https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 2, Pages 631–652 (2021)
DOI: 10.31801/cfsuasmas.793098
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: September 10, 2020; Accepted: March 17, 2021

FIXED POINT THEOREMS TO GENERALIZED $\mathtt{F}_{\Re}$- CONTRACTION MAPPINGS WITH APPLICATIONS TO NONLINEAR MATRIX EQUATIONS

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Abstract. In the present paper, we introduce the notion of generalized $\mathtt{F}_{\Re}$-contraction and establish some fixed point results for such mappings, which extend and generalize the result of Alam and Imdad [1], Sawangsup et al. [23] and many others. Our results reveal that the assumption of M-closedness of underlying binary relation is not a necessary condition for the existence of fixed points in relational metric spaces. We also derive some N-order fixed point theorems from our main results. As an application of our main result, we find a solution to a certain class of nonlinear matrix equations.

1. Introduction

It is widely known that the Banach contraction principle (BCP) [7] is the first metric fixed point theorem and one of the most powerful and versatile result in the field of nonlinear analysis. It asserts that every contraction mapping on a complete metric space possesses a unique fixed point. Several extensions of this principle were considered by many authors to various generalized contractions and different type of spaces (see [1], [3], [4], [5], [6], [8], [10], [12], [18], [20], [21], [26]). Wardowski [26] generalized the Banach contraction principle by introducing the notion of $\mathtt{F}$-contraction on metric spaces. The result of Wardowski was further extended and generalized by several authors (see [10], [11], [12], [17], [19], [27] and references therein) by improving the condition of $\mathtt{F}$-contraction .

2020 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Keywords and phrases. Fixed point, contraction mapping, binary relation, k-continuous mapping.

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Another important generalization of the BCP was obtained by Alam and Imdad [1] in 2015. They generalized the BCP to complete metric spaces endowed with an arbitrary binary relation. Subsequently, Sawangsup et al. [23] introduced the notion of $\mathtt{F}_{\Re}$-contraction in relational metric space by modifying the condition of $\mathtt{F}$-contraction. They also introduced the notion of $\mathtt{F}_{\Re^N}$- contraction and established some multidimensional fixed point results of N-order.

In the present paper, we improve the idea of Sawangsup et al. [23] by introducing the notion of generalized $\mathtt{F}_{\Re}$-contraction mappings and prove some fixed point results for such mappings. Our results generalize the result of Alam and Imdad [1], Wardowski [26], Sawangsup et al. [23] and many others in the existing literature. We also introduce the notions of multidimensional generalized $\mathtt{F}_{\Re^N}$-contraction and $\mathtt{F}_{\Re^N}$-graph contraction and prove some multidimensional results for the existence of fixed points of N-order. Our results do not force the underlying binary relation to be M-closed for the existence of fixed points in relational metric spaces. Moreover, we furnish some examples to demonstrate the usefulness of our main results. As an application, we apply our result to find a solution of a class of non-linear matrix equations.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of positive integers, the set of non-negative integers, the set of real numbers and the set of positive real numbers, respectively.

Definition 1. *[26] Let $\mathcal{F}$ denotes the family of all functions $\mathtt{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:*

($\mathtt{F}_1$) $\mathtt{F}$ is strictly increasing, i.e., for all $\varrho, \mu \in \mathbb{R}^+$ such that $\varrho < \mu, \mathtt{F}(\varrho) < \mathtt{F}(\mu)$;
($\mathtt{F}_2$) for each sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of positive numbers we have $\lim_{n\to\infty} \varrho_n = 0$ iff $\lim_{n\to\infty} \mathtt{F}(\varrho_n) = -\infty$;
($\mathtt{F}_3$) there exists $k \in (0, 1)$ such that $\lim_{\varrho\to 0^+} \varrho^k \mathtt{F}(\varrho) = 0$.

Example 2. *[26] Let $\mathtt{F}_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3, 4$ by:*

(i) $\mathtt{F}_1(\varrho) = \log(\varrho)$ for all $\varrho > 0$;
(ii) $\mathtt{F}_2(\varrho) = \varrho + \log(\varrho)$ for all $\varrho > 0$;
(iii) $\mathtt{F}_3(\varrho) = -\frac{1}{\sqrt{\varrho}}$ for all $\varrho > 0$;
(iv) $\mathtt{F}_4(\varrho) = \log(\varrho^2 + \varrho)$ for all $\varrho > 0$.

Definition 3. *[26] Let (X, d) be a metric space and $M : X \to X$ be a mapping. The mapping M is said to be a $\mathtt{F}$-contraction if there exists $\tau > 0$ and $\mathtt{F} \in \mathcal{F}$ such that*

$$d(M\nu, M\rho) > 0 \Longrightarrow \tau + \mathtt{F}(d(M\nu, M\rho)) \leq \mathtt{F}(d(\nu, \rho)), \quad \nu, \rho \in X.$$

We accept the following relation-theoretic notations and definitions in our subsequent discussions.

Definition 4. *[1] Let X be a non-empty set. A binary relation $\Re$ on X is a subset of $X \times X$. We say that ν relates to ρ under $\Re$ if and only if $(\nu, \rho) \in \Re$.*

Definition 5. *[1] Let $\Re$ be a binary relation on X. If either $(\nu, \rho) \in \Re$ or $(\rho, \nu) \in \Re$ then we say ν and ρ are $\Re$-comparable and we denote it by $[\nu, \rho] \in \Re$.*

Definition 6. *[1] A binary relation $\Re$ defined on a non-empty set X is called*

(a) reflexive if $(\nu, \nu) \in \Re$ for all $\nu \in X$,
(b) irreflexive if $(\nu, \nu) \notin \Re$ for all $\nu \in X$,
(c) symmetric if $(\nu, \rho) \in \Re$ implies $(\rho, \nu) \in \Re$,
(d) antisymmetric if $(\nu, \rho) \in \Re$ and $(\rho, \nu) \in \Re$ implies $\nu = \rho$,
(e) transitive if $(\nu, \rho) \in \Re$ and $(\rho, z) \in \Re$ implies $(\nu, z) \in \Re$,
(f) complete, connected or dichotomous if $[\nu, \rho] \in \Re$ for all $\nu, \rho \in X$,
(g) weakly complete, weakly connected or trichotomous if $[\nu, \rho] \in \Re$ or $\nu = \rho$ for all $\nu, \rho \in X$.

