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ON H-CONTINUITY OF MULTIFUNCTIONS DEFINED FROM A PRODUCT SPACE TO A PRODUCT SPACE

M. AKDAĞ

Cumhuriyet University, Faculty of Arts and Science, Department of Mathematics, 58140 Sivas-TURKEY
E-mail: makdag@cumhuriyet.edu.tr

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ABSTRACT

The purpose of the present paper is to investigate some properties of H-continuous (Almost H-continuous) multifunctions defined from a product space to a product space. The relations between the strongly closed graph of a multivalued functions and H-upper semicontinuity of multivalued functions are also investigated.

1. INTRODUCTION

The H-continuity of single valued function has been studied by P. E. Long and T. R. Hamlett [1,1975].The study for multivalued functions of H-continuity has been taken over by V. Popa [2] and R. E. Smithson [3]. Moreover, the H-almost upper semi continuity of single valued functions has been extended to multivalued functions by Y. Küçük and M. Akdağ [4].

In this paper, I studied H-almost upper semicontinuity and H-upper semi continuity and C-upper semi continuity of multivalued functions defined from a product space to a product space. Also, I obtained some relations between strongly closed graph and H-upper semicontinuity of multivalued functions.

2. PRELIMINARIES

A multivalued function $F:X\ \ Y$ is a function $F:X\ \ P(Y)\backslash\{\emptyset\}$ where $P(Y)$ is the power set of Y. For a multivalued function F, the upper and the lower inverse of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$, respectively where $F^+(B)=\{\ x\in X \mid F(x)\subset B\ \}$ and $F^-(B)=\{\ x\in X \mid F(x)\ \ B\ \ \ \emptyset\ \}$ [5].

The graph $G(F)$ of a multivalued function $F:X\ \ Y$ is the subset $\{(x,y) \mid x\in X, y\in F(x)\}$ of $X\times Y$. A multivalued function $F:X\ \ Y$ has a closed graph (strongly closed graph) if and only if for each $(x,y) \in X \times Y \setminus G(F)$, there exist open sets U and V containing x and y, respectively such that $(U \times V)\ \ G(F)=\emptyset$ $((U \times \overline{V})\ \ G(F)=\emptyset)$ [6].

A subset A of a space X is a quasi H-closed set (or H-set) if for every open cover $\vartheta=\{U_\alpha \mid \alpha \in \Delta\}$ of A there exist a finite subcover $\{U_1, U_2, \ldots, U_n\}$ of ϑ such that $A \subset \bigcup_{i=1}^{n} cl(U_i)$ [7]. If X is an H-set, then X is H-closed [7]. A space X is H-closed if it is Hausdorff and H-set [7]. A Hausdorff space X is locally H-closed if for each $x \in X$, there exists a H-closed neighbourhood of X [8]. A space X is C-compact if every closed subset of X is an H-set [9]. A space X is HC-space if every H-set of X is a closed set [5].

Let X and Y be two topological spaces and F be a multivalued function from X to Y. Then F is said to be H-upper semicontinuous (H-almost upper semicontinuous) at $x \in X$ if for any H-set V with $F(x)\ \ V=\emptyset$, there exists a neighbourhood U of x such that for $x \in U$, $F(x)\ \ U=\emptyset$, $(F(x)\ \ \overset{o}{\overline{U}}=\emptyset)$. If for every point x in X, F is H-upper semicontinuous (H-almost upper semicontinuous), then F is H-upper semicontinuous (H-almost upper semicontinuous) at X [5].

A multifunction F is said to be C-upper semicontinuous at $x \in X$ if for any compact set V with $F(x)\ \ V=\emptyset$, there exists a neighbourhood U of x such that for $x_o \in U$, $F(x_o)\ \ V=\emptyset$ [10]

A multifunction F is said to be point closed (point compact) if for each $x \in X$, F(x) is closed (compact).

