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# ON THE DUALITY OF GENERALISED EULER FORMULA FOR EUCLIDEAN HYPERSURFACES

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ABSTRACT

In order to define the generalised Euler formula in a dual manner, we studied the angles between two hyperplanes in $R^{n+1}$ and we obtained that the Gauss curvature can be expressed by the normal curvature and its dual form.

## I. INTRODUCTION

In Euclidean space $R^{n+1}$ of dimension $n+1$ we consider an n-dimensional hypersurface M given by a local coordinate system $\{u^1, u^2, \ldots, u^n\}$. Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be an orthogonal coordinate system of $R^{n+1}$. We assume that the $x_i$'s are $C^\infty$- functions of $u^\alpha$'s and that $1 \leq i \leq n+1$, $1 \leq \alpha \leq n$. Let X be a vector whose orthogonal components are $(x_1, \ldots, x_{n+1})$, then the hypersurface M can be characterized by a vector function

$$X = X(u^\alpha), \ \alpha = 1, \ldots, n. \tag{I.1}$$

Let us denote by N the unit normal vector field of the hypersurface M, then it satisfies the conditions $<N,N> = 1$ and $<N, \frac{\partial X}{\partial u^\alpha}> = O$. Now let us introduce an orthonormal frame in $R^{n+1}$ by $e_i$, and using this frame we can write that

$$N = \sum_{i=1}^{n+1} N_i \, e_i \tag{I.2}$$

and that

$$\frac{\partial X}{\partial u^k} = \sum_{i=1}^{n+1} (x_k)_i \, e_i, \ k = 1, \ldots, n, \tag{I.3}$$

where $N = N_i(u^\alpha)$, $\alpha = 1, \ldots, n$, $1 \leq i \leq n+1$.

## II. PRELIMINARIES

Let v denote a tangent vector of the tangent space $T_M(m)$ at the point m of hypersurface M. In this direction the curvature $\frac{1}{R}$ of the hypersurface M is defined by

$$\frac{1}{R} = - < v, \frac{\partial N}{\partial u^k} > = h_{\alpha\beta}\, u^\alpha\, u^\beta \qquad \text{(II.1)}$$

where $h_{\alpha\beta}$ is the second fundamental tensor of M and defined as

$$h_{\alpha\beta} = < N, \frac{\partial^2 X}{\partial u^\alpha \partial u^\beta} > = - < \frac{\partial N}{\partial u^\alpha}, \frac{\partial X}{\partial u^\beta} > .$$

The principal curvatures at a point of M are the eigen values of the second fundamental tensor evaluated at this point. Hence they are the roots of the characteristic equation as follows

$$\det\left[h_{\alpha\beta} - \frac{1}{R} g_{\alpha\beta}\right] = (-1)^n \det(g_{\alpha\beta}) \left(\frac{1}{R} - \frac{1}{R_1}\right) \cdots \left(\frac{1}{R} - \frac{1}{R_n}\right)$$

$$= (-1)_n \det(g_{\alpha\beta}) \Big\{ \frac{1}{R^n} - \frac{1}{R^{n-1}}\left(\sum_{i_1=1}^{n} \frac{1}{R_{i_1}}\right) + \frac{1}{R^{n-2}}\left(\sum_{i_1 < i_2} \frac{1}{R_{i_1} R_{i_2}}\right)$$

$$+ \ldots + (-1)^{n-1} \frac{1}{R}\left(\sum_{i_1 < \ldots < i_{n-1}} \frac{1}{R_{i_1} \ldots R_{i_{n-1}}}\right) + (-1)^n \frac{1}{R_1 \ldots R_n} \Big\} = 0 \qquad \text{(II.2)}$$

where $g_{\alpha\beta}$'s are the coefficients of the first fundamental form of the hypersurface M. The principal directions always exist and we can find an orthonormal system of principal directions.

Now let $\theta_\alpha$ denote the angles between the direction v and the principal directions, where $\alpha$ runs from 1 to n. If we denote the principal directions by $t_1, \ldots, t_n$, then $\theta_1 \sphericalangle (t_1, v), \ldots, \theta_n = \sphericalangle (t_n, v)$.