Definition 7. *[1] Let X be a non-empty set and $\Re$ be a binary relation on X. A sequence $\{\nu_n\} \in X$ is called $\Re$-preserving if*

$$(\nu_n, \nu_{n+1}) \in \Re, \quad \text{for all } n \in \mathbb{N}_0.$$

Definition 8. *[1] Let (X, d) be a metric space and $\Re$ be a binary relation on X. If for any $\Re$-preserving sequence $\{\nu_n\}$ on X such that*

$$\{\nu_n\} \xrightarrow{d} \nu,$$

*there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu] \in \Re$, for all $k \in \mathbb{N}_0$, then the binary relation $\Re$ is called d-*self-closed *on X.*

Definition 9. *[1, 22] Let X be a non-empty set and M be a self-mapping on X. A binary relation $\Re$ is called M-closed, if for $\nu, \rho \in X$ with*

$$(\nu, \rho) \in \Re \implies (M\nu, M\rho) \in \Re$$

and the mapping M is also called comparative mapping on X, under binary relation $\Re$.

Definition 10. *[14] Let $\Re$ be a binary relation on X and $M : X \to X$ be a mapping. We denote the relational graph of mapping M under the binary relation $\Re$ on X, by $G(M; \Re)$ and defined as:*

$$G(M; \Re) = \{(\nu, M\nu) \in \Re : \nu \in X\}.$$

Definition 11. *[14] Let $\Re$ be a binary relation on X and $M : X \to X$ be a self-mapping. By $X(M; \Re)$, we denotes the set of all those $\nu \in X$ for which $(\nu, M\nu) \in G(M; \Re)$, that is,*

$$X(M; \Re) = \{\nu \in X : (\nu, M\nu) \in G(M; \Re)\}.$$

The above Definition 11 is equivalent to the Definition 2.12 of Shukla and Rodríguez-López [25] which states that $X(M;\Re)$ is a set of all those points ν in X for which $(\nu, M\nu) \in \Re$, that is,

$$X(M;\Re) = \{\nu \in X : (\nu, M\nu) \in \Re\}.$$

Definition 12. *[14] Let (X, d) be a metric space, $\Re$ be a binary relation on X and $M : X \to X$ be a mapping. A binary relation $\Re$ is called M_G-d-closed if the following condition holds:*

$$(\nu, \rho) \in G(M;\Re), \quad d(M\nu, M\rho) \leq d(\nu, \rho) \implies (M\nu, M\rho) \in G(M;\Re).$$

Remark 13. *We notice that the condition of M_G-d-closedness is weaker than the condition of M-closedness. The following example illustrates this fact.*

Example 14. *Let $X = [0, 1]$ equipped with usual metric $d(\nu, \rho) = |\nu - \rho|$. Let a binary relation $\Re$ and a self-map M on X be defined as $\Re = \{(0, 0), (1, 0), (1, 1), (1/3, 1)\}$ and*

$$M(\nu) = \begin{cases} \nu/4, & \text{if } \nu \in [0, 1/3], \\ 1, & \text{if } \nu \in (1/3, 1]. \end{cases}$$

Then $G(M;\Re) = \{(0, 0), (1, 1)\}$ and for each $(\nu, \rho) \in G(M;\Re)$, we have $d(M\nu, M\rho) = d(\nu, \rho)$ and $(M\nu, M\rho) \in G(M;\Re)$. Hence the binary relation $\Re$ is M_G-d-closed. But $\Re$ is not M-closed in X because $(1/3, 1) \in \Re$ and $(M1/3, M1) = (1/12, 1) \notin \Re$.

Definition 15. *[2] Let (X, d) be a metric space and $\Re$ be a binary relation on X. A self-mapping M on X is called* $\Re$-continuous mapping *at point $\nu \in X$ if for any $\Re$-preserving sequence $\{\nu_n\}$ such that $\{\nu_n\} \xrightarrow{d} \nu$, we have $\{M(\nu_n)\} \xrightarrow{d} M(\nu)$. Moreover, M is called* $\Re$-continuous *if it is $\Re$-continuous at each point of X.*

By above definition, it is clear that every continuous mapping is $\Re$-continuous and under universal relation the definition of $\Re$-continuity coincides with the definition of continuity.

Definition 16. *[16] A self-mapping M of a metric space (X, d) is called k- continuous, $k = 1, 2, 3 \ldots,$ at a point $\nu \in X$ if $\{M^k \nu_n\} \to M\nu$, whenever $\{\nu_n\}$ is a sequence in X such that $\{M^{k-1}\nu_n\} \to \nu$ in X. Moreover, M is called k-continuous if it is k-continuous at each point of X.*

It is obvious by the definition of k-continuity that every continuous mapping M of a metric space (X, d) is k-continuous and the notion of continuity coincides with the notion of 1-continuity. However, k-continuity of a function (for $k \geq 2$) does not imply the continuity of the function (see Example 1.2 in [16]).

Definition 17. *[13] Let (X, d) be a metric space endowed with a binary relation $\Re$. A mapping $M : X \to X$ is called $(\Re, k)$-continuous at a point $\nu \in X$ if whenever $\{\nu_n\}$ is $\Re$-preserving sequence in X such that $\{M^{k-1}\nu_n\} \xrightarrow{d} \nu$, we have*

$\{M^k(\nu_n)\} \xrightarrow{d} M\nu$. Moreover, if M is a $(\Re, k)$-continuous at each point of X then M is called $(\Re, k)$-continuous.

