-B)\,k + (A-B)\,(p-\alpha)}\,.$$

The result is sharp with the extermal function

$$f(z) = z^p - \frac{(A-B)(p-\alpha)}{(1-B)k + (A-B)(p-\alpha)} z^{p+k}.$$

Putting $\alpha = 0$ in Theorem 2 and Corollary 1 we get:

Corollary 2. If a function f(z) defined by (1.4) is in the class $P_k(p, A, B)$, then for $|z| = r$

$$r^p - \frac{(A-B)p}{(1-B)k + (A-B)p} r^{p+k} \leq |f(z)| \leq r^p + \frac{(A-B)p}{(1-B)k + (A-B)p} r^{p+k}$$

and

$$p\, r^{p-1} - \frac{(A-B)p(p+k)}{(1-B)k + (A-B)p} r^{p+k-1} \leq |f'(z)|$$

$$\leq p\, r^{p-1} + \frac{(A-B)p(p+k)}{(1-B)k + (A-B)p} r^{p+k-1}$$

The result is sharp, with the extermal function

$$f(z) = z^p - \frac{(A-B)p}{(1-B)k + (A-B)p} z^{p+k} \quad (z = \pm r).$$

Corollary 3. If $f(z) \in P_k(p, A, B)$, then the disc U is mapped by f(z) onto a domain that contains the disc

$$|w| < \frac{(1-B)k}{(1-B)k + (A-B)p}.$$

The result is sharp with the extermal function

$$f(z) = z^p - \frac{(A-B)p}{(1-B)k + (A-B)p} z^{p+k}.$$

4. INTEGRAL OPERATORS

Theorem 3. Let c be a real number such that $c > -p$. If $f(z) \in P_k(p, A, B, \alpha)$, then the function F(z) defined by (1.5) also belongs $P_k(p, A, B, \alpha)$.

Proof: Let $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$. Then from the representation of F(z), it follows that

$$F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}| z^{p+n},$$

where

$$|b_{p+n}| = \left(\frac{c+p}{c+p+n}\right) |a_{p+n}|.$$

Therefore using Theorem 1 for the coefficients of F(z) we have

$$\sum_{n=k}^{\infty} [(1-B) n + (A-B) (P-\alpha)] |b_{p+n}|$$

$$= \sum_{n=k}^{\infty} [(1-B) n + (A-B) (P-\alpha)] \left(\frac{c+p}{c+p+n}\right) |a_{p+n}|$$

$$\leq (A-B) (p-\alpha)$$

since $\frac{c+p}{c+p+n} < 1$ and $f(z) \in P_k (p, A, B, \alpha)$. Hence $F(z) \in P_k (p, A, B, \alpha)$

Theorem 4. Let c be a real number such that $c > -p$. If $F(z) \in P_k (p, A, B, \alpha)$, then the function f(z) defined by (1.5) is p–valent in $|z| < R^*$, where

$$R^* = \inf_{n \geq k \geq 2} \left[\left(\frac{c+p}{c+p+n}\right) \left[\frac{(1-B) n + (A-B) (P-\alpha)}{(A-B) (p-\alpha)}\right] \left(\frac{p}{p+n}\right) \right]^{\frac{1}{n}}.$$

The result is sharp.

Proof: Let $F(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$. It follows then from (1.5) that

$$f(z) = \frac{z^{1-c}}{c+p} [z^c F(z)]' = z^p - \sum_{n=k}^{\infty} \left(\frac{c+p+n}{c+p}\right) |a_{p+n}| z^{p+n}.$$

In order to obtain the required result it sufficies to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p \text{ in } |z| < R^*.$$

Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| -\sum_{n=k}^{\infty} (p+n) \left(\frac{c+p+n}{c+p} \right) |a_{p+n}| z^n \right|$$

$$\leq \sum_{n=k}^{\infty} (p+n) \left(\frac{c+p+n}{c+p} \right) |a_{p+n}| \; |z|^n.$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p, \text{ if } \sum_{n=k}^{\infty} (p+n) \left(\frac{c+p+n}{c+p} \right) |a_{p+n}| \; |z|^n < p. \tag{4.1}$$

But Theorem 1 confirms that

$$\sum_{n=k}^{\infty} p \left[\frac{(1-B)\, n + (A-B)\,(p-\alpha)}{(A-B)\,(p-\alpha)} \right] |a_{p+n}| \leq p.$$