The curvature $\frac{1}{R}$ in this direction v can be expressed in terms of the principal curvatures $\frac{1}{R_i}$, $i = 1, \ldots, n$, by means of Euler's formula

$$\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i} \sin^2\theta_i. \tag{II.3}$$

Now let us define a kind of normal curvature which we will denote by $\overline{R}$ and will be thought as a dual corresponding of R. This will be defined at the image point of m under the normal projection in the direction v of M. This concept has been defined by A. Mannheim (see [1] and [4]). From this dual viewpoint the Euler formula may be constructed as

$$\overline{R} = \sum_{i=1}^{n} R^*_i \sin^2\theta_i \tag{II.4}$$

where $R^*_i$ shows the dual principal curvature corresponding to $\frac{1}{R_i}$.

Denoting by $v_i$ the rectangular components of the unit vector v we write that

$$v = \sum_{i=1}^{n} v_i e_i. \tag{II.5}$$

Also we have that $\cos\theta_i = \langle v, e_i \rangle$, $i = 1, \ldots, n$. On multiplying both members of (II.5) by $e_k$ we find that $v_k = \langle v, e_k \rangle$ and that $v_i = \cos\theta_i$, $i = 1, \ldots, n$. Consequently we have

$$v = \sum_{i=1}^{n} e_i \cos\theta_i \text{ or } \sum_{i=1}^{n} \cos^2\theta_i = 1. \tag{II.2}$$

## III. ANGLES BETWEEN HYPERPLANES IN $R^{n+1}$

Let us consider two n–dimensional tangent vectors $T_1{}^n$, $T_2{}^n$ which are n–planes in euclidean space $R^{n+1}$ and $t_1$ and $t_2$ be the tangent vectors of the normal sections of $T^n{}_1$ and $T^n{}_2$ with hypersurface M. Also define the angles between the vectors $t_1$, $t_2$ and any vector in tangent space $T_M(m)$. To find this angles we will follow the procedure which has been given by H. Gluck [2]. The angle between a pair of lines in euclidean space $R^{n+1}$ is the smaller of the two possible angles between any vectors parallel to these lines. The angle between a line and a hyperplane (that will be consider as a tangent vector to M) is the smallest angle between this line and any line in hyperplane. This is the same as the angle between a line and its orthogonal projection in hyperplane, or $\pi / 2$ in case this orthogonal projection degenerates to a point. Let us consider now a pair of hyperplanes of n–dimensional $T_1^n$ and $T_2^n$ in $R^{n+1}$. Suppose that among all pairs of lines, one from $T_1^n$ and one from $T_2^n$, the lines $t_1$ and $t_2$ make the smallest possible angle, $w_1$, with each other. Let $T_1^{n-1}$ and $T_2^{n-1}$ be the orthogonal complements of $t_1$ and $t_2$ in $T_1^n$ and $T_2^n$, respectively. Then it is easily seen that $t_1$ is orthogonal not only to $T_1^{n-1}$ but also to $T_2^{n-1}$, and similarly $t_2$ is orthogonal not olny to $T_2^{n-1}$ but also to $T_1^{n-1}$. If we iterate this procedure with $T_1^{n-1}$ and $T_2^{n-1}$ in the roles of $T_1^n$ and $T_2^n$, we get another angle $w_2 = w_1$. Doing this n–times we get n angles $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \pi / 2$. This angles depend only on $T_1^n$ and $T_2^n$, and not on the various choices possible during the above procedure, and these angle are called the principal angles between the hyperplane $T_1^n$ and $T_2^n$. If we choose two orthonormal bases $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ for the subspaces $V_1^n$ and $V_2^n$ parallel to $T_1^n$ and $T_2^n$ such that $<u_i, v_i> = \cos w_i$ for $1 \leq i \leq n$ and $<u_i, v_j> = 0$ for $i \neq j$. Note that the orthogonal projection of $v_i$ into $V_2^n$ is $(\cos w_i)\, v_i$ and the orthogonal projection of $v_i$ into $V_i^n$ is $(\cos w_i) u_i$. Suppose that it is desired to find a single angle which might reasonably be called the angle between $T_1^n$ and $T_2^n$. If one is forced to choose from among the principal angles, one would have to select the largest principal angle $w_n$ for such a role, in order to insure that $T_1^n$ and $T_2^n$ are parallel if and only if the angle between them is zero. To arrive at the right definition carefully consider the case $n = 1$.