3. PRODUCT SPACES

Let (X_α, τ_α) and $(Y_\alpha, \hbar_\alpha)$ be topological spaces, $(\Pi X_\alpha, \tau)$ be product space and also let us denote that $F(x)=\{F_\alpha(x_\alpha)\}$ such that $x=\{x_\alpha\}_{\alpha \in \Delta}$, $F_\alpha : X_\alpha \ \ Y_\alpha$ and $F : \Pi X_\alpha \ \ \Pi Y_\alpha$.

Lemma 3.1. Let $f:(X,\tau) \to (Y, \hbar)$ a continuous function. If A is a H-set of X, then f(A) is H-set of Y.

Proof: Let A be a H-set in X and let $\circ=\{U_\alpha \mid \alpha \in \Delta\}$ be an open cover of f(A).Then $f(A) \subset \bigcup_{\alpha \in \Delta} U_\alpha$ and $A \subset f^{-1}(f(A)) \subset f^{-1}(\bigcup_{\alpha \in \Delta} U_\alpha) = \bigcup_{\alpha \in \Delta} f^{-1}(U_\alpha)$. Since f is continuous, $f^{-1}(U_\alpha)$ is an open set in X for each $\alpha \in \Delta$. Thus $\circ\ =\{f^{-1}(U_\alpha) \mid \alpha \in \Delta\}$ is an open cover of A. Since A is an H-set, there exists a finite cover $\{U_1, U_2, \ldots, U_n\}$ of A such that

$A \subset \bigcup_{i=1}^{n} \overline{f^{-1}(U_i)}$. Therefore, since f is continuous,

$$f(A) \subset f(\bigcup_{i=1}^{n} \overline{f^{-1}(U_i)}) \subset \bigcup_{i=1}^{n} f(\overline{f^{-1}(U_i)}) \subset \bigcup_{i=1}^{n} \overline{f(f^{-1}(U_i))} \subset \bigcup_{i=1}^{n} \overline{U}_i.$$

Thus f(A) is an H-set in Y.

Lemma 3.2. If f:X Y is continuous, onto, open and X is locally H-closed Hausdorff space, then Y is locally H-closed.

Proof: Let $y \in Y$. Then there is a $x \in \{f^{1}(y)\} \subset X$. Since X is locally H-closed, there exists a quasi H-closed neighbourhood K of x. Hence there is an open set U in X such that $x \in U \subset K$. Since f is continuous and open, f(K) is a quasi H-closed neighbourhood of y from Lemma 3. 1. Thus Y is locally H-closed.

Lemma 3.3. If $\prod X_\alpha$ is locally H-closed, then X_α is locally H-closed for each $\alpha \in \Delta$.

Proof: Since the α-th projection function P_α: $\prod X_\alpha$ X_α is continuous, open and from Lemma 3. 2., the proof is clear.

Lemma 3. 4. For each $\alpha \in \Delta$, multifunctions $F_\alpha : X_\alpha \; Y_\alpha$ is point closed if and only if multifunction $F : \Pi X_\alpha \; \Pi Y_\alpha$ is point closed where $F(x)=\{F_\alpha(x_\alpha)\}$ for $x=\{x_\alpha\}$.

Proof: Let $x \in \Pi X_\alpha$ with $x=\{x_\alpha\}$, $\alpha \in \Delta$. Then $F(x)= \{F_\alpha(x_\alpha)\} = \overline{\{F_\alpha(x_\alpha)\}}$ $=\prod \overline{\{F_\alpha(x_\alpha)\}} = \prod \overline{F_\alpha(x_\alpha)} = \overline{\prod F_\alpha(x_\alpha)} = \overline{F(x)}$

Lemma 3.5. If Y_α is H-closed, then $\prod Y_\alpha$ is H-closed for each $\alpha \in \Delta$ [7].

Lemma 3. 6. Let $\{X_\alpha | \alpha \in \Delta\}$ and $\{Y_\alpha | \alpha \in \Delta\}$ be two families of topological spaces. F_α: X_α Y_α be a multifunction for each $\alpha \in \Delta$. If F_α is strongly closed for each $\alpha \in \Delta$, then G(F) which is the graph of F is strongly closed .

