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ISOMETRIC DEFORMATION OF SURFACES IN KÄHLER PLANE PRESERVING THE MEAN CURVATURE FUNCTION

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ABSTRACT

In this paper, we study the condition that a surface in Kahler plane K^2 admits a nontrivial isometric deformation without umbilic points preserving the mean curvature function. Also, we show that the Gaussian curvature for such surface must be identically zero if it is constant.

1. INTRODUCTION

O. Bonnet [1] proved that a surface of constant mean curvature can be isometrically deformed preserving the mean curvature. S.S. Chern [2] has studied such a deformation for surfaces of non-constant mean curvature in E^3 and he gave an interesting criterion for its existence. W. Scherrer [6] and R.A. Tribuzy [7] have found another necessary and sufficient condition for the existence of such deformation.

Recently, A. Gervasio Colares and K. Kenmotsu [3] have classified surfaces with constant Gaussian curvature which admit nontrivial isometric deformations preserving the mean curvature function.

The purpose of this paper is to study the condition that makes the deformation of surfaces without umbilic points in Kahler plane K^2 preserving the mean curvature function.

In this section we shall summarize some results which are useful in this paper.

. The Kahler plane K^2 is defined as a four dimensional space E^4 together with the endomorphism $J : V(E^4) \rightarrow V(E^4)$, where $J^2 = -1$ such that

$\langle u, v \rangle = \langle Ju, Jv \rangle$ for all $u, v \in V(E^4)$

and

$J \circ \nabla(u) = \nabla \circ J(u)$,

where $V(E^4)$ is the vector space of E^4 and ∇ is the covariant differentiation on K^2 [4,5].

Let $M \subset K^2$ be an oriented surface. By the field of orthonormal moving frames of K^2, we mean the orthonormal moving frames $\{m, v_1, v_2, v_3, v_4\}$ of $E^4, m \in M$, such that

$J(v_1) = v_3, J(v_2) = v_4, v_1, v_2 \in T_m(M)$,

where $T_m(M)$ is the tangent space to M at m. For the dual frame $\{\omega^i, i = 1,2,3,4\}$ we have

$$\omega^3 = \omega^4 = 0, \tag{1}$$

$\nabla m = \omega^1 v_1 + \omega^2 v_2$,

and

$$\nabla v_1 = \omega_1^j v_j, \quad i, j = 1,2,3,4. \tag{2}$$

Since $J \circ \nabla v_1 = \nabla \circ J v_1$, then

$$\omega_1^2 = \omega_3^4, \quad \omega_1^4 = \omega_2^3. \tag{3}$$

The structure equations are:

$d\omega^1 = \omega^2 \wedge \omega_2^1, \quad d\omega^2 = \omega^1 \wedge \omega_1^2$

and

$$d\omega_1^j = \omega_1^k \wedge \omega_1^j, \quad i, j, k = 1,2,3,4. \tag{4}$$

By exterior differentiation of (1) and using Cartan's lemma and (3), we obtain

$\omega_1^3 = a_{11} \omega^1 + a_{12} \omega^2$,

$\omega_1^4 = a_{12} \omega^1 + a_{22} \omega^2$,

and

$$\omega_2^4 = a_{22} \omega^1 + a_{23} \omega^2, \tag{5}$$

where and $a: M \to R$ The Gaussian curvature K and the mean curvature H at each $m \in M$ are given by

$K = a_{11} a_{22} - a_{12}^2 + a_{12} + a_{23} - a_{22}^2$,

$$H = (a_{11} + a_{22}) + (a_{12} + a_{23}). \tag{6}$$

From structure equations (5) and (6), we have

$\nabla \omega_1^2 = -K \omega^1 \wedge \omega^2$.

For fixed v_1, v_2, v_3 and v_4 there exist some functions x , y such that

$$\omega_1^3 = (H+x)\omega^1 + y\omega^2,$$

$$\omega_2^3 = y\omega^1 + (H-x)\omega^2 = \omega_1^4,$$

and

$$\omega_2^4 = (H-x)\omega^1 + \frac{(H-x)^2}{y}\omega^2. \qquad (7)$$

where

$$y = \frac{-H + \sqrt{H^2 - 4(H-x)^2}}{2}.$$

Since the Gaussian curvature K is written by

$$K = (H-x)(H-x) - y^2 + y\frac{(H-x)^2}{y} - (Hx)^2,$$

we have that $H^2 - K = x^2 + y^2$, which is positive. Therefore from [3] we can write

$$\omega_2^3 = (\sqrt{H^2 - K}\cos\alpha\,)\omega^1 + (\sqrt{H^2 - K}\sin\alpha\,)\omega^2,$$

$$\omega_2^3 = (\sqrt{H^2 - K}\sin\alpha\,)\omega^1 + (H - \sqrt{H^2 - K}\cos\alpha\,)\omega^2,$$

and   (8)

$$\omega_2^4 = (H - \sqrt{H^2 - K}\cos\alpha\,)\,\omega^1 + \frac{(H - \sqrt{H^2 - K}\cos\alpha\,)^2}{\sqrt{H^2 - K}\sin\alpha)}\omega^2.$$