Then there is just one principal angle $w_1$ between the lines $t_1$ and $t_2$ and it coincides with the ordinary angle $\theta$ between these lines. This angle $\theta$, lying between 0 and $\pi / 2$, has the following property. If U is any measurable subset of $t_1$ with one–dimensional measure s(U), then the orthogonal projection of U into $t_2$ is also measurable and has one–dimensional measure $(\cos\theta)\, s\,(U)$ in $t_2$. Similarly, if $U'$ is a measurable subset of $t_2$ with measure $s(U')$, then the orthogonal projection of $U'$ into $t_1$ has measure $(\cos\theta)\, s\,(U')$ in $t_1$. Thus the angle $\theta$ between $t_1$ and $t_2$ may be defined as that angle between 0 and $\pi / 2$ whose cosine is the reduction factor for one–dimensional measure under orthogonal projection of $t_1$ into $t_2$, then (1x1) matrix of the orthogonal projection of $V^1_1$ into $V^1_2$ has a determinant whose absolute value is $\cos\theta$. So we can give the following definition directly.

**Definition III.1.** Let $T_1^n$ and $T_2^n$ be hyperplanes in $R^{n+1}$. Let the number p, $0 \leq p \leq 1$, be the reduction factor for n–dimensional measure under orthogonal projection of $T_1^n$ into $T_2^n$. Then the unique angle $\theta$, $0 \leq \theta \leq \pi/2$ such that $\cos\theta = p$, will be called the angle between $T_1^n$ and $T_2^n$. The following theorem gives us the relation between the angle $\theta$ and the principal angles $w_1, \ldots, w_n$.

**Theorem III.1.** Let $T_1^n$ and $T_2^n$ be hyperplanes in $R^{n+1}$, and let $w_1 \leq w_2 \leq \ldots \leq w_n$ be the principal angles between them. Then the angle $\theta$ between $T_1^n$ and $T_2^n$ is given by $\cos\theta = \cos w_1 \ldots \cos w_n$. For a proof of this theorem see the paper [2]. To give a practical technique for computing the angle between two hyperplanes we will express the following theorem:

**Theorem III.2.** Let $T_1^n$ and $T_2^n$ be hyperplanes in $R^{n+1}$, and let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be arbitrary bases for $V_1^n$ and $V_2^n$, respectively. Then the angle $\theta$ between the hyperplanes is given by the formula

$$\cos\theta = \frac{|\det(u_i . v_j)|}{\sqrt{\det(u_i . u_j)}\ \sqrt{\det(v_i . v_j)}} . \qquad \text{(III.1.)}$$

This formula is the generalisation of the formula known for one dimensional two vectors in a vector space. Now we will give a classical concept which is called Dupin indicatrix.

**Definition III.2.** Dupin indicatrix $I_m$ at each point m in M is the subset of $T_M(m)$ consisting of all vectors z such that $\langle Sz, z\rangle = \pm 1$ and $Sz = \bar{D}_z N$, where S is the weingarten map and $\bar{D}$ is the natural connection defined on $R^{n+1}$, [3]. Now let $t_1, \ldots, t_n$ be an orthonormal set of eigen vectors of the map $S^*$ which will be assumed as dual corresponding of the weingarten map S. Then $z = \sum_{j=1}^{n} a^j t_j$ and we write

$$\langle S^*z, z\rangle = \langle \sum_{j=1}^{n} a^j S^* t_j, \sum_{j=1}^{n} a^j t_j)\rangle = \sum_{j=1}^{n} (a^j)^2 \langle S^* t_j, t_j\rangle$$

$$= \sum_{j=1}^{n} (a^j)^2 \frac{1}{R^*_1 \ldots \hat{R}^*_j \ldots R^*_n} \qquad \text{(III.2)}$$

where $\hat{R}^*_j$ indicates that the $R^*_j$ is omitted as an argument.

