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ON SEPERATION AXIOM C-D_1

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1. INTRODUCTION

In 1997, Caldas [1] has introduced a new seperation axiom semi-D_1 which is situated between semi-T_0 and semi-T_1 due to Maheshwari and Prasad [5]. In 1996, Hatır, Noiri and Yüksel [2] defined C-sets and C-continuity in topological spaces to obtain a decomposition of continuity. Quite recently, Jafari [3] has used the C-sets to define and investigate C-T_2 spaces, C-compact spaces and C-connected spaces. In this paper, we define cD-sets as the difference set of C-sets and use these sets to define C-D_1-spaces, cD-compact spaces and cD-connected spaces. We also investigate the relationship between these spaces and C-continuity (or C-irresoluteness).

2. PRELIMINARIES

Throughout this paper X and Y denote topological spaces on which no seperation axiom is assumed. Let A be a subset of a space X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively.

We shall recall some definitions used in the sequel.

Definition 2.1. A subset A of a space X is said to be

(a) *semi-open* [4] if $A \subset Cl(Int(A))$,
(b) α^*-*set* [2] if $Int(Cl(Int(A))) = Int(A)$,
(c) C-*set* [2] if $A = O \cap F$, where O is open and F is an α^*-set.

Remark 2.1. Semi-open sets and C-sets are independent. A set {a, b} in [2, Example 3.1] is a C-set but it is not semi-open. A set {a, b} in Example 3.1 (below) is semi-open but it is not a C-set.

Definition 2.2. A function $f:X \to Y$ is said to be *C-continuous* [2] (resp. *semi-continuous* [4]) for each open set V of Y, $f^{1}(V)$ is a C-set (resp. *semi-open* in X.

3. C-D_1 SPACES

Definition 3.1. A subset S of a space X is called a *c-difference* (briefly cD-set) (resp. D-set [6], sD-set [1]) if there exist two C-sets (resp. open sets, semi-open sets) O_1, O_2 in X such that $O_1 \neq X$ and $S = O_1 \backslash O_2$.

Remark 3.1. Every proper C-set is a cD-set, but the converse is false as the following example shows.

Example 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, (a, d\}, \{A, b, d\}, \{a, c, d\}\}$. Then {a, b} is a cD-set but it is not a C-set.

Definition 3.2. A topological space X is C-D_0 (resp. C-D_1) if for x, $y \in X$ such that $x \neq y$ there exists a cD-set of X containing x but not y or (resp. and) a cD-set containing y but not x.

A topological space X is C-T_0 (resp. C-T_1) if for x, $y \in X$ such that $x \neq y$ there exists a C-set of X containing x but not y or (resp. and) a C-set containing y but not x.

Definition 3.3. A topological space X is C-D_2 (resp. C-T_2 [3]) if for x, $y \in X$ such that $x \neq y$ there exist disjoint cD-sets (resp. C-sets) S_1 and S_2 such that $x \in S_1$ and $y \in S_1$.

Remark 3.2. The following implications hold:

a) If X is T_i, then X is C-T_i, for i = 0, 1, 2.
b) If X is C-T_i, then X is C-D_i, for i = 0, 1, 2.
c) If X is C-D_i, then X is C-D_{i-1}, for i = 1, 2.
d) If X is C-T_i, then X is C-T_{i-1}, for i = 1, 2.

Theorem 3.1. A topological space X is C-D_0 if and only if it is C-T_0.

Proof. The sufficiency is Remark 3.2 (b).

Necessity: Let X be C-D_0. Then for each pair of distinct points x, y $\in$ X, at least one of x, y, say x, belongs to a cD-set S but y $\notin$ S. Let S $\in$ $O_1 \backslash O_2$, where $O_1 \neq X$ and O_1 and O_2 are C-sets. Then X $\in$ O_1 and for y $\notin$ S we have two cases:

(1) y $\notin$ O_1 ; (2) y $\in$ O_1 and y $\in$ O_2

In case (1): O_1 contains x but doesn't contain y.
In case (2): O_2 contains y but doesn't contain x. Thus X is C-T_0.

Theorem 3.2. If a topological space X is C-D_1, then it is C-T_0.

Proof. This follows from Remark 3.2 and Theorem 3.1.

Theorem 3.3. If f: X $\rightarrow$ Y is a semi-continuous (resp. C-continuous surjection and S is a D-set in Y, then f^{-1} (S) is a sD-set (resp. cD-Set) in X

Proof. We prove only the first case being the second similar. Let S be a D-set of Y. Then there are open sets O_1 and O_2 in Y such that $S = O_1 \backslash O_2$ and $O_1 \neq Y$. By the semi-continuity of f, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are semi-open in X. Since $O_1 \neq Y$ and f is surjective, we have $f^{-1}(O_1) =$ X. Hence $f^{-1}(S) = f^{-1}(O_1) \backslash f^{-1}(O_2)$ is a sD-set.

A space X is said to be *semi*-D_1 [1] if for any pair of distinct points x and y of X, there exist sD-sets U and V of X such that x $\in$ U, y $\notin$ U, x $\notin$ V and y $\in$ V.