By the definition of $(\Re, k)$-continuity, it is clear that every $\Re$-continuous mapping is a $(\Re, k)$-continuous mapping and both the definitions coincide for $k = 1$. Also every k-continuous mapping is $(\Re, k)$-continuous and for universal relation the definition of $(\Re, k)$-continuity is equivalent to the definition of k-continuity introduced by Pant and Pant in [16].

Remark 18. *Every continuous, k-continuous and $\Re$-continuous mapping is a $(\Re, k)$-continuous mapping but converse may not be true. The following example illustrates that $(\Re, k)$-continuity does not imply $\Re$-continuity and k-continuity as well.*

Example 19. *Let $X = [-1, 2]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$. Let $\Re = \{(\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N}\}$ be a binary relation on X and M be a self-mapping on X, defined as*

$$M(\nu) = \begin{cases} 1/3, & \text{if } \nu \in [-1, 0], \\ 1/2, & \text{if } \nu \in (0, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

Clearly, M is not a continuous mapping in X and the sequence $\{\nu_n\} = \{\frac{1}{2^n}\}, n \in \mathbb{N}$ is $\Re$-preserving in X as $(\nu_n, \nu_{n+1}) \in \Re$, for all $n \in \mathbb{N}$. Since $\{\nu_n\} \to 0$ as $n \to \infty$ then $\{M\nu_n\} \to 1/2 \neq M0$. Hence, M is not a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, \ldots$,

$$M^k(\nu) = \begin{cases} 1/2, & \text{if } \nu \in [-1, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

Since $M^k(\nu)$ is continuous everywhere in X, except at $\nu = 1$. Also, there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. So M is obviously a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1+\frac{1}{n}\}, n \in \mathbb{N}$, $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$ yields M is not a k-continuous mapping in X.

Hence, the mapping M is a $(\Re, k)$-continuous mapping in X, but M is neither a continuous nor a k-continuous and also not a $\Re$-continuous mapping in X.

Definition 20. *[2] Let (X, d) be a metric space and $\Re$ be a binary relation on X. If every $\Re$-preserving Cauchy sequence converges in X, then we say that (X, d) is $\Re$-complete .*

Every complete metric space is $\Re$-complete under an arbitrary binary relation $\Re$ and both the definitions coincide under the universal relation.

Definition 21. *[15] Let $\Re$ be a binary relation on a non-empty set X and $\nu, \rho \in X$. A path of length $k \in \mathbb{N}$ in $\Re$ from ν to ρ is a finite sequence $\{z_0, z_1, \ldots, z_k\} \subseteq X$ satisfying the following conditions:*

(1) $z_0 = \nu$ and $z_k = \rho$;
(2) $(z_i, z_{i+1}) \in \Re$ for all $i \in \{0, 1, 2, \dots, k-1\}$.

We denote by $\gamma(\nu; \rho; \Re)$, the family of all paths in $\Re$ from ν to ρ.

3. Main Results

Firstly, we introduce the notion of generalized $\mathtt{F}_\Re$-contraction mapping and $\mathtt{F}_\Re$-graph contraction mapping. Then, we will state our main results.

Definition 22. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and A is any non-empty subset of $X(M; \Re)$. Then, the mapping M is called a generalized $\mathtt{F}_\Re$-contraction with respect to A, if for each $\nu, \rho \in A$ with $(\nu, \rho) \in \Re$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(M\nu, M\rho) > 0 \implies \tau + \mathtt{F}(d(M\nu, M\rho)) \leq \mathtt{F}(d(\nu, \rho)). \tag{1}$$

If we take $A = X(M; \Re)$ in the above definition then we get the following definition, which is a special case of the Definition 22.

Definition 23. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. A self-mapping M on X is called a generalized $\mathtt{F}_\Re$-contraction with respect to $X(M; \Re)$ or $\mathtt{F}_\Re$-graph contraction, if for each $\nu, \rho \in X(M; \Re)$ with $(\nu, \rho) \in \Re$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(M\nu, M\rho) > 0 \implies \tau + \mathtt{F}(d(M\nu, M\rho)) \leq \mathtt{F}(d(\nu, \rho)). \tag{2}$$

Clearly condition (1) and condition (2) is weaker than the condition of $\mathtt{F}_\Re$-contraction due to Sawangsup et al. [23].

Now, we state our first result for a generalized $\mathtt{F}_\Re$-contraction mapping in a relational metric space.

Theorem 24. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose $M : X \to X$ be a mapping and there exists a non-empty subset A of $X(M; \Re)$ such that the following conditions hold:*

(a) $M(A) \subseteq A$,
(b) M is $(\Re, k)$-continuous mapping or $\Re$ is d-self closed,
(c) M is a generalized $\mathtt{F}_\Re$-contraction with respect to A,
(d) there exists $\mathrm{Y} \subseteq A$ such that $M(A) \subseteq \mathrm{Y} \subseteq A$ and (Y, d) is $\Re$-complete.

Then, for each $\nu_0 \in A$, there exists a Picard sequence $\{\nu_n\}$ of M, starting from $\nu_1 = \nu_0$ which converges to the fixed point of M.

