Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 153-163 (1997)

ON SOME CONFIGURATIONAL PROPOSITIONS CONNECTED WITH THE MINOR FORMS OF DESARGUES

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(Received Sept. 9, 1997; Accepted Oct. 14, 1997)

ABSTRACT

In this paper, we have studied connection among some restricted dual Pappus Theorems, some speciel forms of Desargues Theorem stated for quadrangles and minor forms of Desargues Theorem stated for triangles, in a projective plane, by using algebraic properties associated with these configurational propositions.

1. INTRODUCTION

Let $\pi = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane with the incidence relation I and coordinatized by a set R with relative to a coordinatizing quadrangles (X, Y, O, E) the points not on XY (the line joining X, Y) and the lines not on Y are written, respectively, as their pairs of coordinates (x, y), [m, k], (x, y, m, k $\in$ R), the points $\neq$ Y on XY and the line $\neq$ XY on Y, respectively as (m), [x] with (x, y) I [x], (m) I [m, k] and O = (0, 0), E = (1, 1), X = (0), OY = [0], EY = [1]; OX = [0, 0], OE = [1, 0], (0, k) I [0 k], (1,m) I [m, 0], (for special elements 0, 1 $\in$ R).

Using the notation in [1], a ternary operation t may be defined on the set R as follows:

(x, y) I [m, k] $\Leftrightarrow$ y = T(m, x, k), for all x, y m, k $\in$ R.

The system (R, T) is called a ternary ring (TR) on π and π can be coordinatized by this ternary ring (R,T) [1].

There different binary operations denoted by +, . , * may be defined on R as follows [1, 3].

$a + b = T(1, a, b)$,
$a * b = T(a, 1, b)$ and
$a . b = T(a, b, 0)$, for all $a, b \in R$.

In [3], it has been studied several minor forms of Desargues Theorem stated for triangles in a Hall ternary ring and several algebraic results associated with these minor froms.

By a configuration we mean, in this paper, a finite partial plane [7, p.22], thus the hypothesis of a configurational proposition constitutes a configuration which will be called the incomplete configuration of the proposition; while the incidence in the conclusion is called the missing incidence. Adding it to the incomplete configration gives the complete configuration of the proposition.

We start with some known properties which will be used in this paper.

Property $\mathcal{D}$ (Desargues theorem stated for quadrangles) [2]: Let ABCD and A′B′C′D′ be two quadrangles in π such that AA′, BB′, CC′ and DD′ pass through a point P and the points $AB \wedge A'B'$, $BC \wedge B'C'$, and $AD \wedge A'D'$ on a line d, then $CD \wedge C'D'$ also lies on d. The point P is called the center of perspectivity and the line d is called the axis of perspectivity for these quadrangles.

(It is denoted that the line through any points A and B as AB or $A \vee B$; the intersection point of any lines α and β as $\alpha\beta$ or $\alpha \wedge \beta$).

Propertiey $\mathcal{D}$(M, N): Let X, Y, O and E be a set of reference points in π and M,N be two points in π.

If two pairs of corresponding (and opposite) sides of the quadrangles in Property $\mathcal{D}$ meet in M, then the other two pairs meet in N.

One can easily observe that Property $\mathcal{D}$ is equivalent to its dual form and the usual Desargues configuration in π for triangles with P as center and d as axis implies Property $\mathcal{D}$. (A, B, C) - **Dual Pappus Theorem** ((A, B, C) DPT) [4]: Let A, B, C be three distinct points of

π; α, β, γ I A and α', β', γ' I B be any lines of π. If $\alpha\beta' \vee \alpha'\beta$ I C and $\alpha\gamma' \vee \alpha'\gamma$ I C then $\beta\gamma' \vee \beta'\gamma$ I C.