Proof: Let $(x,y) \in G(F)$. Then there is a $\beta \in \Delta$ such that $y_\beta \notin F_\beta(x_\beta)$. Since $G(F_\beta)$ is strongly closed, there exist two open sets U_β and V_β, respectively in X_β and in Y_β such that $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$ and $F_\beta(U_\beta) \;\; \overline{V_\beta} = \emptyset$. If we take $U=U_\beta \times \prod_{\alpha \neq \beta} X_\alpha$ and $V=V_\beta \times \prod_{\alpha \neq \beta} Y_\alpha$, then U and V are open sets in ΠX_α and ΠY_α, respectively and $x \in U$, $y \in V$ and $F(U) \;\; \overline{V} = \emptyset$. Indeed:

$$F(U) \;\; V=(F_\beta(U_\beta)) \times \prod_{\alpha \neq \beta} Y_\alpha) \;\; (\overline{V_\beta} \times \prod_{\alpha \neq \beta} Y_\alpha)=(F_\beta(U_\beta) \;\; \overline{V_\beta}) \times \prod_{\alpha \neq \beta} Y_\alpha = \emptyset.$$

Thus G(F) is strongly closed.

Lemma 3. 7. If for each $\alpha\in\Delta$, X_α is HC-closed, then $\prod X_\alpha$ is HC-space.

Proof: Let $G=\prod G_\alpha \subset \prod X_\alpha$ be a quasi H-closed set. From Lemma 3.1. and since α-th projection P_α is continuous for each $\alpha\in\Delta$, $P_\alpha(G)=G_\alpha\subset X_\alpha$ is a quasi H-closed set. Since X_α is HC-space, G_α is closed in X_α that is $G_\alpha=G_\alpha$. Thus $G=\prod\limits_{\alpha\in\Delta} G_\alpha=\prod\limits_{\alpha\in\Delta}\overline{G_\alpha}=\overline{\prod\limits_{\alpha\in\Delta} G_\alpha}=\overline{G}$ and $\prod X_\alpha$ is HC-space.

Theorem 3. 8. Let $\{X_\alpha|\ \alpha\in\Delta\}$ and $\{Y_\alpha|\ \alpha\in\Delta\}$ be two families of topological spaces. If for each $\alpha\in\Delta$, F_α:X_α Y_α is H-u.s.c.(C-u.s.c.), point compact (point closed) and Y_α is locally H-closed Hausdorff space (locally compact Hausdorff) then multifunction F: $\prod X_\alpha$ $\prod Y_\alpha$ is H-u.s.c.(C-u.s.c.).

Proof: Since for each $\alpha\in\Delta$, F_α is H-u.s.c.(C-u.s.c.) from the Propositions 4.11 and 4.12 in [11] and the Proposition 14 in [10], $G(F_\alpha)$ is strongly closed .From Lemma 3. 4., G(F) is strongly closed. Thus F is H-u.s.c.(C-u.s.c.)

Theorem 3. 9. If multifunction F: $\prod X_\alpha$ $\prod Y_\alpha$ is H-almost u.s.c., then for each $\alpha\in\Delta$, multifunctions F_α: X_α Y_α is H-almost u.s.c..