Proof. Let A be a non-empty subset of $X(M; \Re)$ and $\nu_0 \in A$. Then by virtue of subset A, we have $(\nu_0, M\nu_0) \in \Re$. If $\nu_0 = M\nu_0$ then the proof is complete. So in view of condition (a), there exists a point say ν_1 in A such that $\nu_1 = M\nu_0$. Again, since $\nu_1 \in A$ so $(\nu_1, M\nu_1) \in \Re$. If $\nu_1 = M\nu_1$ then ν_1 is a fixed point of M and the proof is complete. Therefore $\nu_1 \neq M\nu_1$ and by assumption (a), there exists a point say $\nu_2 \in A$ such that $\nu_2 = M\nu_1$. Continuing this process again and again, we get a $\Re$-preserving Cauchy sequence of points $\{\nu_n\}$ in A such that

$$\nu_{n+1} = M\nu_n \quad \text{and} \quad (\nu_n, \nu_{n+1}) \in \Re, \qquad \text{for all } n \in \mathbb{N}_0.$$

We denote $\zeta_n = d(\nu_{n+1}, \nu_n),\ n \in \mathbb{N}_0$ and assume that $\nu_{n+1} \neq \nu_n$ for $n \in \mathbb{N}$. Then $\zeta_n > 0$, for $n \in \mathbb{N}$ and

$$\mathtt{F}(\zeta_n) \leq \mathtt{F}(\zeta_{n-1}) - \tau \leq \mathtt{F}(\zeta_{n-2}) - 2\tau \leq \cdots \leq \mathtt{F}(\zeta_0) - n\tau. \tag{3}$$

From (3), we get $\lim_{n\to\infty} \mathtt{F}(\zeta_n) = -\infty$ and together with $(\mathtt{F}_2)$, we have

$$\lim_{n\to\infty} \zeta_n = 0. \tag{4}$$

From $(\mathtt{F}_3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \zeta_n^k \mathtt{F}(\zeta_n) = 0. \tag{5}$$

By (3), the following inequality holds

$$\zeta_n^k \mathtt{F}(\zeta_n) - \zeta_n^k \mathtt{F}(\zeta_0) \leq \zeta_n^k (\mathtt{F}(\zeta_0) - n\tau) - \zeta_n^k \mathtt{F}(\zeta_0) = -\zeta_n^k n\tau \leq 0, \tag{6}$$

for all $n \in \mathbb{N}$. Making $n \to \infty$ in (6) and using (5), we obtain

$$\lim_{n\to\infty} n\zeta_n^k = 0. \tag{7}$$

From (7), we observe that there exists $n_1 \in \mathbb{N}$ such that $n\zeta_n^k \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\zeta_n \leq \frac{1}{n^{1/k}}, \tag{8}$$

for $n \geq n_1$. In order to prove that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy, consider $m, n \in \mathbb{N}$ with $m > n > n_1$. From (8) and triangle inequality, we get

$$d(\nu_m, \nu_n) \leq \zeta_{m-1} + \zeta_{m-2} + \cdots + \zeta_n < \sum_{i=n}^{\infty} \zeta_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

Now it follows, from the above inequality and by the convergence of $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$, that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy in A. Since $\{\nu_n\}_{n\in\mathbb{N}} \subseteq M(A) \subseteq \mathrm{Y}$ therefore $\{\nu_n\}_{n\in\mathbb{N}}$ is a $\Re$-preserving Cauchy sequence in Y. Since (Y, d) is a $\Re$-complete metric space so there exists a point say $\nu^* \in \mathrm{Y} \subseteq A$ such that $\lim_{n\to\infty} \nu_n = \nu^*$.

We now assume that M is a $(\Re, k)$-continuous mapping. Since the sequence $\{\nu_n\} = \{M^{k-1}(\nu_{n-k+1})\}$ converges to ν^* then $(\Re, k)$-continuity of M implies that $\{M^k(\nu_{n-k+1})\}$ converges to $M(\nu^*)$. Hence, from the above we conclude that $M(\nu^*) = \nu^*$, that is, ν^* is a fixed point of the function M.

Alternately, we assume that $\Re$ is d-self-closed. Since $\{\nu_n\}$ is a $\Re$-preserving sequence in A such that

$$\{\nu_n\} \xrightarrow{d} \nu^*$$

and $\nu^* \in A$, therefore by assumption of d-self-closedness, there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu^*] \in \Re$ for all $k \in \mathbb{N}_0$. From contraction condition (22), we obtain

$$\mathtt{F}\big(d(\nu_{n_k+1}, M\nu^*)\big) = \mathtt{F}\big(d(M\nu_{n_k}, M\nu^*)\big) \le \mathtt{F}\big(d(\nu_{n_k}, \nu^*)\big) - \tau$$
$$\implies d(\nu_{n_k+1}, M\nu^*) < d(\nu_{n_k}, \nu^*) \to 0 \text{ as } k \to \infty,$$

which yields $\nu_{n_k+1} \xrightarrow{d} M(\nu^*)$, that is, M has a fixed point at ν^* in X. □

The following example illustrates our Theorem 24.

Example 25. *Let $X = (-1, 2]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$. Let $\mathcal{L} = \big\{\big(\frac{1}{4^n}, \frac{1}{4^{n+1}}\big) : n \in \mathbb{N}\big\}$ and $\Re = \{(0,0), (0,1), (1,1), (0, \frac{3}{2}), (0, \frac{1}{4}), (1, \frac{1}{6}), (\frac{1}{4}, \frac{1}{6}), (\frac{1}{6}, \frac{1}{6})\} \cup \mathcal{L}$ be a binary relation on X. We define a self-mapping M on X as*

$$M(\nu) = \begin{cases} \frac{1}{4}, & \text{if } \nu \in (-1, 0], \\ \frac{1}{6}, & \text{if } \nu \in (0, 1], \\ \nu, & \text{if } \nu \in (1, 2], \end{cases}$$

then it is easy to see that $X(M; \Re) = \{0, \frac{1}{4}, \frac{1}{6}, 1\}$. Suppose that $A = \{0, \frac{1}{4}, \frac{1}{6}\} \subset X(M; \Re)$ and $Y = \{1/4, 1/6\}$. Then clearly $Y = M(A) \subseteq A$ and Y is $\Re$-complete. Since $\{\nu_n\} = \{\frac{1}{4^n} : n \in \mathbb{N}\}$ is a $\Re$-preserving sequence in X and $\{\nu_n\} \to 0$ but $\{M\nu_n\} \to \frac{1}{6} \neq M0$. Therefore, M is neither a continuous nor a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, \dots$,

$$M^k(\nu) = \begin{cases} \frac{1}{6}, & \text{if } \nu \in (-1, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