One can easily prove that (A, B, C) - DPT implies (X, Y, Z) - DPT, where (X, Y, Z) is any permutation of the points A, B, C. In each (A, B, C) - DPT exactly 4 of 9 lines are arbitrary and positions of the remaining lines depend on these four lines. This leads us to:

Definition 1.1. An (A, B, C) - DPT is called an (A, B,C) - Restricted Dual Pappus Theorem ((A, B, C) - RDPT) if the number of its arbitrary lines is less than 4.

Minor forms of Desargues arise when one requires, one, two and finally three vertices of one triangle to lie on the sides of the other triangle [7, p.26]. We shall denote these propositions by D_1, D_2, D_3, respectively. Another special form of D_1, denote by $D^1{}_1$, may be formulated as follows: ABC and A′B′C′ be any two triangles, with A I B′C′, A′ I BC, and this triangles are perspective from the point M, then these triangles are perspective from the line d.

Theorem 1.2. [3]: D_1 and the validity of the following conditions, in every coordinatizing (R, T), are equivalent for all a, b, c in R:

(i) $a * b = a + b$.

(ii) $a(1 + b) = a + ab$.

(iii) $T(a, b, c) = ab * c$.

Theorem 1.3. [3]: $D^1{}_1$ holds in π if and only if one of the following conditions is valid in every coordinatizing (R,T) for all a, b $\in$ R:

(i) $a(1 + 1) = a + a$.

(ii) $T(a, b, ab) = ab * ab$.

Theorem 1.4. [3]: D_2 holds in π if and only if one of the following conditions is valid in every coordinatizing (R,T) for all a, b $\in$ R:

(i) $a * a = a + a$.

(ii) $a + b = 0 \Rightarrow a * b = 0$.

(iii) $a + b = 1 \Rightarrow a * b = 1$.

Theorem 1.5. [3]: D_3 holds in π if and only if one of the following conditions is valid in every coordinatizing (R,T) for all a, b $\in$ R:

(i) $a * a = 1 \Rightarrow a + a = 1$.

(ii) $a + a = 1 \Rightarrow a * a = 1$.

2. MAIN RESULTS

Proposition 2.1. The following statemens are equivalent:

(i) The operations + and * are equivalent in the ternary ring (R,T), that is $a \dotplus b = a * b$, for all a, b $\in$ R.

(ii) ((0, b), (∞), (0)) - RDPT holds in π whenever $\alpha = [0, b]$, $\beta = [a, b]$, $\gamma = [1, b]$ and $\alpha' = [0]$, $\beta' = [a]\gamma' = [1]$ for all a, b $\in$ R - {0} $a \neq 1$.

(iii) The property $\mathcal{D}((\infty), (0))$ holds in π for two quadrangles ABCD and A'B'C'D' with the point (0, b), as center, and the line [0, b], as common side, whenever $C' = (a, a+b)$ and $D' = (1, a * b)$ for all a b $\in$ R - {0}, $a \neq 1$.

Proof: Using figure 1, let α, β, γ and α', β', γ' generate an incomplete configuration ((0, b), (∞), (0)) - RDPT. Using properties of loops (R, *) and (R, +), it is easily seen that $\alpha\beta' \vee \alpha'\beta = \alpha\gamma' \vee \alpha''\gamma = [0, b]$, $\beta\gamma' = (1, a * b)$ and $\beta'\gamma = (a, a + b)$. So,

$\beta\gamma' \vee \beta'\gamma = (1, a * b) \vee (a, a + b)\ I\ (0) \Leftrightarrow a + b = a * b$, for all a, b $\in$ R. Hence,

$$\text{(i)} \Leftrightarrow \text{(ii)} \tag{1}$$

(i) $\Leftrightarrow$ (iii): Using figure 1. Let ABCD and A'B'C'D' generate an incomplete configuration $\mathcal{D}((\infty), (0))$, where $D = [b] \wedge [a, b]$ for all a, b $\in$ R + {0}, $a \neq 1$. Then $D = (b, T(a, b, b))$ and $AB = A'B' = [0,b]$.