Proof: Let $x_\beta\in X_\beta$ for any $\beta\in\Delta$.and let V_β be a set such that $F_\beta(x_\beta)\subset V_\beta\subset Y_\beta$ and its complement is quasi H-closed. Also we define a set A_β with $A_\beta=\{x\in\prod X_\alpha \mid$ the β-th coordinate of x is $x_\beta\}$ for $x\in A_\beta$ and since $A_\beta\subset\prod X_\alpha$, $x\in\prod X_\alpha$. On the other hand, using the β-th projection P_β: $\prod Y_\alpha$ Y_β we obtain $P^{-1}{}_\beta(V_\beta)=V_\beta\times\prod\limits_{\alpha\neq\beta} Y_\alpha$. Since F is H-almost u.s.c., there exists an open set $U\subset\prod X_\alpha$ such that $F(U)\subset \overset{o}{\overline{V_\beta\times\prod\limits_{\alpha\neq\beta} Y_\alpha}}\subset\overset{o}{\overline{V_\beta}}\times\prod\limits_{\alpha\neq\beta} Y_\alpha$. Since P_β is open, $P_\beta(U_\beta)=U_\beta\subset X_\beta$ is an open set in X_β. Thus $F_\beta(U_\beta)\subset\overset{o}{\overline{V_\beta}}$ and F_β is H-almost u.s.c. at $x_\beta\in X_\beta$. Since x_β is an arbitrary point, F_β is H-almost u.s.c. on X_β.

Theorem 3. 10. Let for each $\alpha\in\Delta$, Y_α be H-closed, locally H-closed and HC-space. If multifunction F: $\prod X_\alpha$ $\prod Y_\alpha$ is H-u.s.c., then multifunctions F_α:X_α Y_α is H-u.s.c. for each $\alpha\in\Delta$.

Proof: Since Y_α is H-closed, locally H-closed and HC-space from Lemma 3. 4., Lemma 3. 5. and Lemma 3. 7., $\prod\limits_{\alpha\in\Delta} Y_\alpha$ is H-closed, locally H-closed and HC-space for each $\alpha\in\Delta$. Since F is H-u.s.c., F is H-almost u.s.c. [4]. Thus for each $\alpha\in\Delta$, F_α is H-almost u.s.c. from Lemma 3. 7. and from the Proposition 3.13 in [4], F_α is H-u.s.c.

Corollary 3. 11. Let for each $\alpha\in\Delta$, Y_α be H-closed, locally H-closed and HC-space and let F_α be compact point. Then multifunction F: $\prod X_\alpha \quad \prod Y_\alpha$ is H-u.s.c. if and only if for each $\alpha\in\Delta$, multifunctions F_α:$X_\alpha \quad Y_\alpha$ is H-u.s.c.

Proof: **($\Rightarrow$):** From Theorem 3. 10., it is clear.
($\Leftarrow$): From Theorem 3. 8., it is clear.

Theorem 3. 12. If for each $\alpha\in\Delta$ multifunctions F_α:X $\quad Y_\alpha$ has strongly closed graph then multifunction F:X $\quad Y_\alpha$ has strongly closed graph.

Proof: Let $(x,y)\in G(F)$. Then $y\in F(x)$ and for at least a $\beta\in\Delta$, $y_\beta \notin F_\beta(x)$. Hence $(x,y_\beta) \notin G(F)$. Since $G(F_\beta)$ is strongly closed, there exist two open sets U and V in X and in Y, respectively such that $x\in U$, $y\in V$ and $F_\beta(U) \quad V_\beta=\emptyset$. If we choose $V=V_\beta\times\prod_{\alpha\neq\beta} Y_\alpha$, then V is open in $\prod Y_\alpha$ and $y\in V$ and $F(U) \quad \overline{V}=\emptyset$. Indeed:

$F(U) \quad \overline{V}=F(U) \quad (V_\beta\times\prod_{\alpha\neq\beta} Y_\alpha)=[(F_\beta(U) \quad \overline{V_\beta})]\times\emptyset\times[\prod_{\alpha\neq\beta} Y_\alpha \quad F_\alpha(U)]=\emptyset$. Thus G(F) is strongly closed.

Corollary 3.13. If Y_α is locally H-closed Hausdorff (locally compact Hausdorff) and for each $\alpha\in\Delta$, multifunctions F_α:X $\quad Y_\alpha$ is compact point (closed point) and H-u.s.c.(C-u.s.c.), then multifunction F:X $\quad \prod Y_\alpha$, H-u.s.c. (C-u.s.c.).

Proof. From Theorem 3. 12. and the Proposition 3.13 in [4], it is clear.

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