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SOME PROPERTIES OF LINEAR POSITIVE OPERATORS IN THE WEIGHTED SPACES OF UNBOUNDED FUNCTIONS

TÜLİN COŞKUN

Department of Mathematics, Faculty of Sciences, Ankara University, Ankara, TURKEY.

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ABSTRACT

In this work, existence of Korovkin's theorem in the space of continuous and unbounded functions defined on unbounded sets has been studied.

INTRODUCTION

Korovkin theorem ([7]) which is important in approximation theory, extends the convergence on three functions to the functions which are continuous on $[a,b]$ and bounded on $\mathbb{R}$. Baskakov ([1]) extended the boundedness condition on $\mathbb{R}$ to the unbounded functions.

Let $C(a,b)$ denote the space of all continuous functions on $[a,b]$ and let $B(a,b)$ is the space of all bounded functions on the same interval. Then the Korovkin's theorem can be stated as follows.

Theorem (P.P. Korovkin) If the sequence of positive linear operators $A_n: C(a,b) \to B(a,b)$ satisfy the three conditions

$$\lim_{n\to\infty} \|A_n(1,x) - 1\|_{C(a,b)} = 0 \tag{1}$$

$$\lim_{n\to\infty} \|A_n(t,x) - x\|_{C(a,b)} = 0 \tag{2}$$

$$\lim_{n\to\infty} \|A_n(t^2,x) - x^2\|_{C(a,b)} = 0 \ , \tag{3}$$

then

$$\lim_{n\to\infty} \|A_n(f,x) - f(x)\|_{C(a,b)} = 0$$

for all function $f \in C(a,b)$ for which $|f(x)| \le M_f(1 + x^2)$ hold on $\mathbb{R}$.

Some generalizations of this theorem may be found in [4], [5] and [6]. But all those generalizations use a finite closed interval for convergence. In [2] and [3], Gadjiev defined the weighted spaces C_ρ and B_ρ of real functions defined on the real line and showed that Korovkin theorem does not hold in these spaces. Here $B_\rho := \{f\colon |f(x)| \leq M_f.\rho(x), -\infty < x < \infty, \rho \geq 1$ and ρ unbounded$\}$ and $C_\rho := \{f\colon f \in B_\rho$ and f continuous$\}$ are the spaces of functions which are defined on an unbounded regions. Furthermore in [2] and [3] it was shown that this theorem holds on a some subspace of the space C_ρ. Generalizations of this theorem also appear in [4].

In this work, we extend these results of A. D. Gadjiev for different ρ_1 and ρ_2, and show that a Korovkin's theorem does not hold for a class of positive linear operators, acting from C_{ρ_1} to B_{ρ_2}. We give a proof of this result, different from the proof, given in [2] and [3].

Let us consider the spaces $C_{\rho_1}(\mathbb{R})$ and $B_{\rho_2}(\mathbb{R})$ where $\rho_1 \neq \rho_2$. First we give the following properties, of positive linear operators which are the maps between these spaces:

1. Positive linear operator L, defined on C_{ρ_1} acting from C_{ρ_1} to B_{ρ_2} iff the inequality

$$\|L(\rho_1,x)\|_{\rho_2} \leq M_1$$

holds.

2. Let $L\colon C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ be positive linear operator. Then

$$\|L\|_{C_{\rho_1} \to B_{\rho_2}} = \|L(\rho_1,x)\|_{\rho_2}$$

and therefore for all $f \in C_{\rho_1}$, the inequality

$$\|L(f,x)\|_{\rho_2} \leq \|L(\rho_1,x)\|_{\rho_2} \|f\|_{\rho_1}$$

holds.

3. Let

$$A_n\colon C_{\rho_1} \to B_{\rho_2}$$

be positive linear operators for all $n \in \mathbb{N}$. Suppose that there exist $M > 0$ such that for all $x \in \mathbb{R}$, $\rho_1(x) \le M\rho_2(x)$. If

$$\lim_{n\to\infty} \|A_n(\rho_1,x) - \rho_1(x)\|_{\rho_2} = 0 ,$$

then the sequence of norms $\|A_n\|_{C_{\rho_1}\to B_{\rho_2}}$ is uniformly bounded.

Remark. From the above condition $\rho_1(x) \le M\rho_2(x)$ we have $C_{\rho_1} \subset C_{\rho_2} \subset B_{\rho_2}$.