C = DX $\wedge$ [1, b] = (x, T(a, b, b)), since CC′ = [1, b], CD I X and the equation x + b = T(a, b, b) has unique solution in the loop (R, +). Therefore, A = (b, b), B = (x, b), C = (x, T(a, b, b)), D = (b, T(a, b, b)), A′ = (1, b), B′ = (a, b) C′ = (a, a + b), D′ = (1, a * b).

Hence,

(i) $\Leftrightarrow$ (iii) (2)

(1) and (2) imply (ii) $\Leftrightarrow$ (iii).

In the following seven propositions, proofs are quite similar to that of the above Proposition 2.1. Therefore we only state these propositions and give a related figure for each of them.

Proposition 2.2. The following statements are equivalent:

(i) T (a, b, c) = ab * c, for all a, b, c $\in$ R.

(ii) ((0, c), (∞), (0)) - RDPT holds in π, whenever α = [0, c], β = [ab, c] γ = [a, c] and α' = [0], β' = [b], γ' = [1] for all a, b, c $\in$ R - {0}, b $\neq$ 1.

(iii) The property $\mathcal{D}$((∞), (0)) holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, c) as center and the line [0, c] as common side, where C′ = (b, T(a, b, c)) and D′ = (1, ab * c) for all a, b, c $\in$ R - {0}, b $\neq$ 1.

(See figure 2).

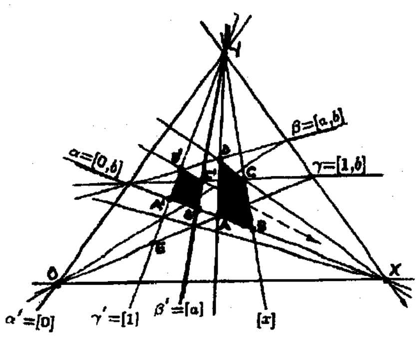

Fig. 1

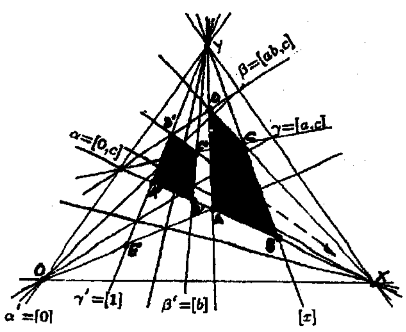

Fig. 2

Corollary 1: D_1 and validity of the following conditions, in each coordinatizing (R, T), are equivalent:

(i) Property $\mathcal{D}((\infty), (0))$ holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, b) as center and the line [0, b] as common side, whenever C = (a, a + b) and D′ = (1, a * b) for all a, b $\in$ R - {0}, a $\neq$ 1.

(ii) Property $\mathcal{D}((\infty), (0))$ holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, c) as center and the line [0, c] as common side, where C′ = (b, T(a, b, c)) and D′ = (1, ab * c) for all a, b, c $\in$ R - {0}, a $\neq$ 1.

(iii) ((0, b), (∞), (0))-RDPT holds in π, whenever α = [0], β' = [a], γ' = [1] for all a, b $\in$ R - {0}, a $\neq$ 1.

(iv) ((0, c), (∞), (0))-RDPT holds in π whenever α = [0, c], β = [ab, c], γ = [a, c] and α' = [0], β' = [b], γ' = [1] for all a, b, c $\in$ R - {0}, b $\neq$ 1.

Sketch of proof: This corollary can be obtained by Proposition 2.1, Proposition 2.2 and Theorem 1.1.

Proposition 2.3. The following statements are equivalent:

(i) a(1 + 1) = a + a, for all a $\in$ R.

(ii) ((0, 0), (∞), (0))-RDPT holds in π, whenever α = [0, 0], β = [a + a, 0], γ = [a, 0] and α' = [0], β' = [1 + 1] γ' = [1] for all a $\in$ R - {0}.

(iii) Property $\mathcal{D}((\infty), (0))$ holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, 0) as center, where C′ = (1 + 1, a(1 + 1)) and D′ = (1, a + a), for all a $\in$ R - {0}.