As $M^k(\nu)$ is continuous everywhere in X, except $\nu = 1$ and there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. Then, it is obvious by Definition 17 that M is a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$, we have $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$ which implies M is not a k-continuous mapping in X. Now, we will prove that M is a generalized $\mathtt{F}_\Re$-contraction mapping with respect to A. For this, we take $\tau = 1, \mathtt{F} \in \mathcal{F}$ given by $\mathtt{F}(\varrho) = \varrho + ln(\varrho), \varrho > 0$ and $\nu, \rho \in A$ with $(\nu, \rho) \in \Re$ such that $d(M\nu, M\rho) > 0$, we have only one choice for such (ν, ρ) in $\Re$, that is, $(\nu, \rho) = (0, 1/4)$. Then from (1), we obtain

$$\frac{d(M\nu, M\rho)}{d(\nu, \rho)} e^{[d(M\nu, M\rho) - d(\nu, \rho)]} = \frac{d(M0, M\frac{1}{4})}{d(0, \frac{1}{4})} e^{[d(M0, M\frac{1}{4}) - d(0, \frac{1}{4})]} = \frac{1}{3} e^{-\frac{1}{6}} < e^{-1}.$$

Hence, all the assumptions of Theorem 24 are hold and M has infinite fixed points in X.

Remark 26. *It is noticeable that the binary relation used in the Example 25 is not M-closed even though M has infinite fixed points in X, which reveals that the assumption of M-closedness of the underlying binary relation is not a necessary* *condition for the existence of fixed points in relational metric spaces. Thus in Example 25, the fixed point results of Sawangsup et al. [23], Alam and Imdad [1], Samet and Turinici [22] and many others does not work but our result is still valid therein.*

Remark 27. *We also notice that, the binary relation $\Re$ used in Example 25 is not one of the earlier known standard binary relation such as reflexive , symmetric, transitive, anti-symmetric, complete or weakly complete. Therefore, theorems contained in [1, 2, 7, 10, 11] can not be apply in the above example. Thus, Theorem 24 extends all the classical results to an arbitrary binary relation.*

We get the following corollary as a direct consequence of Theorem 24 by taking $\tau = \log \frac{1}{\varrho}$ and $\mathtt{F} = \log \nu$ in Theorem 24.

Corollary 28. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose $M : X \to X$ be a mapping and there exists a non-empty subset A of $X(M; \Re)$ such that the following conditions hold:*

(a) $M(A) \subseteq A$,
(b) M is $(\Re, k)$-continuous mapping or $\Re$ is d-self closed,
(c) there exists $\varrho \in [0, 1)$ such that

$$d(M\nu, M\rho) \leq \varrho\ d(\nu, \rho), \text{ for all } \nu, \rho \in A \text{ such that } (\nu, \rho) \in \Re.$$

(d) there exists $\mathrm{Y} \subseteq A$ such that $M(A) \subseteq \mathrm{Y} \subseteq A$ and (Y, d) is $\Re$-complete.

Then M has a fixed point in X.

Now we prove fixed point theorem for $\mathtt{F}_{\Re}$-graph contraction mappings in relational metric spaces.

Theorem 29. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and $X(M; \Re)$ be a non-empty set such that the following conditions are satisfied:*

(a) $\Re$ is M_G-d-closed;
(b) M is $(\Re, k)$-continuous or $\Re$ is d-self closed;
(c) M is $\mathtt{F}_{\Re}$-graph contraction on X,
(d) there exists $\mathrm{Y} \subseteq X(M; \Re)$ such that $M(X(M; \Re)) \subseteq \mathrm{Y} \subseteq X(M; \Re)$ and (Y, d) is $\Re$-complete.

Then, for each $\nu_0 \in X(M; \Re)$, there exists a Picard sequence $\{\nu_n\}$ of M, starting from $\nu_1 = \nu_0$ which converges to the fixed point of M.

Proof. Suppose $X(M; \Re)$ be a non-empty and ν_0 be any point in $X(M; \Re)$. Then by virtue of $X(M; \Re)$, we have $(\nu_0, M\nu_0) \in \Re$. If $\nu_0 = M\nu_0$ then ν_0 is a fixed point of M and the proof is completed. Therefore, we assume that $\nu_0 \neq M\nu_0$ and $M\nu_0 = \nu_1$ (say). Now as $(\nu_0, \nu_1) = (\nu_0, M\nu_0) \in G(M; \Re)$ and M is a $\mathtt{F}_{\Re}$-graph contraction, we have

$$d(M\nu_0, M\nu_1) \leq d(\nu_0, \nu_1). \tag{9}$$

In view of assumption (a) and from condition (9), we get $(M\nu_0, M\nu_1) = (\nu_1, M\nu_1) \in \Re$. Again, if $\nu_1 = M\nu_1$ then the proof is complete, otherwise there exists a point say ν_2 in X, such that $\nu_2 = M\nu_1$ and $\nu_1 \neq \nu_2$. Continuing this process again and again, we get a $\Re$-preserving Cauchy sequence of points $\{\nu_n\}$ in X such that

$$\nu_{n+1} = M\nu_n \quad \text{and} \quad (\nu_n, \nu_{n+1}) \in \mathbb{R}, \quad \text{for all } n \in \mathbb{N}_0.$$

If we take $\nu_n = \nu_{n+1}$ for some $n \in \mathbb{N}$, then ν_n is called fixed point of M. Therefore, we assume that $\nu_n \neq \nu_{n+1}$ for $n \in \mathbb{N}$, that is, $d(\nu_n, \nu_{n+1}) \neq 0$ for $n \in \mathbb{N}$. Now proceeding the proof of Theorem 24, we get the conclusion. □

The following example illustrates the utility of Theorem 29.

