Commun. Fac. Sci. Univ. Ank. Series A_1
V. 44. pp. 95-102 (1995)

ON THE INVERSE SCATTERING PROBLEM FOR A DISCRETE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

G. SH. GUSEINOV and H. TUNCAY

(Received Dec. 14, 1994; Accepted Dec. 7, 1995)

SUMMARY

In this paper the necessary and sufficient conditions for the existence of the solution of the inverse scattering problem for a discrete one-dimensional Schrödinger equation (second order difference equation on the whole axis) are obtained.

1. INTRODUCTION

A formal solution of the inverse problem of scattering theory for a discrete one-dimensional Schrödinger equation (second order difference equation on the whole axis) was given in the articles [1]–[4] under the assumption that the coefficients of the difference equation converge rapidly enough to their corresponding limits. In this paper we identify a natural class of coefficients of the difference equation, finding necessary and sufficient conditions for solvability in this class of the inverse scattering problem. A similar problem for the one-dimensional continuous Schrödinger equation was throughly investigated by L.D. Faddeev [5]; see also [6], [7]. The inverse scattering problem for a second order difference equation on a semiaxis (for an infinite Jacobi matrix) was studied in [8].

2. DIRECT SCATTERING PROBLEM

Consider the infinite system of equations

$$a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1} = \lambda y_n,\ n = \pm 1, \pm 2, \ldots \tag{1}$$

where $\{y_n\}_{-\infty}^{+\infty}$ is the solution sought, λ is a complex parameter and $\mathrm{Im} b_n = 0$, $a_n > 0$, $n = 0, \pm 1, \pm 2, \ldots$ $\sum\limits_{-\infty}^{\infty} |n| (|1-a_n| + |b_n|) < \infty.$ (2)

We denote by $l^2(-\infty, \infty)$ the Hilbert space of sequence $y = \{y_n\}_{-\infty}^{+\infty}$

such that $\sum_{-\infty}^{\infty} | y_n |^2 < \infty$, with inner production $(x, y) = \sum_{-\infty}^{\infty} x_n \bar{y}_n$ (the bar over a number or a function here and below denotes complex conjugation). By L we denote the minimal closed linear operator generated in $l^2(-\infty, \infty)$ by the operation $(ly)_n = a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1}$. From (2) it follows that the operator L is selfadjoint.

Theorem 1. Under condition (2) the operator L has a double continous spectrum filling the segment $[-2, -2]$ and a finite number of simple real discrete eigenvalues lying outside the continuous spectrum. If $b_n \equiv 0$, the eigenvalues occur in symmetrical pairs with respect to the point $\lambda = 0$.

In the equation (1) we shall put $\lambda = 2\text{Cos } z$, where $z = \xi + i\tau$. Wherever ξ appears in what follows it will denote only real parameters.

Theorem 2. Under condition (2) equation (1) with $\lambda = 2\text{Cos } z$ has the unique solutions $\{f_n(z)\}_{-\infty}^{+\infty}$ and $\{g_n(z)\}_{-\infty}^{+\infty}$ regular in the half plane $\text{Im } z > 0$, continuous up to the real axis and representable in the forms

$$f_n(z) = \alpha_n e^{inz} (1 + \sum_{m=1}^{\infty} A_{nm} e^{imz}), \quad g_n(z) = \beta_n e^{-inz} (1 + \sum_{-\infty}^{m=-1} B_{nm} e^{-imz}) \tag{3}$$

in this connection we have the equalities

$$a_n = \frac{\alpha_{n+1}}{\alpha_n} = \frac{\beta_n}{\beta_{n+1}}, \quad b_n = A_{n1} - A_{n-1,\,1} = B_{n,\,-1} - B_{n+1,\,-1} \tag{4}$$

and for A_{nm} and B_{nm} the estimates

$$| A_{nm} | \leq ((n)\, \sigma_1 \left(n + \left[\frac{m}{2}\right]\right), \quad | B_{nm} | \leq D(n)\, \sigma_2 \left(n + \left[\frac{m}{2}\right] + 1\right),$$

where

$$\sigma_1(n) = \sum_{p=n}^{\infty} (| 1-a_p | + | b_p |), \quad \sigma_2(n) = \sum_{-\infty}^{p=n} (| 1-a_p | + | b_p |),$$

$[\cdot]$ denoting the integral part; C(n) and D(n) denote nonnegative functions on an integral argument n, C (n) being a function that is monotone nonincreasing, bounded as $n \to \infty$ and in general increasing as $n \to -\infty$ and D (n) a monotone nondecreasing function bounded as $n \to -\infty$ and in general increasing as $n \to \infty$.