(See figure 3).

Proposition 2.4. The following statements are equivalent:

(i) T(a, b, ab) = ab * ab, for all a, b $\in$ R.

(ii) ((0, ab), (∞), (0))-RDPT holds in π, whenever α = [0, ab], β = [ab, ab] γ = [a, ab] and α' = [0], β' = [b] γ' = [1] for all a, b $\in$ R - {0}, b $\neq$ 1.

(iii) Property $\mathcal{D}$((∞), (0)) holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, ab) as center, and the line [0, ab], a common side, where C′ = (b, T(a, b, ab)) and D′ = (1, ab * ab), for all a, b $\in$ R - {0}, b $\neq$ 1.

(See figure 4).

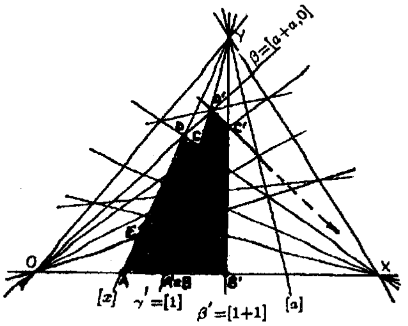

Fig. 3

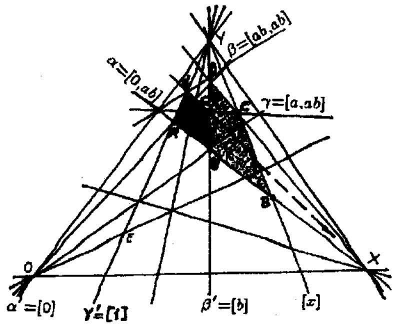

Fig. 4

The following corollary can be obtained by Proposition 2.3, Proposition 2.4 and Theorem 1.2.

Corollary 2: D^1_1 holds in π if and only if one of the following conditions are valid in every coordinatizing (R, T):

(i) Property $\mathcal{D}$((∞), (0)) holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, 0) as center and the line [0, 0] as common side, whenever C′ = (1 + 1, a(1 + 1)) and D′ = (1, a + a) for all a $\in$ R - {0}.

(ii) ((0, ab), (∞), (0))-RDPT holds in π, whenever α = [0, ab], β = [a + a, 0], γ = [a, 0] and α' = [0], β' = [1 + 1], γ' = [1] for all a, b $\in$ R - {0}, b $\neq$ 1.

(iii) ((0, 0), (∞), (0))-RDPT holds in π, whenever α = [0, 0], β = [a + a, 0], γ = [a, 0] and α' = [0], β' = [1 + 1], γ' = [1] for all a $\in$ R - {0}.

(iv) Property $\mathcal{D}$((∞), (0)) holds in π for quadrangles ABCD and A′B′C′D′ with the point (0, ab) as center, and the line [0, ab], as common side, where C′ = (b, T(a, b, ab)) and D′ = (1, ab * ab), for all a, b $\in$ R - {0}, b $\neq$ 1.

Proposition 2.5. The following statements are equivalent:

(i) a * a = 1 $\Rightarrow$ a + a = 1, for all a $\in$ R.

(ii) Property $\mathcal{D}$((∞), (0, a)) holds in π for the quandrangles ABCD and A′B′C′D′ with the point (0), as center. Where A = (1, a), B = B′ = (0), C = (a), D = (1, 1) and A′ = (a, a), C′ = (1), D′ = (a, 1) for all a $\in$ R - {0, 1}.

(iii) ((∞), (1, 1), (0, a))-RDPT holds in π whenever α = [1], β = [∞], γ =[a] and α' = [a, a], β' = [0, 1], γ' = [1, 0] for all a $\in$ R - {1}.

(See figure 5).

Proposition 2.6. The following statements are equivalent:

(i) a + a = 1 $\Rightarrow$ a * a = 1, for all a $\in$ R.