Example 30. *Let $X = (-1, 3]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$ and $\mathcal{P} = \big\{(\frac{1}{n}, \frac{1}{n+1}) : n \in \mathbb{N}\big\}$. Let a binary relation $\Re$ and a self-map M on X is defined as $\Re = \big\{(0,0), (0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{8}), (\frac{1}{8}, \frac{1}{8}), (1, \frac{1}{8}), (1, 2)\big\} \cup \mathcal{P}$ and*

$$M(\nu) = \begin{cases} \frac{1}{6}, & \textit{if } \nu \in (-1, 0], \\ \frac{1}{8}, & \textit{if } \nu \in (0, 1], \\ 2, & \textit{if } \nu \in (1, 3]. \end{cases}$$

Then, clearly $X(M; \Re) = \{0, \frac{1}{6}, \frac{1}{8}, 1\}$ and $G(M; \Re) = \{(0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{8}), (\frac{1}{8}, \frac{1}{8}),\ (1, \frac{1}{8})\}$. For each $(\nu, \rho) \in G(M; \Re)$, we have $d(M\nu, M\rho) \leq d(\nu, \rho)$ and $(M\nu,\ M\rho) \in G(M; \Re)$ which yields the binary relation $\Re$ on X is M_G-d-closed. However, $\Re$ is not M-closed in X as $(0, 0) \in \Re$ but $(M0, M0) = (\frac{1}{6}, \frac{1}{6}) \notin \Re$. Since $\{\nu_n\} = \{\frac{1}{n}\}$, $n \in \mathbb{N}$ is a $\Re$-preserving sequence in X as $(\nu_n, \nu_{n+1}) \in \Re$ and $\{\nu_n\} \to 0$ then $\{M\nu_n\} \to \frac{1}{8} \neq M0$. Thus, M is neither a continuous nor a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, ...,$

$$M^k(\nu) = \begin{cases} \frac{1}{8}, & \textit{if } \nu \in (-1, 1], \\ 2, & \textit{if } \nu \in (1, 3]. \end{cases}$$

As $M^k(\nu)$ is continuous everywhere in X, except $\nu = 1$ and there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. So M is obviously a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1 + \frac{1}{n}\}$, $n \in \mathbb{N}$, $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$, yields M is not a k-continuous mapping in X. Hence, the mapping M is a $(\Re, k)$-continuous mapping in X, but M is neither a continuous nor a k-continuous and also not a $\Re$-continuous mapping in X. Now, we will show that M is a generalized $\mathtt{F}_{\Re}$-graph contraction mapping with $\tau = 1$ and $\mathtt{F} \in \mathcal{F}$ defined by

$$\mathtt{F}(\varrho) = \varrho + ln(\varrho), \ \textit{for all } \varrho > 0.$$

For any $\nu,\rho \in X(M;\Re)$ with $(\nu,\rho) \in \Re$ and $d(M\nu, M\rho) > 0$, we have only one choice for $(\nu,\rho) = (0, \frac{1}{6})$ in $\Re$. Then from (23),

$$\frac{d(M\nu, M\rho)}{d(\nu,\rho)}\ e^{\{d(M\nu,M\rho)-d(\nu,\rho)\}} = \frac{d(M0, M\frac{1}{6})}{d(0,\frac{1}{6})}\ e^{\{d(M0,M\frac{1}{6})-d(0,\frac{1}{6})\}} = \frac{1}{4}e^{-\frac{1}{8}} < e^{-1}.$$

This yields M is a $\mathtt{F}_{\Re}$-graph contraction with $\tau = 1$. Hence, all the conditions of Theorem 29 are hold and M has two fixed points at points $\nu = \frac{1}{8}$ and $\nu = 2$.

A generalized version of relation-theoretic contraction principle due to Alam and Imdad [1] is derived from Theorem 29 by taking $\tau = \log\frac{1}{k}$ and $\mathtt{F} = \log\nu$ in Theorem 29.

Corollary 31. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and $X(M;\Re)$ be a non-empty set such that the following conditions are satisfied:*

(a) $\Re$ is M_G-d-closed,
(b) M is $(\Re, k)$-continuous or $\Re$ is d-self-closed,
(c) there exists $k \in [0, 1)$ such that

$$d(M\nu, M\rho) \leq k\ d(\nu,\rho),\ \text{for all } \nu,\rho \in X(M;\Re) \text{ with } (\nu,\rho) \in \Re.$$

(d) there exists $\mathrm{Y} \subseteq X(M;\Re)$ such that $M(X(M;\Re)) \subseteq \mathrm{Y} \subseteq X(M;\Re)$ and (Y, d) is $\Re$-complete.

Then M has a fixed point.

Remark 32. *We notice that Theorem 24 and Theorem 29 remain valid if we replace the assumption of $(\Re; k)$-continuity of M either by continuity of M, k-continuity of M or $\Re$-continuity of M (without altering the rest of the hypothesis).*

The following theorem guarantees the uniqueness of fixed points of Theorem 29 in a relational metric space.

Theorem 33. *In addition to the hypothesis of Theorem 29, suppose that $\Re$ is a transitive relation on X and $\gamma(\nu,\rho,\Re)$ is non-empty, for all $\nu,\rho \in X(M;\Re)$. Then, M has a unique fixed point in $X(M;\Re)$.*

Proof. Let ν^* and ρ^* be two distinct fixed points of M in $X(M;\Re)$ then $\nu^* = M\nu^*, \rho^* = M\rho^*$. Since $\gamma(\nu^*,\rho^*,\Re)$ is non-empty, there is a path (say $\{z_0, z_1, \ldots, z_k\}$) of some finite length k in $\Re$ from ν to ρ, so that

$$z_0 = \nu^*,\ z_k = \rho^*,\ (z_i, z_{i+1}) \in \Re, \quad \text{for each } i = 0, 1, 2, \ldots, k-1.$$