For Im $z \geq 0$ the following formulas hold:

$$f_n(z) = e^{inz}\,[1 + o(1)],\ n \to \infty \quad g_n(z) = e^{-inz}\,[1 + o(1)],\ n \to -\infty \tag{5}$$

For $\xi \neq k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ the pairs $\{f_n(\xi)\}_{-\infty}^{\infty}$, $\{f_n(-\xi)\}_{-\infty}^{\infty}$ and $\{g_n(\xi)\}_{-\infty}^{\infty}$, $\{n_n(-\xi)\}_{-\infty}^{\infty}$ constitute two fundamental systems of solutions of (1) with $\lambda = 2\mathrm{Cos}\ \xi$.

We have the relations

$$f_n(\xi) = b(\xi)\, g_n(\xi) + a(\xi)\, g_n(-\xi),$$

$$g_n(\xi) = -b(-\xi)\, f_n(\xi) + a(\xi)\, f_n(-\xi). \tag{6}$$

The functions $a(\xi)$ and $b(\xi)$ are defined for all

$$\xi \in \mathbf{R}^{+} = (-\infty, \infty) \setminus \{k\pi\colon k = 0, \pm 1, \pm 2, \ldots\}$$

and are continuous. Moreover,

$$a(\xi + 2\pi) = a(\xi),\ b(\xi + 2\pi) = b(\xi),\ \overline{a(\xi)} = a(-\xi),$$

$$\overline{b(\xi)} = b(-\xi),\ |a(\xi)|^2 - |b(\xi)|^2 = 1,\ \xi \in \mathbf{R}^{*}.$$

The function $a(\xi)$ can be continued analytically into the half-plane Im $z > 0$ and as $\tau \to \infty$

$$a(z) = (\prod_{-\infty}^{\infty} a_p)^{-1} + o(1),\ z = \xi + i\tau.$$

For Im $z > 0$ the function $f_n(z)$ decreases exponentially when $n \to \infty$ and $g_n(z)$ does so when $n \to -\infty$. If $a(z_0) = 0$ for some z_0 with Im$z_0 > c$ then the solutions $\{f_n(z_0)\}_{-\infty}^{\infty}$ and $\{g_n(z_0)\}_{-\infty}^{-\infty}$ are linearly dependent; consequently for $\lambda = 2\mathrm{Cos}\ z_2$ equation (1) has a solution in the space $l^2(-\infty, \infty)$ and therefore, $\lambda_0 = 2\mathrm{Cos}\ z_0$ is an eigenvalue of the operator L. The converse also true: if the number $\lambda_0 = 2\mathrm{Cos}\ z_0$ is an eigenvalue of L for some z_0 with Im $z_0 > 0$, then $a(z_0) = 0$.

Since the eigenvalues of the operator L are real and form a finite set, the function a(z) can have only a finite number of zeros in the half-strip

$$\Pi_{+} = \{z = \xi + i\tau\colon -\frac{\pi}{2} \leq \xi \leq \frac{3\pi}{2},\ \tau > 0\}$$

which will be lay on half–lines $\operatorname{Re} z = 0$ and $\operatorname{Re} z = \pi$ $(\operatorname{Im} z > 0)$. Denote them by $z_j = i\tau_j$, $j = 1, \ldots, N_0$, $z_j = \pi + i\tau_j$, $j = N_0 + 1, \ldots, N$ so that $a(z_j) = 0$ and

$$f_n(z_j) = c_j g_n(z_j), \quad j = 1, \ldots, N$$

where c_j, $j = 1, \ldots, N$ are certain nonzero real constants. The zeros of $a(z)$ are simple and moreover, the following formula is valid:

$$\dot{a}(z_j) = -ic_j \sum_{-\infty}^{\infty} g^2_n(z_j) = -\frac{i}{c_j} \sum_{-\infty}^{\infty} f^2_n(z_j), \ j = 1, \ldots, N,$$

where the dot over the function indicates the derivative with respect to z.

Dividing both parts of equalities (6) to $a(\xi)$, we get for $\xi \neq k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ the following solutions of the equation (1):

$$u_n^-(\xi) = t(\xi) f_n(\xi) = r^-(\xi) g_n(\xi) + g_n(-\xi),$$

$$u_n^+(\xi) = t(\xi) g_n(\xi) = r^+(\xi) f_n(\xi) + f_n(-\xi),$$

where

$$r^-(\xi) = \frac{b(\xi)}{a(\xi)}, \ r^+(\xi) = -\frac{b(-\xi)}{a(\xi)}, \ t(\xi) = \frac{1}{a(\xi)}.$$

These solutions satisfy in virtue of (5) the asymptotic formulas:

$$u^-_n(\xi) = t(\xi) e^{in\xi} + o(1) \quad n \to \infty$$

$$u^-_n(\xi) = r^-(\xi) e^{-in\xi} + e^{in\xi} + o(1) \quad n \to -\infty$$

$$u^+_n(\xi) = t(\xi) e^{-in\xi} + o(1) \quad n \to -\infty$$

$$u^+_n(\xi) = r^+(\xi) e\, n\xi + e - n\xi + o(1) \quad n \to \infty$$

and are called the eigenfunctions of left $(u^-_n(\xi))$ and right $(u^+_n(\xi))$ scattering problems. The coefficients $r^-(\xi)$ $r^+(\xi)$ and $t(\xi)$ are called respectively the left and right reflection coefficients and passing coefficient.

