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RESTRICTIVE SEMIGROUPS OF HOLOMORPHIC ENDOMORPHISMS ON RIEMANN SURFACES

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ABSTRACT

Let R be a Riemann surface and X be any nonempty subset of R. E(R,X) denotes the semigroup, under composition, of all holomorphic selfmaps of R which carry X into X and is referred to as a restrictive semigroup of holomorphic endomorphisms. Let R_1, R_2 be Riemann surfaces which have bounded nonconstant holomorphic functions and X, Y be any subsets of R_1, R_2, respectively. If ϕ : E(R_1 , X) - E(R_2 , Y) is an isomorphism of semigroups, then there exist a conformal (or anticonformal) isomorphism $\psi : X \rightarrow Y$ such that $\phi(f) = \psi \circ f \circ \psi^{-1}$ for every $f \in E(R_1, X)$.

1. INTRODUCTION

Let R be a Riemann surface and X be a nonempty subset of R. The semigroup, under composition, of all holomorphic selfmaps of R which carry X into X is denoted by E(R,X) and is referred to as a restrictive semigroup. E(R,X) is clearly a subsemigroup of E(R) and it coincides with E(R) precisely when X is all of R.

A well-known theorem by L. Bers states that two plane domains are conformally (or anticonformally) equivalent if and only if their rings of holomorphic functions are isomorphic [1]. This result has been generalized to Riemann surfaces and its nonempty subsets [3, 4, 5]. A. Eremenko has shown that if R_1 and R_2 are two Riemann surfaces which have bounded nonconstant holomorphic functions and E(R_1) and E(R_2) are the semigroups of all holomorphic endomorphisms of R_1 and R_2, respectively, then any isomorphism of E(R_1) with E(R_2) induces a conformal (or anticonformal) isomorphism R_1 with R_2 [2].

2. ISOMORPHISIMS BETWEEN RESTRICTIVE SEMIGROUPS

Definition 1. R_1, R_2 are two Riemann surfaces and X,Y be any nonempty subsets of R_1, R_2, respectively. A function ψ: X $\to$ R_2 is said to be holomorphic if for each point P $\in$ X there exists an open neighborhood U_P of P and a holomorphic function φ_P: $U_p \to R_2$ such that φ_P and ψ coincide on $U_P \cap X$. This is equivalent to assuming that there is a single open set U $\supset$ X and a holomorphic functions φ: U $\to$ R_2 such that $\varphi|X = \psi$. ψ: X $\to$ Y is said to be a conformal (anticonformal) mapping if ψ is holomorphic (or antiholomorphic, i.e., $\overline{\psi}$ is holomorphic), one-to-one and onto [3].

Let R_1, R_2 be Riemann surfaces which have bounded nonconstant holomorphic functions and X,Y be any nonempty subsets of R_1, R_2, respectively. It is immediate that each conformal (or anticonformal) mapping ψ: $R_1 \to R_2$ which carries X onto Y induces an isomorphism ϕ: $E(R_1,X) \to E(R_2,Y)$ such that $\phi(f) = \psi \circ f \circ \psi^{-1}$, $f \in E(R_1,X)$.

The purpose of this paper is to prove the following theorem. So we generalize the Eremenko's result to nonempty subsets of Riemann surfaces.

Theorem: Let R_1, R_2 be Riemann surfaces which have bounded nonconstant holomorphic functions and X,Y be any nonempty subsets of R_1, R_2, respectively. Suppose that ϕ: $E(R_1,X) \to E(R_2,Y)$ is an isomorphism of semigroups of holomorphic endomorphisms, then there exists a conformal (or anticonformal) isomorphism ψ: X $\to$ Y such that $\phi(f) = \psi \circ f \circ \psi^{-1}$, for each $f \in E(R_1,X)$.

Proof. We denote the constant mapping which maps R_1 to P $\in$ X by c_P and denote the set of all constant endomorphisms by $C(R_1,X)$ the subsemigroup of $E(R_1,X)$. Then $c_P(P') = P$ for all $c_P \in C(R_1,X)$ and $P' \in R_1$;

$$f \circ c_P = c_{f(P)} \text{ and } c_P \circ f = c_P \text{ for all } f \in E(R_1,X).$$

We first prove that ϕ: $C(R_1,X) \to C(R_2,Y)$, i.e., ϕ maps constants to constants. Let $c_P \in C(R_1,X)$, P $\in$ X. For any Q $\in$ Y there exists an $f \in E(R_1,X)$ such that $\phi(f) = c_Q$ since ϕ is onto. Hence, for all $Q' \in Y$

$$\phi(c_P)(Q) = \phi(c_P) \circ \phi(f)(Q') = \phi(c_P \circ f)(Q') = \phi(c_P)(Q'),$$

which shows that $\phi(c_P) \in C(R_2,Y)$. Thus, we can define $\psi: X \to Y$ by

$$\phi(c_P) = c_{\psi(P)} \text{ for all } P \in X \text{ (i.e., } \psi(P) = Q).$$

Then ψ is one-to-one, because $\psi(P) = \psi(P')$ implies that $\phi(c_P) = \phi(c_{P'})$, which leads to $P = P'$. Further, ψ is onto, beacuse for any $Q \in Y$, if we take $f \in E(R_1,X)$ such that $\phi(f) = c_Q$, then we can show in the same way as above that $f \in C(R_1,X)$, or $f = c_P$ for some $P \in X$, and hence, $Q = \psi(P)$.