Hence (4.1) will be satisfied if

$$(p+n) \left(\frac{c+p+n}{c+p} \right) |a_{p+n}| \; |z|^n \leq p \left[\frac{(1-B)\, n + (A-B)\,(p-\alpha)}{(A-B)\,(p-\alpha)} \right] |a_{p+n}|,$$

$$n \geq k \geq 2$$

or if

$$|z| \leq \left\{ \left(\frac{c+p}{c+p+n} \right) \left[\frac{(1-B)\, n + (A-B)\,(P-\alpha)}{(A-B)\,(p-\alpha)} \right] \left(\frac{p}{p+n} \right) \right\}^{\frac{1}{n}}$$

$$n \geq k \geq 2.$$

Therefore f(z) is p–valent in $|z| < R^*$. Sharpness follows if we take

$$F(z) = z^p - \frac{(A-B)\,(p-\alpha)}{(1-B)\, n + (A-B)\,(p-\alpha)} z^{p+n}, \; n \geq k \geq 2.$$

5. RADIUS OF CONVEXITY

Theorem 5. If $f(z) \in P_k(p, A, B, \alpha)$, then f(z) is p– valently convex in the disc $|z| < R_p$, where

$$R_p = \inf_{n \geq k \geq 2} \left\{ \left[\frac{(1-B)\, n + (A-B)\,(p-\alpha)}{(A-B)\,(p-\alpha)} \right] \left(\frac{p}{p+n} \right)^2 \right\}^{\frac{1}{n}}.$$

The result is sharp.

Proof: In order to establish the required result it sufficies to show that

$$\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| \leq p \text{ for } |z| < R_p.$$

Let $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}|\, z^{p+n}$. Then we have

$$\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| = \left| \frac{-\sum_{n=k}^{\infty} n\,(p+n)\, |a_{p+n}|\, z^n}{p - \sum_{n=k}^{\infty} (p+n)\, |a_{p+n}|\, z^n} \right|$$

$$\leq \frac{\sum_{n=k}^{\infty} n\,(p+n)\, |a_{p+n}|\, |z|^n}{p - \sum_{n=k}^{\infty} (p+n)\, |a_{p+n}|\, |z|^n}.$$

Therefore $\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| \leq p$ if

$$\sum_{n=k}^{\infty} n\,(p+n)\, |a_{p+n}|\, |z|^n \leq p^2 - \sum_{n=k}^{\infty} p\,(p+n)\, |a_{p+n}|\, |z|^n$$

or if

$$\sum_{n=k}^{\infty} \left(\frac{p+n}{p} \right)^2 |a_{p+n}|\, |z|^n \leq 1. \qquad (5.1)$$

From Theorem 1, we have

$$\sum_{n=k}^{\infty} \frac{(1-B)\,n + (A-B)\,(p-\alpha)}{(A-B)\,(p-\alpha)} \mid a_{p+n} \mid \leq 1.$$

Hence (5.1) will be satisfied if

$$\left(\frac{p+n}{p}\right)^2 \mid z \mid^n \leq \left[\frac{(1-B)\,n + (A-B)\,(p-\alpha)}{(A-B)\,(p-\alpha)}\right]$$

or if

$$\mid z \mid \leq \left\{ \left[\frac{(1-B)\,n + (A-B)\,(P-\alpha)}{(A-B)\,(p-\alpha)}\right] \left(\frac{p}{p+n}\right)^2 \right\}^{\frac{1}{n}}, \; n \geq k,\; k \geq 2.$$

Therefore **f(z)** is p–valently convex in the disc $\mid z \mid < R_p$. The result is sharp with the extremal function f(z) defined by (2.3).

6. CLOSURE PROPERTIES

In this section we show that the class P_k (p, A, B, α) is closed under "arithmetic mean" and "convex linear combinations".

Theorem 6. If $f_j(z) = z^p - \sum_{n=k}^{\infty} \mid a_{p+n_j} \mid z^{p+n}$, j = 1, 2,..., m.

If $f_j(z) \in P_k$ (p, A, B, α) for each j = 1, 2,..., m, then the function h(z) $= z^p - \sum_{n=k}^{\infty} \mid b_{p+n} \mid z^{p+n}$ also belongs to P_k (p, A, B, α), where $b_{p+n} = \frac{1}{m} \sum_{j=1}^{\infty} a_{p+n_j}$.