Define the positive numbers (the normalizing factors) M^+_j and M^-_j by the formulas

$$(M^+_j)^{-2} = \sum_{-\infty}^{\infty} f^2_n(z_j) \quad (M^-_j)^{-2} = \sum_{-\infty}^{\infty} g^2_n(z_j).$$

Definition: The collection of quantities $\{r^+(\xi), z_j, M^+_j, j = 1, \ldots, N\}$ and $\{r^-(\xi), z_j, M^-_j, j = 1, \ldots, N\}$ we call respectively the right and left scattering data for equation (1).

3. INVERSE SCATTERING PROBLEM

The inverse scattering problem for equation (1) consists in recovering the coefficients a_n and b_n on the basis of the right or left scattering data and in finding necessary and sufficient conditions which an arbitratily chosen collection $\{r(\xi), z_j, M_j, j = 1, \ldots, N\}$ should satisfy in order that it be the right (left) scattering data for some equation of the form (1) with coefficients satisfying (2).

In solving of the inverse scattering problem a major role plays the following:

Theorem 3. The quantities α_n, A_{nm}, β_n, B_{nm} in formula (3) satisfy the equations

$$F_{2n+m} + A_{nm} + \sum_{k=1}^{\infty} A_{nk} F_{k+m+2n} = 0, \quad m = 1, 2, 3, \ldots, \tag{7}$$

$$\alpha_n^{-2} = 1 + F_{2n} + \sum_{k=1}^{\infty} A_{nk} F_{k+2n}, \tag{8}$$

$$\Phi_{2n+m} + B_{nm} + \sum_{-\infty}^{k=-1} B_{nk} \Phi_{k+m+2n} = 0, \quad m = -1, -2, -3, \ldots \tag{9}$$

$$\beta_n^{-2} = 1 + \Phi_{2n} + \sum_{-\infty}^{k=-1} B_{nk} \Phi_{k+2n},$$

where

$$F_m = \sum_{j=1}^{N} M^{+}_j e^{imz_j} + \frac{1}{2\pi} \int_{-\pi}^{\pi} r^{+}(\xi) e^{im\xi} d\xi,$$

$$\Phi_m = \sum_{j=1}^{N} M^{-}_j e^{-imz_j} + \frac{1}{2\pi} \int_{-\pi}^{\pi} r^{-}(\xi) e^{-im\xi} d\xi.$$

Equations (7) and (9) can be regarded as equations for A_{nm} and B_{nm} respectively. These are the fundamental equations of the inverse problem and are called the Gelfand– Levitan or Marchenko type equations. The main result of this paper is the following:

Theorem 4. For a given collection $\{r^{+}(\xi), z_j, M^{+}_j, j = 1, \ldots, N\}$, where $-\infty < \xi < \infty$; $z_j = i\tau_j$, $\tau_j > 0$, $j = 1, \ldots, N_0$ and are distinct; $z_j = \pi + i\tau_j$ $\tau_j > 0$, $j = N_0 + 1, \ldots, N$ and are distinct; $M_j^{+} > 0$, $j = 1, \ldots, N$, to be the right scattering data of some equation (1) with coefficients a_n and b_n satisfying condition (2), it is necessary and sufficient that the following conditions be satisfied:

1) The function $r^+(\xi)$ is continuous on the entire real axis $-\infty < \xi < \infty$, $r^+(\xi + 2\pi) = r^+(\xi)$, $\overline{r^+(\xi)} = r^+(-\xi)$, $|r^+(\xi)| \leq 1$ and $|r^+(\xi)| < 1$ for $\xi \neq k\pi$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$' if $|r^+(k\pi)| = 1$, then $r^+(k\pi) = -1$; there exists a positive number $C > 0$ such that the lower bound $1 - |r^+(\xi)| \geq C \sin^2(\xi - k\pi)$ holds.

2) The function $za(z)$, where

$$a(z) = \exp\left\{ -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\log(1 - |r^+(\xi)|^2)}{\operatorname{Sin}(\xi - z)} d\xi \right\} \prod_{p=1}^{N} \frac{\operatorname{Sin}(z - z_j)}{\operatorname{Sin}(z + z_j)}$$

is continuous in the closed upper half-plane.