Now let $f \in E(R_1,X)$ and $P, P' \in X$ such that $f(P) = P'$ and $\psi(P) = Q \in Y$. Then for all $Q' \in Y$,

$$\begin{aligned}\phi(f)(\psi(P)) &= \phi(f)(Q) &&= [\phi(f) \circ (c_Q)](Q') \\ &= [\phi(f) \circ c_{\psi(P)}](Q') &&= [\phi(f) \circ \phi(c_P)](Q') \\ &= \phi(f \circ c_P)(Q') &&= \phi(c_{P'})(Q') \\ &= c_{\psi(P')}(Q') &&= c_{\psi \circ f(P)}](Q').\end{aligned}$$

Hence $\phi(f) \circ \psi = \psi \circ f$ so $\phi(f) = \psi \circ f \circ \psi^{-1}$.

Now we show that ψ is continuous. Firstly, we give following definition:

Definition 2: Let f be an element of $E(R,X)$. f is called a good element if for any iterate f^n of f, $f^n(R)$ is relatively compact image in R.

If f is a good element, then the existence of a fixed point in R follows from relatively compactness of the image. Every element of E (R,X) which is different from identity has at most one fixed point in R. If R is a hyperbolic Riemann surface, i.e., the universal covering of R, is the unit disk U, then there exists a Riemannian metric on R which is called Poincaré metric. Denote by ρ the distance in the Poincaré metric in R. The invariant form of the Schwarz lemma states that $\rho(f(P), f(Q)) \leq \rho (P,Q)$ for every P and Q. If f(R) is relatively compact, then f cannot be a covering so f strictly decreases the Poincaré distance. It follows that the sequence $f(R) \supset f^2(R) \supset \ldots$ has one point of intersection and this point P is the unique attractive fixed point of f in R. (Attractive means that $|f'(P)| < 1$.

The derivative at a fixed point does not depend on the choice of local coordinate.).

Now let $f \in E(R_1,X)$ be a good element. Then f has a fixed point $P_0 \in X$ and f is univalent in a neighborhood of this fixed point and

$$\bigcap_{n\in N} f^n(R_1) = \{P_0\}.$$

Eremenko showed that $\{f^n(R_1)\}$ forms a fundamental set of neighborhoods of P_0 [2]. Now let $Q_0 = \psi(P_0) \in Y$. Since f is good, $\phi(f) = g$ is a good element in $E(R_2,Y)$ which fixes Q_0. We also have $\psi(f^n(R_1) \cap X) = g^n(R_2) \cap Y$. So ψ maps a fundamental set of neighborhoods of P_0 to a fundamental set of neighborhoods of Q_0, in the relative topologies. Thus ψ is continuous.

Next, we show that ψ is conformal (or anticonformal). Let

$$P(f) = \{h \in E(R_1,X) \mid h \circ f = f \circ h,\ f \in E(R_1,X)\}.$$

This is a semigroup of $E(R_1,X)$. Denote by S the group of all linear self-maps of the field C, i.e.,

$$S = \{z \to \lambda z \mid \lambda \in C^* = C \setminus \{0\}\}.$$

The group S is isomorphic to the multiplicative group C^*. There exists a neighborhood $O_1 \subset R_1$ of P_0 and a local coordinate $F: (O_1,P_0) \to (C,0)$ which conjugates $P(f)$ to some subsemigroup $S_1 \subset S$. In other words $s(h) = F \circ h \circ F^{-1} \in S$ if $h \in P(f)$ and $h \to s(h)$ is an isomorphism of semigroups $P(f) \to S_1$. Similarly consider a local coordinate $G: (O_2,Q_0) \to (C,0)$, $Q_0 \in O_2 \subset R_2$, which conjugates $P(g)$ to a subsemigroup $S_2 \subset S$. If S_1 and S_2 are considered as subsets of C^*, then they contain some punctured neighborhoods of 0.

Lemma. Let S_1 and S_2 be subsemigroups of the multiplicative group C^* both containing some punctured neighborhoods of 0. If V is a continuous injective mapping in a neighborhood of 0 which conjugates S_1 to S_2, then

$$V(z) = az^A \bar{z}^B, \tag{1}$$

where $a \in C^*$, $A, B \in C$ and $A - B = \pm 1$ [2].

Note that V given by (1) is differentiable (as a function from R^2 to R^2) and nondegenerate in C*. It is differentiable and nondegenerate at 0 iff A + B = 1. In the latter case V is conformal (or anticonformal) because A + B = 1 and A - B = ±1 imply A = 1 or B = 1.

Now the function $V_{P_0} = G \circ \psi \circ F^{-1}$ maps a neighborhood of 0 to some neighborhood of 0 and conjugates S_1 to S_2. According to the Lemma, for arbitrary $P \in O_1 \setminus \{P_0\}$ the function V_P is differentiable and nondegenerate. Therefore $\psi \mid X \cap (O_1 \setminus \{P_0\})$ is differentiable and nondegenerate. So V_P is conformal (or anticonformal) this implies that ψ is conformal (or anticonformal).

REFERENCES

[1] BERS, L. *On rings of analytic functions*, Bull. Amer. Math. Soc., 54(1948), 311-315.

[2] EREMENKO, A., *On the characterization of a Riemann surface by its semigroup of endomorphisms*, Trans. Amer. Math. Soc., Vol. 338, No.1 (1993), 123-131.

[3] MINDA, C.D., *Rings of holomorphic and meromorphic functions on subsets of Riemann surfaces*, Journal of the Indian Math. Soc., 40(1976), 75-85.

[4] RUDIN, W., *An algebraic characterization of conformal equivalence*, Bull. Amer. Math. Soc., 61(1955), 543.

[5] ŞERBETÇİ, A., and ÖZKIN, İ.K., *On the rings of analytic functions on any subset of an open Riemann surface*, Jour. Inst. Math. and Comp. Sci. (Math. Ser.), Vol.3, No.1 (1990), 15-20.