We note that α in (8) depends on the frame v_1, v_2, v_3 and v_4 Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ and $\tilde{v}_4$ be another orthonormal moving frames of K^2 such that $J(\tilde{v}_1) = \tilde{v}_3, J(\tilde{v}_2) = \tilde{v}_4$ and denote $(\tilde{v}_1 + i\tilde{v}_2) = \exp(i\theta)(v_1, iv_2)$. For any tensor field α of (0,1)-type, we define its corvariant derivatives $\alpha_{i,j}$ as follows

$$D\alpha_i = d\alpha_i + \sum_s \alpha_s \omega_1^s = \sum_s \alpha_{1,j}\omega^j , \quad 1 \le i \le 2$$

and

$$\Delta\alpha = \alpha_{1,2} + \alpha_{2,2}.$$

By exterior differentiation of (8) and using the relation

$$D\alpha = d\alpha + 2\omega_1^2 = \alpha_1\omega^1 + \alpha_2\omega^2,$$

we obtain

$$\sqrt{H^2 - K}\,D\alpha = (\sqrt{H^2 - K}\,)_2\,\omega^1 - (\sqrt{H^2 - K}\,)_1\,\omega^2 + \cos\alpha(H_1\omega^2 + H_2\omega^1)$$

$$- \sin\alpha(H_1\omega^1 - H_2\omega^2), \qquad (9)$$

where H_i and $(\sqrt{H^2-K})_i$, $(i=1,2)$ are exterior derivatives of the scalar functions H and $\sqrt{H^2-K}$, respectively. Let us consider the 1-forms

$$W_1 = \frac{H_1\omega^1 - H_2\omega^2}{\sqrt{H^2-K}},$$

$$W_2 = \frac{H_1\omega^2 + H_2\omega^1}{\sqrt{H^2-K}},$$

(10)

Using the *-operator [4] , the relation (9) becomes

$$D\alpha = -*\log\sqrt{H^2-K} + W_2\cos\alpha - W_1\sin\alpha.$$

(11)

Exterior derivative of (10) gives

$$dW_1 = \frac{1}{(\sqrt{H^2-K})^2}\left\{\sqrt{H^2-K}(dH_1\omega^1 - dH_2\omega^2) - (H_1\omega^1 - H_2\omega^2)\wedge d(\sqrt{H^2-K})\right\}.$$

and hence,

$$dW_1 = \frac{1}{(\sqrt{H^2-K}}\left\{(\log\sqrt{H^2-K})_1 H_2 + (\log\sqrt{H^2-K})_2 H_1\right.$$

$$\left. -2H_{12}]\omega^1\wedge\omega^2 - 2\sqrt{H^2-K}W_2\wedge\omega_1^2\right\}.$$

Similarly,

$$dW_2 = \frac{1}{(\sqrt{H^2-K}}\left\{[-(\log\sqrt{H^2-K})_1 H_1 + (\log\sqrt{H^2-K})_2 H_2\right.$$

$$\left. + H_{11} - H_{22}]\omega^1\wedge\omega^2 + 2\sqrt{H^2-K}W_1\wedge\omega_1^2\right\},$$

(12)

where $H_{i,j}$'s are the second covariant derivative of H. Using (12) and exterior differentiation of (11), we obtain

$$-2A\sin\alpha + B\cos\alpha + P = 0,$$

(13)

where

$$A = H_{12}\sqrt{H^2-K} - H_2(\sqrt{H^2-K})_1 - H_1(\sqrt{H^2-K})_2 ,$$

$$B = (H_{22} - H_{11})\sqrt{H^2-K} + H_1(\sqrt{H^2-K})_1 - 2H_2(\sqrt{H^2-K})_2 ,$$

(14)

$$P = (H^2-K)(\Delta\log\sqrt{H^2-K} - 2K) - |gradH|^2 ,$$

and the operator Δ is the Laplacian for the induced metric of M.