Theorem 3.4. If y is a D_1-space and f : X $\rightarrow$ Y a is semi-continuous (resp. C-continuous) bijection, then X is a semi-D_1 (resp. C-D_1) space.

Proof. We prove only the first case being the second is analogous. Suppose that Y is a D_1-space. Let x and y be any pair of distinct points in X. Since f is injective and Y is D_1-space, there exist D-sets S_X and S_y

of Y containing f(x) and f(y), respectively, such that $f(y) \notin S_x$, $f(x) \notin S_y$. By Theorem 3.3, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are sD-sets in X containing x and y respectively, such that $y \notin f^{-1}(S_x)$ and $x \notin f^{-1}(S_y)$. This implies that X is a semi-D_1 space.

Definition 3.4. A function $f : X \to Y$ is called *C-irresolute* if for every C-set A in Y, its inverse image $f^{-1}(A)$ is C-set in X.

Theorem 3.5. If $f : X \to Y$ is a C-irresolute surjecton and S is a cD-set of Y, then $f^{-1}(S)$ is a cD-set of X.

Proof. Suppose that S is a cD-set of Y. Then there are C-sets O_1 and O_2 in Y such that $S = O_1 \backslash O_2$ and $O_1 \neq Y$. By the C-irresoluteness of f, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are C-sets in X. Since $O_1 \neq Y$, we have $f^{-1}(O_1) \neq X$. Hence $f^{-1}(S) = f^{-1}(O_1) \backslash f^{-1}(O_2)$ is a cD-set.

Theorem 3.6. A space X is C-D_1 if and only if for each pair of distinct points x and y of X, there exist a C-irresolute surjection f of X onto a C-D_1 space Y such that $f(x) \neq f(y)$.

Proof. Necessity: Take the identity function on X.

Sufficiency: Let x and y be any pair of distinct points in X. By hypothesis, there exists a C-irresolute surjection f of X onto a C-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist cD-sets S_x and S_y in Y such that $f(x) \in S_x$, $f(y) \notin S_x$; $f(y) \in S_y$, $f(x) \notin S_y$. Since f is C-irresolute and surjective, by Theorem 3.5, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are cD-sets in X such that $x \in f^{-1}(S_x)$, $y \notin f^{-1}(S_x)$; $y \in f^{-1}(S_y)$, $x \notin f^{-1}(S_y)$. Therefore, X is a C-D_1 space.

We can give the following notions:

Definition 3.5. A filterbase **B** is called cD-*convergent* (resp. D-*convergent*) to a point $x \in X$ if for any cD-set (resp. D-set) A containig x, here exists $B \in \mathbf{B}$ sùch that $B \subset A$.

Theorem 3.7. If function $f : X \to Y$ is C-continuous and surjective, then for each point $x \in X$ and each filterbase **B** on X cD-convengent to x, the filterbase f(**B**) is D-convergent to f(x).

Proof. Let $x \in X$ and **B** be any filterbase cD-convergent to x. Since f is a C-continuous surjection, by Theorem 3.3, for each D-set $V \subset Y$ containing f(x), $f^{-1}(V) \subset X$ is a cD-set containing x. Since **B** is cD-convergent o x, then there exists $B \in$ **B** such that $b \subset f^{-1}(V)$; hence $f(B) \subset V$. It follows that f(**B**) is d-convergent to f(x).

Corollary 3.1. If a function $f : X \to Y$ is C-irresolue and surjective, then for each point $x \in X$ and each filterbase **B** onX cD-convergent to x, filterbase f(**B**) is cD-convergent to f(x).

We can give the following notions:

Definition 3.6. A space X is called cD-*compact* (resp. D-*compact*) if every cover of X by cD-sets (resp. D-sets) has a finite subcover.

Theorem 3.8. Let a function $f : X \to Y$ be C-continuous and surjective. If X is cD-compact, then Y is D-compact.

Proof. Let γ be an cover of Y by D-sets. Since f is C-continuous and surjective, by Theorem 3.3, $f^{-1}(\gamma) = \{f^{-1}(V_1)|V \in \gamma\}$ is a cover of X by cD-sets. Since X is cD-compact, there exists a finite subcover $\{f^{-1}(V_1), ..., f^{-1}(V_n)\}$ of $f^{-1}(\gamma)$. Therefore, $\{V_1, ..., V_n\}$ is a finite subcover of g. Hence Y is D-compact.

Corollary 3.2. Let $f : X \to Y$ be a C-irresolute surjection. If X is cD-compact, then Y is cD-compact.

We can also give the following notion.

Definition 3.7. A space X is called cD-*connected* (resp. D-*connected*) if X can not be expressed as the union of two nonempty disjoint cD-sets (resp. D-sets).

Theorem 3.9. If $f : X \to Y$ is a C-continuous surjection and X is cD-connected, then Y is D-connected.

Proof. Straightforward.

Corollary 3.3. If $f : X \to Y$ is a C-irresolute surjection and X is cD-connected, then Y is cD-connected.

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