Commun. Fac. Sci. Univ. Ank. Series A1
V. 50 pp. 55-63 (2001)

A SUBSET OF THE SPACE OF THE ENTIRE SEQUENCES

K.C. RAO[1], N. SUBRAMANIAN[2]

[1]*Dept. of Mathematics, Mohamed Sathak Engineering College, Kilakarai, India.*
[2]*Dept. of Mathematics, Shanmugha Arts, Science, Techn. and Research Acad., Tanjore, India.*

(Received June 27, 2001; Accepted Nov. 08, 2001)

ABSTRACT

Let Γ denote the space of all entire sequences. Let $\wedge$ denote the space of all analytic sequences. This paper is devoted to a study of the general properties of Sectional space Γ_s of Γ.

KEYWORDS

Sectional sequence spaces, entire sequences, analytic sequences.

AMS SUBJECT CLASSIFICATION : 46A45.

1. INTRODUCTION

A complex sequence whose kth term is x_k will be denoted by (x_k) or x. A sequence $x = (x_k)$ is said to be analytic if $\sup_{(k)} |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\wedge$. A sequence x is called entire sequence $\lim_{k \to \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be de denoted by Γ.

Let $\Gamma_s = \{x = (x_k) : \xi = (\xi_k) \in \Gamma\}$.
where $\xi_k = x_1 + x_2 + ... + x_k$ for $k = 1,2,3,...$.
and $\wedge_s = \{y = (y_k) : \eta = (\eta_k) \in \wedge\}$.
where $\eta_k = y_1 + y_2 + ... + y_k$ for $k = 1,2,3,...$.
Then Γ_s and $\wedge_s$ are metric spaces with the metric
$d(x, y) = \sup_{(k)} \left\{ |\xi_k - \eta_k|^{1/k} : k = 1,2,3,... \right\}$.
Let $\sigma(\Gamma)$ denote the vector space of all sequences $x = (x_k)$ such that

$\{\xi_k / k\}$ is an entire sequence. We recall that cs_0 denotes the vector space of all sequences $x = (x_k)$ such that
$\{\xi_k\}$ is a null sequence. Let $\Phi = \{\text{all finite sequences}\}$.
$\delta^{(n)} = (0,0,\ldots,1,0,0,\ldots)$, 1 in the n[th] place and zero's elsewhere. An FK-space X is said to have AK-property if $(\delta^{(n)})$ is a Schauder basis for X.
If X is a sequence space, we define the β-dual X^β of X by

$$X^\beta = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for every } x \in X \right\}.$$

Remark :

$$x = (x_k) \in \sigma(\Gamma) \Leftrightarrow \left\{ \frac{x_1 + x_2 + \ldots + x_k}{k} \right\} \in \Gamma.$$

$$\Leftrightarrow \left| \frac{|x_1 + x_2 + \ldots + x_k|}{k} \right|^{1/k} \to 0 \text{ as } k \to \infty.$$

$$\Leftrightarrow |x_1 + x_2 + \ldots + x_k|^{1/k} \to 0 \text{ as } k \to \infty, \text{ because } k^{1/k} \to 1 \text{ as } k \to \infty$$

$$\Leftrightarrow (x_k) \in \Gamma_s.$$

Hence $\Gamma_s = \sigma(\Gamma)$, the Cesáro space of order 1.
In this paper we investigate

(i) set-inclusion between Γ_s and Γ,
(ii) AK-property possessed by Γ_s,
(iii) Solidity of Γ_s as a linear space,
(iv) β-dual of Γ_s,
(v) Relation between Γ_s and $\Gamma \cap cs_0$,
(vi) the Cesáro space of order $\alpha > 0$ of the entire sequences.

Proposition 1. $\Gamma_s \subset \Gamma$.

Proof.
Let $x \in \Gamma_s$.

$$\Rightarrow \xi \in \Gamma$$

$$\Rightarrow |\xi_k|^{1/k} \to 0 \quad \text{as} \quad k \to \infty.$$

(1.1)

But $x_k = \xi_k - \xi_{k-1}$.

Hence $|x_k|^{1/k} \le |\xi_k|^{1/k} + |\xi_{k-1}|^{1/k}$

$\le |\xi_k|^{1/k} + |\xi_{k-1}|^{1/k-1}$

$\to 0$ as $k \to \infty$ by using (1.1)

$\Rightarrow x \in \Gamma.$

$\Rightarrow \Gamma_s \subset \Gamma.$

Note : The above inclusion is strict.

Take the sequence $\delta^{(1)} \in \Gamma$. We have

$\xi_1 = 1$

$\xi_2 = 1 + 0 = 1$

$\xi_3 = 1 + 0 + 0 = 1$

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$\xi_k = 1 + 0 + 0 + ... + = 1$

$\to k -$ terms $\leftarrow$

and so on.