(ii) Property $\mathcal{D}$((∞), (0, a)) holds in π for the quandrangles ABCD and A′B′C′D′ with the point (0), as center, and the lines [0, a] and [∞] as common sides, where A = (a, a), B = B′ = (0), C = (1), D = (a, 1), A′ = (1, a), C′ = (a), D′ = (1, 1), for all a $\in$ R - {0, 1}.

(iii) ((∞), (0, 0), (0, a))-RDPT holds in π, whenever α = [0] = α', β = [1], γ =[∞], β' = [a, 0], γ' = [1, 0], for all a $\in$ R - {0}.

(See figure 6).

The following theorem can be obtained by Proposition 2.5, Proposition 2.6 and Theorem 1.4.

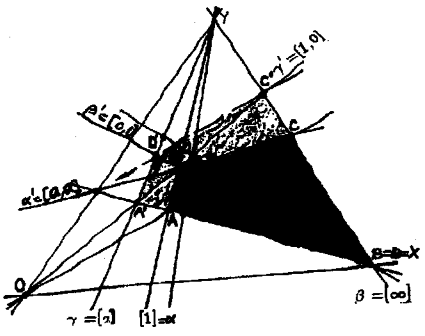

Fig. 5

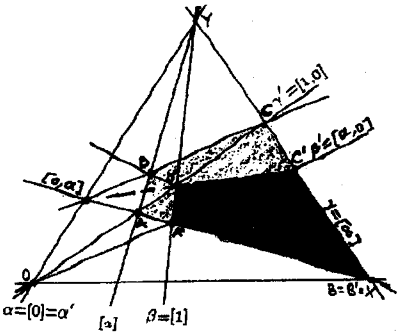

Fig. 6

Corollary 3: D_3 holds in π if and only if one of the following conditions are valid in every coordinatizing (R, T):

(i) The property $\mathcal{D}((\infty), (0, a))$ holds in π for quadrangles ABCD and A′B′C′D′ with the point (0), as center, where A = (1, a), B = B′ = (0), C = (a), D = (1, 1), and A′ = (a, a), C′ = (1), D′ = (a, 1) for all a $\in$ R - {0, 1}.

(ii) ((∞), (1, 1), (0, a))-RDPT holds in π, whenever α = [1], β = [∞], γ = [a] and α' = [a, a], β' = [0, 1], γ' = [1, 0], for all a $\in$ R - {1}.

(iii) ((∞), (0, 0), (0, a))-RDPT holds in π, whenever α = α' = [0], β = [1], γ = [∞] and β' = [a, 0], γ' = [1, 0] for all a $\in$ R - {0}.

Proposition 2.7. The following statements are equivalent:

(i) a + b = 0 $\Rightarrow$ a * b = 0, for all a, b $\in$ R.

(ii) ((∞), (0, 0), (0, b))-RDPT holds in π, whenever α = [0] = α', β = [0, 0], γ =[a, 0], β' = [∞], γ' = [1], for all a $\in$ R - {0}.

(iii) Property $\mathcal{D}((\infty), (0, b))$ holds in π, for the quandrangles ABCD and A′B′C′D′ with the point (0), as center, where A = (0) = A′, B = (a, b), C = (a, 0), D = (1) and B′ = (1, b), C′ = (1, 0), D′ = (a), for all a, b $\in$ R - {0, 1}.

(See figure 7).

Proposition 2.8. The following statements are equivalent:

(i) $a + b = 1 \Rightarrow a * b = 1$, for all $a, b \in R$.

(ii) $((\infty), (0, 0), (0, b))$-RDPT holds in π, whenever $\alpha = [0] = \alpha'$, $\beta = [\infty]$, $\gamma = [1]$, $\beta' = [1, 0]$, $\gamma' = [a, 0]$, for all $a, b \in R - \{0\}$.

(iii) Property $\mathcal{D}((\infty), (0, b))$ holds in π, for the quandrangles ABCD and A′B′C′D′ with the point (0), as center, where $A = (a, b) = B$, $B' = (0)$, $C = (1)$, $D = (a, 1)$, $A' = (1, b)$, $C' = (a)$, $D' = (1, 1)$, for all $a, b \in R - \{0\}$, $a \neq 1$.