Theorem 1. Let φ_1 and φ_2 be two continuous functions, monotone increasing on real axis such that $\lim_{x\to\pm\infty} \varphi_1 = \lim_{x\to\pm\infty} \varphi_2 = \pm\infty$ and that $\rho_1(x) \le M\rho_2(x)$ ($M > 0$ is arbitrary constant) for all $x \in \mathbb{R}$ where

$$\rho_k(x) = 1 + \varphi_k^2(x) \quad , \quad k = 1, 2$$

and

$$\lim_{x\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = a \neq 0.$$

Then, there exist a sequence

$$A_n : C_{\rho_1} \to B_{\rho_2}$$

of positive linear operators satisfying the following three conditions

$$\lim_{n\to\infty} \|A_n(\varphi_1^\nu,x) - \varphi_1^\nu(x)\|_{\rho_2} = 0 \quad , \quad \nu = 0, 1, 2, \tag{4}$$

but on the other side there exist $f^* \in C_{\rho_1} \subset B_{\rho_2}$ such that

$$\limsup_{n\to\infty} \|A_n(f^*,x) - f^*(x)\|_{\rho_2} \neq 0.$$

Proof. For given functions φ_1 and φ_2, let $(A_n)_{n\in\mathbb{N}}$ be the sequence of operators defined as follows:

$$A_n(f,x) := \begin{cases} f(x) + \dfrac{\rho_2(x)}{2\rho_2(n)} \left[\dfrac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x) \right] & ; \quad 0 \le x \le n \\ f(x) & ; \quad x \notin [0,n] \end{cases}$$

Without loss of generality we can assume that $\varphi_1(0) = 0$ and $\varphi_2(0) = 0$ since $\overline{\varphi}_1(x) := \varphi_1(x) - \varphi_1(0)$ implies $\overline{\varphi}_1(0) = 0$ whenever $\varphi_1(0) \neq 0$.

It is obvious that A_n's are linear. Furthermore, since for all $x \in [0, n]$, $\rho_2(x) \le \rho_2(n)$ and therefore $1 - \frac{\rho_2(x)}{2\rho_2(n)} \ge 1 - \frac{1}{2} > 0$, we obtain

$$A_n(f,x) = f(x) + \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x) \right]$$

$$= f(x) \left[1 - \frac{\rho_2(x)}{2\rho_2(n)} \right] + \frac{\rho_2(x)}{2\rho_2(n)} \frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f(x+1) - f(x) \ge 0$$

for all $x \in [0, n]$ and for $f \ge 0$. That means A_n's are positive. Since φ_1 monotonic, by using the fact $\varphi_1^2(x) \le \varphi_1^2(x+1)$, we get the following inequality

$$A_n(\rho_1, x) = \rho_1(x) + \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} (1 + \varphi_1^2(x+1)) - (1 + \varphi_1^2(x)) \right]$$

$$\le \rho_1(x) \le M\rho_2(x).$$

By Property 1 above, we have $A_n(\rho_1, x) \in B_{\rho_2}$. Thus

$$A_n : C_{\rho_1} \to B_{\rho_2}$$

is positive linear operator. Next we will show that this sequence of operators satisfy three conditions in (4). Since for $x \in [0, n]$

$$A_n(1, x) = 1 + \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} - 1 \right]$$

we have

$$\frac{|A_n(1, x) - 1|}{\rho_2(x)} = \frac{1}{2\rho_2(n)} \left| \frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} - 1 \right| < \frac{1}{2\rho_2(n)} .$$

That means

$$\lim_{n\to\infty} \|A_n(1, x) - 1\|_{\rho_2} = 0 .$$

Also, since

$$\frac{|A_n(\varphi_1, x) - \varphi_1(x)|}{\rho_2(x)} = \frac{1}{2\rho_2(n)} |\varphi_1(x)| . \left| \frac{\varphi_1(x)}{\varphi_1(x+1)} - 1 \right|$$

and by the monotonicity of φ_1, $\left|\frac{\varphi_1(x)}{\varphi_1(x+1)} - 1\right| < 1$ we obtain

$$\|A_n(\varphi_1, x) - \varphi_1(x)\|_{\rho_2} \le \sup_{x\in[0,n]} \frac{1}{2\rho_2(n)} |\varphi_1(x)| \le \frac{\varphi_1(n)}{2\rho_2(n)}$$

and therefore

$$\lim_{n\to\infty} \|A_n(\varphi_1, x) - \varphi_1(x)\|_{\rho_2} = 0.$$

Finally,

$$A_n(\varphi_1^2, x) - \varphi_1^2(x) = \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} \varphi_1^2(x+1) - \varphi_1^2(x) \right] = 0.$$

Therefore all conditions of theorem are satisfied.