By transitivity of $\Re$, we get

$$(\nu^*, z_1) \in \Re, \quad (z_1, z_2) \in \Re, \ldots, (z_{k-1}, \rho^*) \in \Re \implies (\nu^*, \rho^*) \in \Re.$$

The condition (23) implies that

$$\tau + \mathtt{F}(d(\nu^*,\rho^*)) = \tau + \mathtt{F}(d(M\nu^*, M\rho^*)) \leq \mathtt{F}(d(\nu^*,\rho^*))$$

which is not possible. Thus, M has a unique fixed point in $X(M;\Re)$. □

4. Multidimensional results for the existence of fixed points of N-order

In this section, we drive some multidimensional results or N-order fixed point theorems from our main results by using very simple tools. Let $\Re$ be a binary relation on X and we denote by $\Re^N$ the binary relation on the product space X^N defined by:

$$\big((\nu_1,\nu_2,\ldots,\nu_N),(\rho_1,\rho_2,\ldots,\rho_N)\big)\in\Re^N \iff \begin{array}{l}(\nu_1,\rho_1)\in\Re,(\nu_2,\rho_2)\in\Re,\\ (\nu_3,\rho_3)\in\Re,\ldots,(\nu_N,\rho_N)\in\Re.\end{array}$$

Suppose $M: X^N\to X$ is a mapping and by $X^N(M;\Re^N)$, we denote the set of all points $(\nu_1,\nu_2,\ldots,\nu_N)\in X^N$ such that

$$\left(\begin{array}{r}(\nu_1,\nu_2,\ldots,\nu_N),\big(M(\nu_1,\nu_2,\ldots,\nu_N),M(\nu_2,\nu_3,\ldots,\nu_N,\nu_1)\\ ,\ldots,M(\nu_N,\nu_1,\ldots,\nu_{N-1})\big)\end{array}\right)\in\Re^N,$$

that is,

$$\big(\nu_i, M(\nu_i,\nu_{i+1},\ldots,\nu_N,\nu_1,\nu_2,\ldots,\nu_{i-1})\big)\in\Re,\ \text{for each } i\in\{1,2,\ldots,N\}.$$

In addition, we denote by $\mathcal{S}_M^N: X^N\to X^N$ the mapping

$$\mathcal{S}_M^N(\nu_1,\nu_2,\ldots,\nu_N)=\left(\begin{array}{r}M(\nu_1,\nu_2,\ldots,\nu_N),M(\nu_2,\nu_3,\ldots,\nu_N,\nu_1)\\ ,\ldots,M(\nu_N,\nu_1,\ldots,\nu_{N-1})\end{array}\right),$$

for all $(\nu_1,\nu_2,\ldots,\nu_N)\in X^N$.

Definition 34. *[24] Let $\Re$ be a binary relation defined on a non-empty set X and $(\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\in X^N$. Then $(\nu_1,\nu_2,...,\nu_N)$ and $(\rho_1,\rho_2,..., \rho_N)$ are $\Re^N$-comparative if either $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big)\in\Re^N$ or $\big((\rho_1,\rho_2,...,\rho_N),(\nu_1,\nu_2,...,\nu_N)\big)\in\Re^N$. We denote it by $\big[(\nu_1,\nu_2,\ldots,\nu_N),(\rho_1, \rho_2,\ldots,\rho_N)\big]\in\Re^N$.*

Definition 35. *[24] Let X be a non-empty set and $\Re$ be a binary relation on X. A sequence $\{(\nu_n^1,\nu_n^2,\ldots,\nu_n^N)\}\subset X^N$ is called $\Re^N$-preserving if*

$$\big((\nu_n^1,\nu_n^2,\ldots,\nu_n^N),(\nu_{n+1}^1,\nu_{n+1}^2,\ldots,\nu_{n+1}^N)\big)\in\Re^N \textit{ for all } n\in\mathbb{N}.$$

Definition 36. *[23] Let $M: X^N\to X$ be a mapping. A binary relation $\Re$ on X is called M_N-closed, if for any $(\nu_1,\nu_2,\ldots,\nu_N),(\rho_1,\rho_2,\ldots,\rho_N)\in X^N$,*

$$\left\{\begin{array}{c}(\nu_1,\rho_1)\in\Re\\ (\nu_2,\rho_2)\in\Re\\ .\\ .\\ .\\ (\nu_N,\rho_N)\in\Re\end{array}\right\}\Rightarrow\left\{\begin{array}{c}\big(M(\nu_1,\nu_2,\ldots,\nu_N),M(\rho_1,\rho_2,\ldots,\rho_N)\big)\in\Re\\ \big(M(\nu_2,\nu_3,\ldots,\nu_1),M(\rho_2,\rho_3,\ldots,\rho_1)\big)\in\Re\\ .\\ .\\ .\\ \big(M(\nu_N,\nu_1,\ldots,\nu_{N-1}),M(\rho_N,\rho_1,\ldots,\rho_{N-1})\big)\in\Re\end{array}\right\}.$$

Definition 37. *If $M : X^N \to X$ is a mapping. Then, we denote the relational graph of the mapping M under the binary relation $\Re^N$ on X^N, by $G^N(M;\Re^N)$ and defined as:*

$$G^N(M;\Re^N) = \big\{\big((\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,\dots,\nu_1),\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big)\big) \in \Re^N : (\nu_1,\nu_2,\dots,\nu_N) \in X^N\big\}.$$

Definition 38. *Let (X,d) be a metric space, $\Re$ be a binary relation on X and $M : X^N \to X$ be a mapping. By $X^N(M;\Re^N)$, we denote the set of all those $(\nu_1,\nu_2,\dots,\nu_N) \in X^N$, for which*

$$\left(\begin{array}{r} (\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,...,\nu_1) \\ ,\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big) \end{array} \right) \in G^N(M;\Re^N),$$

that is,

$$X^N(M;\Re^N) = \{(\nu_1,\nu_2,\dots,\nu_N) \in X^N : \big((\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N), M(\nu_2,\nu_3,\dots,\nu_1),\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big)\big) \in G^N(M;\Re^N)\}.$$