Now let Q be a point in the intersection of $I_m$ and $T^n{}_o$, then we illustrate the following figure in dimension 2.

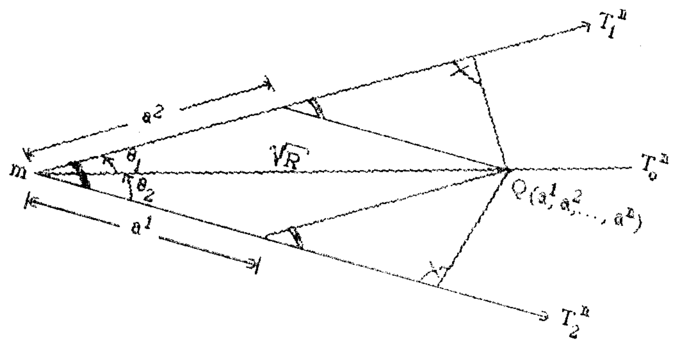


Figure III. 1.

By using figure III.1. for n–dimension we might infer that

$$a^j = \frac{\sqrt{R}\ \sin\ \theta_j}{\sin\ (\sum\limits_j\ \theta_j)}\ ,\ 1 \leq j \leq n, \tag{III.3.}$$

where $\theta_j$'s are defined as in the Theorem III.1. Putting (III.3) into (III.2) we get from (II.4) that

$$\frac{\sin^2 (\Sigma\ \theta_j)}{R} = \frac{\sin^2 \theta_1}{R^*_1\, R^*_2 \ldots R^*_n} + \frac{\sin^2 \theta_2}{R^*_1\, R^*_2 \ldots R^*_n} + \cdots$$

$$+ \frac{\sin^2 \theta_n}{R^*_1 \ldots R^*_{n-1}\, R^*_n} \tag{III.4}$$

## CONCLUSION

For the special case $\sum\limits_j \theta_j = \pi / 2$ we have $R_i = R_i^*$, $1 \leq i \leq n$, so the expression (III.4) changes into (II.3), but there is a slight difference that we will omit it here. (III.4) gives us a dual form of generalised Euler formula. Thus we get a relation between R and $\overline{R}$ by using (III.4) and (II.4). To get this we will use that

$R^*_1 \ldots \hat{R}^*_j \ldots R^*_n = \dfrac{1}{K\, R_j^*}$ , (K is the Gaussian curvature), so we have that

$$R\,\bar{R} = \frac{\sin^2(\sum_j \theta_j)\,(R_1^* \sin^2\theta_1 + \ldots + R^*_n \sin^2\theta_n)}{\dfrac{\sin^2\theta_1}{\dfrac{1}{K\,R^*_1}} + \ldots + \dfrac{\sin^2\theta_n}{\dfrac{1}{K\,R^*_n}}}$$

or $K = \dfrac{\sin^2(\sum_j \theta_j)}{R\,\bar{R}}$. And finally for the special case $\sum_j \theta_j = \pi/2$ we find that

$$K = \frac{1}{R\,\bar{R}} \ . \qquad \text{(III.5)}$$

REFERENCES

[1] BLASCHKE, W., "Kreis and Kugel," Leipzig 1916 p. 118.

[2] GLUCK, H., "Higher curvatures of curves in Euclidean space, II" Monthly, (1967), 1049–1056.

[3] HICKS, N.J., "Notes on differential geometry, Van Nostrand Reinhold Company, London (1974).

[4] SCHAAL, H., "Ein Beitrag zur konstruktiven differential geometrie" LXV Band mit 3 textabbildungen, Wien (1961), 265–269.