Now let g(x) be a function defined on the interval [−1,1] given as follows

$$g(x) := \begin{cases} 2(1 + x) & ; \ -1 \le x \le 0 \\ 2(1 - x) & ; \ 0 < x \le 1, \end{cases}$$

and let us extend g(x) to a function h(x) on $\mathbb{R}$ with period 2.

If f^* is defined by

$$f^*(x) := \varphi_1^2(x) \,.\, h(x)$$

for all $x \in \mathbb{R}$, then we can obtain the following equality

$$\begin{aligned} A_n(f^*, x) &= f^*(x) + \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} f^*(x+1) - f^*(x) \right] \\ &= f^*(x) + \frac{\rho_2(x)}{2\rho_2(n)} \left[\frac{\varphi_1^2(x)}{\varphi_1^2(x+1)} \varphi_1^2(x+1)\, h(x+1) - \varphi_1^2(x)\, h(x) \right] \\ &= f^*(x) + \frac{\rho_2(x)}{2\rho_2(n)} \varphi_1^2(x) \left[h(x+1) - h(x)\right] \end{aligned}$$

for all $x \in [0, n]$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \sup_{x\in[0,n]} \frac{|A_n(f^*, x) - f^*(x)|}{\rho_2(x)} &\ge \frac{\varphi_1^2(n)}{2\rho_2(n)} |h(n+1) - h(n)| \\ &= \frac{\varphi_1^2(n)}{2\rho_2(n)} 2 = \frac{\varphi_1^2(n)}{1 + \varphi_2^2(n)} . \end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{1+\varphi_2^2(n)} = 0$ and $\lim_{n\to\infty} \frac{1+\varphi_1^2(n)}{1+\varphi_2^2(n)} = a$, it follows

$$\limsup_{n\to\infty} \|A_n(f^*, x) - f^*(x)\|_{\rho_2} \geq \lim_{n\to\infty} \frac{1+\varphi_1^2(n)}{1+\varphi_2^2(n)} = a \neq 0.$$

This completes the proof.

This theorem shows that there exists no theorem of Korovkin type for the class of positive linear operators from C_{ρ_1} to B_{ρ_2}. Note that more general statement can be proved for a class of positive linear operators acting from C_{ρ_1} to C_{ρ_2}. We have

Theorem 2. Let ρ_1 and ρ_2 be as in the Theorem 1. Then there exists a sequence $A_n : C_{\rho_1} \to C_{\rho_2}$ of positive linear operators such that

$$\lim_{n\to\infty} \|A_n(\varphi_1^v, x) - \varphi_1^v(x)\|_{\rho_2} = 0 \quad , \quad v = 0, 1, 2,$$

Moreover there exists a function $f^* \in C_{\rho_1}$ such that

$$\limsup_{n\to\infty} \|A_n(f^*, x) - f^*(x)\|_{\rho_2} \neq 0.$$

The positive statements of Korovkin-type theorems for linear positive operators $A_n : C_{\rho_1} \to B_{\rho_2}$ may be proved as in [2] and [3], in some subspace of C_{ρ_2}. Such a type theorem will be given in our another paper.

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REFERENCES

[1] BASKAKOV, V.A., On a sequence of linear positive operators. In "Research in modern constructive functions theory", Moscow (1961).

[2] GADJIEV, A.D., The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P.P. Korovkin. English translated Sov. Math. Dokl. Vol 15, No 5 (1974).

[3] GADJIEV, A.D., On P.P. Korovkin type theorems. Mathem. Zametki, Vol 20, No 5 (1976) (in Russian).

[4] HACISALİHOĞLU, H., HACIYEV, A., Lineer Pozitif Operatör Dizilerinin Yakınsaklığı, Ankara (1995).

[5] HERMANN, T., Approximation of unbounded functions on unbounded interval. Acta Math. Acad. Sci. Hungaria, 29(3-4) (1977).

[6] HSU, L.C., Approximation of non-bounded continuous functions by certain sequences of linear positive operators or polynomials. Studia Math., XXI (1961).

[7] KOROVKIN, P.P., Linear Operators and Approximation Theory. Delhi (1960).