(See figure 8).

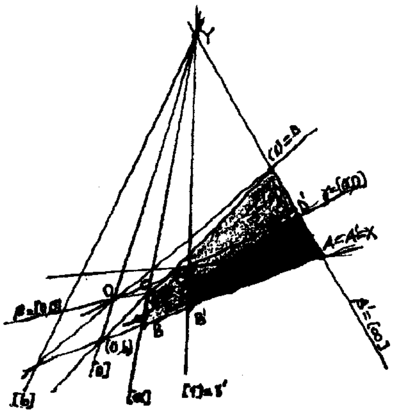

Fig. 7

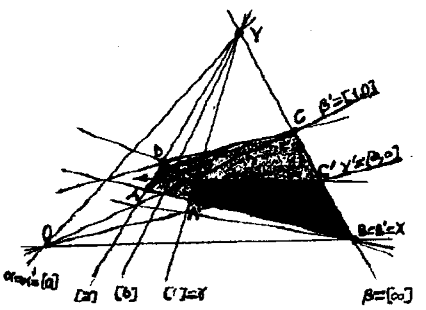

Fig 8

Corollary 4: D_2 holds in π if and only if one of the following conditions is valid in the coordinatizing (R, T):

(i) $((\infty), (0, 0), (0, b))$-RDPT holds in π, whenever $\alpha = [0]$, α', $\beta = [0, 0]$, $\gamma = [a, 0]$, $\beta' = [\infty]$, $\gamma' = [1]$, for all $a \in R - \{0\}$.

(ii) Property $\mathcal{D}((\infty), (0, b))$ holds in π for quadrangles ABCD and A′B′C′D′ with the point (0) as center, where $A = (0) = A'$, $B = (a, b)$, $C = (a, 0)$, $D = (1)$, and $B' = (1, b)$, $C' = (1, 0)$, $D' = (a)$ for all $a, b \in R - \{0, 1\}$.

(iii) $((\infty), (0, 0), (0, b))$-RDPT holds in π, whenever $\alpha = [0] = \alpha'$, $\beta = [\infty]$, $\gamma = [1]$ and $\beta' = [1, 0]$, $\gamma' = [a, 0]$, for all a, b $\in$ R - $\{0\}$.

(iv) Property $\mathcal{D}((\infty), (0, b))$ holds in π for quadrangles ABCD and A'B'C'D' with the point (0) as center, where A = (a, b), B = B' = (0), C = (1), D = (a, 1), and A' = (1, b), C' = (a), D' = (1, 1) for all a, b $\in$ R - $\{0\}$, $a \neq 1$.

REFERENCES

[1] PICKERT, G.: Projective Ebenen, Berlin - Göttingen - Heidelberg, Springer 1955.

[2] NAGAMUNI REDDY, L. and SITARAM, K.: Associativity in a Hall ternary ring, Jour. Ind. Math. Soc. Vol.44, 173-176, 1980.

[3] AL-DHAHIR, M.W., and GHALIEH, K.: On minor forms of Desargues and Pappus, Geom. Ded. 42, 329-344, 1992.

[4] KAYA, R.: On the connection between ternary rings and the restricted dual Pappus theorems-I, Jour. of Fac. of the K.T.Ü. vol. III, Fasc. 6, 49-57, 1980.

[5] OLGUN, Ş. and GÜNALTILI, İ.: On the connection between ternary rings and some configurational propositions-I, Tr. Jour. of Math. 18, 302-310, 1994.

[6] OLGUN, Ş. and GÜNALTILI, İ.: On the connection between ternary rings and some configurational propositions-II, Jour. Inst. Math. and Comp. Sci. (Math. Ser.) vol.7 No. 2, 147-154, 1994.

[7] SKORNYAKOV, L.A.: Projective planes, Amer. Math. Soc. Transl. 99, 1953.