Now $|\xi_k|^{1/k} = 1$ for all k. Hence $\{|\xi_k|^{1/k}\}$ does not tend to zero as $k \to \infty$. So $\delta^{(1)} \notin \Gamma_s$. Thus the inclusion $\Gamma_s \subset \Gamma$ is strict. This completes the proof.

Proposition 2. Γ_s has AK-property.

Proof.

Let $x = (x_k) \in \Gamma_s$ and take $x^{[n]} = (x_1, x_2, ..., x_n, 0, 0, ...)$, for $n = 1,2,3,...$ Hence

$$d(x, x^{[n]}) = \sup_{(k)} \left\{ \left| \xi_k - \xi_k^{(n)} \right|^{1/k} \right\}.$$

$$= \sup\left\{ |\xi_{n+1} - \xi_n|^{\frac{1}{n}+1}, |\xi_{n+2} - \xi_n|^{\frac{1}{n}+2} ... \right\} \to 0 \text{ as } n \to \infty.$$

Therefore, $x^{[n]} \to x$ as $n \to \infty$ in Γ_s. Thus Γ_s has AK. This completes the proof.

Corollary. The set $\{\delta^{(1)}, \delta^{(2)}, ...\}$ is a Schauder basis for Γ_s.

Proposition 3. Γ_s is a linear space over field C of complex numbers.

Proof.

Let $x = (x_k)$ and $y = (y_k)$ belong to Γ_s. Let $\alpha, \beta \in C$. Then $\xi = (\xi_k) \in \Gamma$ and $\eta = (\eta_k) \in \Gamma$. But Γ is a linear space. Hence $\alpha\xi + \beta\eta \in \Gamma$. Consequently $\alpha x + \beta y \in \Gamma_s$. Therefore Γ_s is linear. This completes the proof.

Proposition 4. Γ_s is solid.

Proof.

Let $|x_k| \le |y_k|$ with $y = (y_k) \in \Gamma_s$. So, $|\xi_k| \le |\eta_k|$ with $\eta = (\eta_k) \in \Gamma$. But Γ is solid. Hence $\xi = (\xi_k) \in \Gamma$. Therefore $x = (x_k) \in \Gamma_s$. Hence Γ_s is solid. This completes the proof.

Proposition 5. In Γ_s weak convergence does not imply strong convergence.

Proof.

Assume that weak convergence implies strong convergence in Γ_s. Then we would have $(\Gamma_s)^{\beta\beta} = \Gamma_s$. (see [8]) But

$$(\Gamma_s)^{\beta\beta} = \wedge^{\beta} = \Gamma.$$

By Proposition 1, Γ_s is a proper subspace of Γ. Thus $(\Gamma_s)^{\beta\beta} \ne \Gamma_s$. Hence weak convergence does not imply strong convergence in Γ_s. This completes the proof.

Proposition 6. $\wedge \subset (\Gamma_s)^{\beta} \subset \wedge(\Delta)$.

Proof.

Step 1.

By Proposition 1., we have

$$\Gamma_s \subset \Gamma.$$

Hence

$$\Gamma^{\beta} \subset (\Gamma_s)^{\beta}$$

But

$$\Gamma^{\beta} = \wedge.$$

Therefore

$$\wedge \subset (\Gamma_s)^{\beta}. \tag{6.1}$$

Step 2.

Let $y = (y_k) \in (\Gamma_s)^{\beta}$. Consider

$$f(x) = \sum_{k=1}^{\infty} x_k y_k.$$

where $x = (x_k) \in \Gamma_s$. Take

$$x = \delta^{n} - \delta^{n+1}$$
$$= (0,0,0,\ldots,1, \quad -1,0,0,\ldots)$$
$$n^{\text{th}} \quad (n+1)^{\text{th}} \quad \text{place}$$

where, for each fixed $n = 1,2,3,\ldots$.

$\delta^{(n)} = (0,0,\ldots,1,0,\ldots)$, 1 in the n$^{\text{th}}$ place and zero's elsewhere. Then

$$f(\delta^{n} - \delta^{n+1}) = y_n - y_{n+1}.$$

Hence

$$\begin{aligned}|y_n - y_{n+1}| &= |f(\delta^n - \delta^{n+1})| \\ &\leq \|f\| d(\delta^n - \delta^{n+1}, 0) \\ &\leq \|f\|.1\end{aligned}$$

So, $\{y_n - y_{n+1}\}$ is bounded.