The formula (13) holds at non-umblilic points of any surface K^2. Applying the *- operator to (11), we have

$$\alpha_1\omega^2 - \alpha_2\omega^1 = -W_2 \sin\alpha - W_1 \cos\alpha + d\log\sqrt{H^2 - K}.$$

By exterior differentiation of the above equation and using (11),(12) we have,

$$(H^2 - K)\Delta\alpha = 2A \cos\alpha + B \sin\alpha. \qquad (15)$$

From the equations (13) and (15) we obtain

$$A = B = 0 \text{ iff } \Delta\alpha = P = 0. \qquad (16)$$

THE MAIN RESULTS:

Theorem 1. Let $M \subset K^2$ be a surface without umbilic points. Then, M admits a non-trivial isometric deformation preserving the mean curvature function iff one of the following conditions holds

$$\nabla\left(\frac{\nabla H}{H^2 - K}\right)\langle Z, Z\rangle = 0, \qquad (17)$$

or

$$(H^2 - K)(\Delta\log\sqrt{H^2 - K} - 2K) - |\operatorname{grad} H|^2 = 0,$$

and $\qquad (18)$

$$\Delta\alpha = 0,$$

where,

$$Z = \frac{1}{2}(v_1 - iv_2).$$

Proof. If M admits a non-trivial isometric deformation preserving the mean curvature function, then (10) is completely integrable, and hence the equations (12) and (14) hold for α's. By differentiation of (12) twice with respect to the direction of the deformation, we obtain

$$2A\sin\alpha - B\cos\alpha = 0.$$

Comparing the last equation with the equations (12-14) we obtain $P = \Delta\alpha = 0$ on M and hence $A = B = 0$. From the equation (17) we have

$$4(H^2 K)^{\frac{3}{2}} \nabla\left(\frac{\nabla H}{H^2 - K}\right)\langle Z, Z\rangle = -(B + 2iA) = 0.$$

Conversely, if

$$\nabla\left(\frac{\nabla H}{H^2 - K}\right)\langle Z, Z\rangle = 0,$$

then $A = B = 0$ and hence $P = \Delta\alpha = 0$ implies that the condition (15) holds for M, then (10) is completely integrable, i.e., M admits isometric deformation preserving the mean curvature function. Then the proof is complete.

Theorem 2. Let $M \subset K^2$ be a surface without umbilic points such that the Gaussian curvature K is constant on M. If M admits isometric deformation preserving the mean curvature function, then K must be zero.

Proof. If M is a minimal surface, then H is zero. In case of $H \neq 0$ we consider a tensor field of (0,1)-type defined by $f_i = H_i\Big/_{H^2 - K}$, $i = 1, 2$. Since K is constant, we have

$$f_{i,j} = \frac{\left\{(H^2 - K)H_{i,j} - 2HH_iH_j\right\}}{(H^2 - K)^2}.$$

Using the conditions $A = B = 0$, there exists some scalar function λ with $f_{i,j} = \lambda\delta_{i,j}$.

By taking the trace of these equations, we obtain

$$2\lambda = \sum_i f_{i,j} = \frac{(H^2 - K)\Delta H - 2H|\text{grad}H|^2}{(H^2 - K)^2}$$

hence

$$(H^2 - K)\Delta H - 2H|\text{grad}H|^2 = 2\lambda(H^2 - K)^2.$$

Putting $\lambda = \dfrac{K}{H}$, we have

$$(H^2 - K)\Delta H - 2H|\text{grad}H|^2 = \frac{2K(H^2 - K)}{H},$$

and

$$Hf_{i,j} = K\delta_{i,j}. \tag{19}$$

By exterior differentiation of (19) and making K constant on M, we have

$$H_K f_{i,j} - H_j f_{i,j} + H(f_{i,j,k} - f_{i,k,j}) = 0. \tag{20}$$

From the relations

$$f_{1,2,1} - f_{1,1,2} = Kf_2, f_{2,1,2} - f_{2,2,1} = Kf_1,$$

and using (19),(20), we get $KH_1 = 0$, $i = 1,2$. If $H_i = 0$, then H is constant and $f_{i,j}$ vanish. Then K must be zero.

REFERENCES

[1] O. Bonnet, Memoire sur la theorie des surfaces applicables, J.E.C. Polyt 42(1967),72-29.

[2] S. S. Chern, Deformations of surfaces preserving principal curvature, Diff. Geom. And Complex Analysis, H.E. Rauch Memorial Volume, Springer (1985),155-163.

[3] A. Gervasio and K. Kenmotsu, Isometric deformation of surfaces in $\mathbf{R}^3$ preserving the mean curvature function, Pacific Journal of Mathematics, Vol. 136,No.1 (1989).

[4] H. Kamal, On Riemannian and Complex Submanifolds, Ph.D. Tanta University. Egypt. (1989).

[5] A. Svec, Contribution to the global differential geometry of surfaces, Academia Nakladatelstvi Ceskoslovenske Akademic ved, Prha (1977).

[6] W. Scherrer, Die Grundgleichungen der Flachentheorie II, Comm. Helv. 32 (1975), 73-84.

[7] R. A. Tribuzy, A charaterization of tori with constant mean curvature in space form, Bol. Soc. Bras. Mat., 11 (1980), 259-274.