3) The quantities

$$F_m^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\xi) e^{im\xi} d\xi, \quad \Phi^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{r}(\xi) e^{-im\xi} d\xi$$

where $r^-(\xi) = -r^+(-\xi) \dfrac{a(-\xi)}{a(\xi)}$, for all finite integers N_1 and N_2 satisfy

$$\sum_{m=N_1}^{\infty} |m| \left| F_{m+2}^{(1)} - F_m^{(1)} \right| < \infty, \quad \sum_{-\infty}^{m=N2} |m| \left| \Phi_{m+2}^{(1)} - \Phi_m^{(1)} \right| < \infty.$$

We note that the following properties of the function $a(z)$ (which are used in an essential manner in the proof of this theorem) are implied by conditions 1) and 2) of Theorem 4:

a) The equalities $\overline{a(\xi)} = a(-\xi)$, $|a(\xi)|^{-2} = 1 - |r^+(\xi)|^2$ for $\xi \in \mathbf{R}^*$ holds, where

$$a(\xi) = \lim_{\varepsilon \to +0} a(\xi + i\epsilon), \quad \xi \in \mathbf{R}^*.$$

b) $\lim\limits_{\xi \mapsto k\pi} a(\xi)\, [1 + r^{\pm}(\xi)]\, \operatorname{Sin}(\xi - k\pi) = 0.$

c) $a(z + 2\pi) = a(z)$ and $a(z) = d + o(1)$ as $\operatorname{Im} z \to \infty$, where $d > 0$.

d) The function $[a(z)]^{-1}$ is bounded in some neighborhoods of $k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

In order to find the coefficients a_n and b_n in (1) on the basis of the right scattering data $\{r^+(\xi), z_j, M_j, j = 1, \ldots, N\}$ we have to examine either of the equations (7) or (9) which are constructed only from the scattering data, with unknowns A_{nm} or B_{nm}, respectively. When the conditions of Theorem 4 are satisfied, these equations are uniquely solvable! We have the formula

$$1 + F_{2n} + \sum_{k=1}^{\infty} A_{nk} F_{k+2n} = \sum_{j=1}^{N} M^+_j \, e^{2inz_j} (1 + A_n(z_j))^2$$

$$+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \{ | 1 + r^+(\xi) \, e^{2in\xi} + A_n(-\xi) + r^+(\xi) \, e^{2in\xi} A_n(\xi) |^2$$

$$+ \frac{| 1 + A_n(\xi) |^2}{| a(\xi) |^2} \} \, d\xi \tag{10}$$

where $A_n(z) = \sum_{m=1}^{\infty} A_{nm} e^{imz}$, $\mathrm{Im}\, z \geq 0$. For $1 + \Phi_{2n} + \sum_{-\infty}^{k=-2} B_{nk} \Phi_{k+2n}$ a similar equality holds. From (10) it follows that the expression on the left side of this equality is positive for all n, $n = 0, \pm 1, \pm 2, \ldots$ After this, we define α_n by formula (8) and a_n, b_n by formulas

$$a_n = \frac{\alpha_{n+1}}{\alpha_n}, \; b_n = A_{n1} - A_{n-1,\, 1}, \; n = 0, \pm 1, \pm 2, \ldots$$

It can be proved that condition (2) is satisfied.

When $r^+(\xi) = 0$ the fundamental equation (7) can be solved and by the same taken the coefficients a_n and b_n can be writen in explicit form.

ÖZET

Bu çalışmada bir boyutlu diskret Schrödinger denklemi (tüm eksen üzerinde ikinci derece fark denklemi) için ters saçılma probleminin çözümünün varlığının gerek ve yeter koşulları bulunmuştur.

REFERENCES

[1] K.M. CASE and S.C. CHIU., J. Mathematical Phys. 14 (1973), 1643. MR 48 $\#$ 10394.

[2] K.M. CASE., J. Mathematical Phys. 15 (1975), 143. MR 48, $\#$ 10350.

[3] S.V. MANAKOV, Z. Ekper. Teoret. Fiz. 67 (1974), 543, Soviet Phycs JETP 40 (1974), 269.

[4] HERMAN, FLASCHKA., Progr. Theoret. Phys. 51 (1974), 703.

[5] L.D. FADDEEV., Trudy Mat. Inst. Steklov. 73 (1964), 314; English Transl. Amer. Math. Soc. Transl. (2) 65 (1967), 139. MR 31 # 2446.

[6] ———., Itogi Nauki i Tehniki: Sovremennye Problemy Mat. vyp. 3, VINITI, Moscow, 1974, p. 93 = J. Soviet Math. 5 (1976), 334.

[7] V.A. MARCHENKO., Sturm–Liouville operators and their applications, "Naukova Dumka", Kiev, 1977. (Russian)

[8] G. SH. GUSEINOV., Dokl. Akad. Nauk SSSR 227 (1976), 1289, Soviet Math. Dokl. 17 (1976), 596.