Consequently $\{y_n - y_{n+1}\} \in \wedge$. That is $\{y_n\} \in \wedge(\Delta)$. But $y = (y_n)$ is originally in $(\Gamma_s)^\beta$. Therefore

$$(\Gamma_s)^\beta \subset \wedge(\Delta). \tag{6.2}$$

From (5.1) and (5.2) we conclude that

$$\wedge \subset (\Gamma_s)^\beta \subset \wedge(\Delta).$$

This completes the proof.

Proposition 7. The β-dual space of Γ_s is $\wedge$.

Proof.

Step 1.

Let $y = (y_k)$ be an arbitrary point in $(\Gamma_s)^\beta$. If y is not in $\wedge$, then for each natural number n, we can find an index $k(n)$ such that

$$|y_{k(n)}|^{1/k(n)} > n, \ (n = 1,2,...).$$

Define $x = (x_k)$ by

$$x = 1/n^k \text{ for } k = k(n); \text{ and}$$
$$x = 0 \text{ otherwise.}$$

Then x is in Γ, but for infinitely many k,

$$|y_k x_k| > 1. \tag{7.1}$$

Consider the sequence $z = \{z_k\}$, where

$z_1 = x_1 - s$ with $s = \sum x_k$; and $z_k = x_k (k = 2,3,...)$.

Then z is a point of Γ. Also $\sum z_k = 0$. Hence z is in Γ_s. But, by the equation (7.1) $\sum z_k y_k$ does not converge. Thus the sequence y would not to be in $(\Gamma_s)^\beta$. This contradiction proves that

$$(\Gamma_s)^\beta \subset \wedge. \tag{7.2}$$

Step 2.

By (6.1) of Proposition 6 we have

$$\wedge \subset (\Gamma_s)^\beta. \tag{7.3}$$

From (6.2) and (6.3) it follows that the β-dual space of $(\Gamma_s)^\beta$ is $\wedge$. This completes the proof.

Proposition 8. $(\Gamma_s)^\mu = \wedge$ for $\mu = \alpha, \beta, \gamma, f$.

Step 1

Γ_s has AK by Proposition 2. Hence by Theorem 7.3.9 in [1] we get $(\Gamma_s)^\beta = (\Gamma_s)^f$. But $(\Gamma_s)^\beta = \wedge$. Hence

$$(\Gamma_s)^f = \wedge. \quad \text{(I)}$$

Step 2

Since AK $\Rightarrow$ AD. Hence by Theorem 7.3.9 in [1] we get $(\Gamma_s)^\beta = (\Gamma_s)^\gamma$. Therefore

$$(\Gamma_s)^\gamma = \wedge. \quad \text{(II)}$$

Step 3

Γ_s has normal by Proposition 4. Hence by Theorem in [1] we get

$$(\Gamma_s)^\alpha = (\Gamma_s)^\gamma = \wedge. \quad \text{(III)}$$

From (I), (II) and (III) we have

$$(\Gamma_s)^\alpha = (\Gamma_s)^\beta = (\Gamma_s)^\gamma = (\Gamma_s)^f = \wedge.$$

Proposition 9. Let Y be any FK-space $\supset \Phi$. Then $Y \supset \Gamma_s$ if and only if the sequence $\{\delta^{(k)}\}$ is weakly analytic.

Proof.

The following implications establish the result.

$Y \supset \Gamma_s \Leftrightarrow Y^f \subset (\Gamma_s)^f$, since Γ_s has AD and by using 8.6.1 in [1].

$\Leftrightarrow Y^f \subset \wedge$, since $(\Gamma_s)^f = \wedge$.

$\Leftrightarrow$ for each $f \in Y'$, the topological dual of $Y.f(\delta^{(k)}) \in \wedge$.

$\Leftrightarrow f(\delta^{(k)})$ is analytic.

$\Leftrightarrow$ The sequence $\{\delta^{(k)}\}$ is weakly analytic.

This completes the proof.

Proposition 10. $\Gamma_s = \Gamma \cap cs_0$.

Proof.

By Proposition 1, $\Gamma_s \subset \Gamma$. Also, since every entire sequence (ξ_k) is a null sequence, it follows that (ξ_k) is a null sequence. In other words $(\xi_k) \in cs_0$.

Thus $\Gamma_s \subset cs_0$. Consequently,

$$\Gamma_s \subset \Gamma \cap cs_0. \quad (10.1)$$

On the other hand, if $(x_k) \in \Gamma \cap cs_0$, then

$$f(z)=\sum_{k=1}^{\infty} x_k z^{k-1}$$

is an entire function. But $(x_k)\in cs_0$. So,

$$f(1)=x_1+x_2+...=0.$$

Hence

$$\frac{f(z)}{1-z}=\sum_{k=1}^{\infty} (\xi_k)z^{k-1}$$

is also an entire function. Hence $(\xi_k)\in\Gamma$. So $x=(x_k)\in\Gamma_s$. But (x_k) is arbitrary in $\Gamma\bigcap cs_0$. Therefore

$$\Gamma\bigcap cs_0\subset\Gamma_s. \qquad (10.2)$$

From (10.1) and (10.2) we get

$$\Gamma_s=\Gamma\bigcap cs_0.$$

This completes the proof.