Proof: Since $f_j(z) \in P_k$ (p, A, B, α), it follows from Theorem 1 that

$$\sum_{n=k}^{\infty} [(1-B)\,n + (A-B)\,(p-\alpha)] \mid a_{p+n_j} \mid \leq (A-B)\,(p-\alpha),\; j = 1, 2, \ldots, m.$$

Therefore

$$\sum_{n=k}^{\infty} [(1-B)\,n + (A-B)\,(p-\alpha)] \mid b_{p+n} \mid$$

$$\leq \sum_{n=k}^{\infty} [(1-B)\,n + (A-B)\,(p-\alpha)] \left\{ \frac{1}{m} \sum_{j=1}^{m} |a_{p+n_j}| \right\}$$

$$\leq (A-B)\,(p-\alpha).$$

Hence, by Theorem 1, $h(z) \in P_k\,(p, A, B, \alpha)$.

Theorem 7. Let $f_p(z) = z^p$ and

$$f_{p+n}(z) = z^p - \frac{(A-B)\,(p-\alpha)}{(1-B)\,n + (A-B)\,(p-\alpha)}\, z^{p+n}\ (n \geq k,\ k \geq 2).$$

Then $f(z) \in P_k\,(p, A, B, \alpha)$ if and only if it can be expressed in the in the form $f(z) = \lambda_p f_p(z) + \sum_{n=k}^{\infty} \lambda_n f_{p+n}(z)$, where $\lambda_n \geq 0$ and $\lambda_p + \sum_{n=k}^{\infty} \lambda_n = 1$.

Proof: Let us assume that

$$f(z) = \lambda_p f_p(z) + \sum_{n=k}^{\infty} \lambda_n f_{p+n}(z)$$

$$= [1 - \sum_{n=k}^{\infty} \lambda_n]\, z^p + \sum_{n=k}^{\infty} \lambda_n \left[z^p - \frac{(A-B)\,(p-\alpha)}{(1-B)\,n + (A-B)\,(p-\alpha)}\, z^{p+n}\right]$$

$$= z^p - \sum_{n=k}^{\infty} \frac{(A-B)\,(p-\alpha)}{(1-B)\,n + (A-B)\,(p-\alpha)}\, \lambda_n\, z^{p+n}.$$

Then from Theorem 1 we have

$$\sum_{n=k}^{\infty} [(1-B)\,n + (A-B)\,(p-\alpha)] \left[\frac{(A-B)\,(p-\alpha)\,\lambda_n}{(1-B)\,n + (A-B)\,(p-\alpha)}\right]$$

$$= (A-B)\,(p-\alpha) \sum_{n=k}^{\infty} \lambda_n \leq (A-B)\,(p-\alpha).$$

Hence $f(z) \in P_k\,(p, A, B, \alpha)$.

Conversely, let $f(z) \in P_k(p, A, B, \alpha)$. It follows from Theorem 1 that

$$|a_{p+n}| \leq \frac{(A-B)(p-\alpha)}{(1-B)n+(A-B)(p-\alpha)} \quad (n = k, k+1, \ldots, k \geq 2).$$

Setting

$$\lambda_n = \frac{(1-B)n+(A-B)(p-\alpha)}{(A-B)(p-\alpha)} \, |a_{p+n}|, \; (n = k, k+1, \ldots, k \geq 2).$$

and

$$\lambda_p = 1 - \sum_{n=k}^{\infty} \lambda_n,$$

we have

$$f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$$

$$= z^p - \sum_{n=k}^{\infty} \lambda_n z^p + \sum_{n=k}^{\infty} \lambda_n z^p - \sum_{n=k}^{\infty} \lambda_n \frac{(A-B)(p-\alpha)}{(1-B)n+(A-B)(p-\alpha)} z^{p+n}$$

$$= [1 - \sum_{n=k}^{\infty} \lambda_n] z^p + \sum_{n=k}^{\infty} \lambda_n [z^p - \frac{(A-B)(p-\alpha)}{(1-B)n+(A-B)(p-\alpha)} z^{p+n}]$$

$$= \lambda_p f_p(z) + \sum_{n=k}^{\infty} \lambda_n f_{p+n}(z).$$

This completes the proof of Theorem 7.

Remarks:

(1) Putting $\alpha = 0$ in Theorems 1, 3, 5, 6 and 7, we get the corresponding results obtained by Sarangi and Patil [8].

(2) We observe that our results in Corollary 2 and Corolary 3 improves the results of Sarangi and Patil [8, Theorem 3 and its Corollary].

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