Definition. Let $\alpha>0$ be not an integer. Write $S_n^{(\alpha)}=\sum_{\gamma=1}^{n} A_{n-\gamma}{}^{(\alpha-1)}x_\gamma$, where $A_\mu^{(\alpha)}$ denotes the binomial coefficient

$$\frac{(\mu+\alpha)(\mu+\alpha-1)...(\alpha+1)}{\mu!}.$$

Then $(x_n)\in\sigma^\alpha(\Gamma)$ means that $\left\{\frac{S_n^{(\alpha)}}{A_n^{(\alpha-1)}}\right\}\in\Gamma$.

Proposition 11. Let $\alpha>0$ be a number which is not an integer. Then

$$\Gamma\bigcap\sigma^\alpha(\Gamma)=\theta$$

where θ denotes the sequence (0,0,...,0).

Proof.

Since $(x_n)\in\sigma^\alpha(\Gamma)$ we have

$$\left\{\frac{S_n^{(\alpha)}}{A_n^{(\alpha-1)}}\right\}\in\Gamma.$$

This is equivalent to $(S_n^{(\alpha)})\in\Gamma$. This, in turn, is equivalent to the assertion that

$$f_\alpha(z)=\sum_{n=1}^{\infty} S_n^{(\alpha)}z^{n-1}$$

is an integral (entire) function. Now

$$f_\alpha(z)=\frac{f(z)}{(1-z)^\alpha}.$$

Since α is not an integer, $f(z)$ and $f_\alpha(z)$ cannot both be integral functions, for if one is an integral function, the other has a branch at $z=1$. Hence the assertion holds good. So, the sequence 0=(0,0,...,0) belongs to both Γ and $\sigma^\alpha(\Gamma)$. But this is the only sequence common to both these spaces. Hence

$$\Gamma \cap \sigma^\alpha(\Gamma) = \theta.$$

This completes the proof.

Definition. Fix $k = 0,1,2,\dots$. Given a sequence (x_k), put

$$\xi_{k,p} = \frac{x_{1+k} + x_{2+k} + \dots + x_{p+k}}{p}$$

for $p = 1,2,3,\dots$. Let $(\xi_{k,p} : p = 1,2,3,\dots) \in \Gamma$ uniformly in $k = 0,1,2,\dots$.

Then we call (x_k) an "almost entire sequence". The set of all almost entire sequences is denoted by Δ.

Proposition 12. $\Gamma \cap \sigma^\alpha(\Gamma) = \Delta$. where Δ is the set of all almost entire sequences.

Proof.

Put $k=0$. Then $(\xi_{0,p}) \in \Gamma \Leftrightarrow \left(\frac{x_1 + x_2 + \dots + x_p}{p}\right) \in \Gamma$

$$\Leftrightarrow \left|x_1 + x_2 + \dots x_p\right|^{1/p} \to 0 \quad \text{as} \quad p \to \infty.$$

$$\Leftrightarrow x_1 + x_2 + \dots = 0. \tag{12.1}$$

$$\Leftrightarrow (x_k) \in cs_0.$$

$$\Leftrightarrow \Delta \subset cs_0.$$

Put $k=1$. Then

$$(\xi_{1,p}) \in \Gamma \Leftrightarrow \left(\frac{x_2 + \dots + x_p}{p}\right) \in \Gamma.$$

$$\Leftrightarrow \left|x_2 + x_3 + \dots + x_p\right|^{1/p} \to 0 \quad \text{as} \quad p \to \infty. \tag{12.2}$$

Similarly we get

$$x_3 + x_4 + \dots = 0. \tag{12.3}$$

$$x_4 + x_5 + \dots = 0. \tag{12.4}$$

$$\dots \qquad \dots \qquad \dots$$

and so on.

From (12.1) and (12.2) it follows that

$x_1 = (x_1 + x_2 + \dots) - (x_2 + x_3 + \dots) = 0.$

Similarly we obtain $x_2 = 0$, $x_3 = 0, \dots$ and so on. Hence $\Delta = \theta$ where θ denotes the sequence (0,0,...). Thus we have proved that

$$\Gamma \bigcap \sigma^{\alpha}(\Gamma) = \theta \quad \text{and} \quad \Delta = \theta.$$

In other words, $\Gamma \bigcap \sigma^{\alpha}(\Gamma) = \Delta$. This completes the proof.

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