Definition 39. *Let (X,d) be a metric space, $\Re$ be a binary relation on X and $M : X^N \to X$ be a mapping. A binary relation $\Re$ is called M_G^N-d-closed if for every $\big((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\big) \in G^N(M;\Re^N)$ with*

$$\left\{ \begin{array}{rl} d\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big) & \leq d((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)) \\ d\big(M(\nu_2,\nu_3\dots,\nu_1),M(\rho_2,\rho_3,\dots,\rho_1)\big) & \leq d((\nu_2,\nu_3,\dots,\nu_1),(\rho_2,\rho_3,\dots,\rho_1)) \\ & \vdots \\ d\left(\begin{array}{c} M(\nu_N,\nu_1,...,\nu_{N-1}), \\ M(\rho_N,\rho_1,...,\rho_{N-1}) \end{array} \right) & \leq d\left(\begin{array}{c} (\nu_N,\nu_1,\dots,\nu_{N-1}), \\ (\rho_N,\rho_1,\dots,\rho_{N-1}) \end{array} \right) \end{array} \right\}$$

$$\Longrightarrow \left\{ \begin{array}{c} \big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big) \in G^N(M;\Re^N) \\ \big(M(\nu_2,\nu_3,\dots,\nu_1),M(\rho_2,\rho_3,\dots,\rho_1)\big) \in G^N(M;\Re^N) \\ \cdot \\ \cdot \\ \cdot \\ \big(M(\nu_N,\nu_1,\dots,\nu_{N-1}),M(\rho_N,\rho_1,\dots,\rho_{N-1})\big) \in G^N(M;\Re^N) \end{array} \right\}.$$

Remark 40. *It is obvious from the above definition that the condition of M_G^N-d-closedness is weaker than the condition of M_N-closedness of underlying relation in relational metric spaces.*

Definition 41. *Let X be a non-empty set and $\Re$ be a binary relation on X. A mapping $M : X^N \to X$ is said to be a $(\Re^N,k)$-continuous at $(\nu^1,\nu^2,\dots,\nu^N) \in X^N$ if for any $\Re^N$-preserving sequence $\big\{(\nu_n^1,\nu_n^2,\dots,\nu_n^N)\big\}$ in X^N such that*

$$\big\{M^{k-1}(\nu_n^1,\nu_n^2,...,\nu_n^N),M^{k-1}(\nu_n^2,\nu_n^3,...,\nu_n^1),...,M^{k-1}(\nu_n^N,\nu_n^1,...,\nu_n^{N-1})\big\} \xrightarrow{d} (\nu^1,\nu^2,\dots,\nu^N),$$

we have

$$\big\{M^k(\nu_n^1,\nu_n^2,\dots,\nu_n^N),M^k(\nu_n^2,\nu_n^3,\dots,\nu_n^1),\dots,M^k(\nu_n^N,\nu_n^1,\dots,\nu_n^{N-1})\big\}\xrightarrow{d}$$
$$\big\{M(\nu^1,\nu^2,\dots,\nu^N),M(\nu^2,\nu^3,\dots,\nu^1),\dots,M(\nu^N,\nu^1,\dots,\nu^{N-1})\big\}.$$

Then mapping M is called $(\Re^N,k)$-continuous if it is $(\Re^N,k)$-continuous at each point of X^N.

Lemma 42. *[23] Given $N\geq 2$ and $M: X^N\to X$ be a given mapping. A point $(\nu^1,\nu^2,\dots,\nu^N)\in X^N$ is an N-order fixed point of M if and only if it is a fixed point of $\mathcal{S}_M^N$.*

Lemma 43. *[23] Given $N\geq 2$ and $M: X^N\to X$, a point $(\nu_1,\nu_2,\dots,\nu_N)\in X^N(M;\Re^N)$ if and only if $(\nu_1,\nu_2,\dots,\nu_N)\in X^N(\mathcal{S}_M^N;\Re^N)$.*

Lemma 44. *[23] Let (X,d) be a metric space and $D^N: X^N\times X^N\to\mathbb{R}$ be defined by*

$$D^N(U,V)=\sum_{i=1}^N d(u_i,v_i)$$

for all $U=(u_1,u_2,\dots,u_N)$, $V=(v_1,v_2,\dots,v_N)\in X^N$. Then the following properties hold:

(1) (X^N,D^N) is also a metric space.
(2) Let $\{U_n=(u_n^1,u_n^2,\dots,u_n^N)\}$ be a sequence in X^N and $U=(u_1,u_2,\dots,u_N)\in X^N$. Then $U_N\xrightarrow{D_N}U$ if and only if $\{u_n^i\}\xrightarrow{d}u_i$ for all $i\in\{1,2,3,\dots,N\}$.
(3) If $\{U_n=(u_n^1,u_n^2,\dots,u_n^N)\}$ is a sequence on X^N, then $\{U_n\}$ is a D^N-Cauchy sequence if and only if $\{u_n^i\}$ is a Cauchy sequence for all $i\in\{1,2,3,\dots,N\}$.
(4) (X,d) is complete if and only if (X^N,D^N) is complete.

Definition 45. *Let (X^N,D^N) be a metric space and $\Re$ be a binary relation on X. If every $\Re^N$-preserving Cauchy sequence converges in X^N then we say that (X^N,D^N) is $\Re^N$-complete.*

Every complete metric space is $\Re^N$-complete under any binary relation $\Re^N$ on X^N and both the definitions coincide under the universal relation.

Definition 46. *[23] Let X be a non-empty set and $\Re$ be a binary relation on X. A path of length $k\in\mathbb{N}$ in $\Re^N$ from $(\nu_1,\nu_2,\dots,\nu_N)\in X^N$ to $(\rho_1,\rho_2,\dots,\rho_N)\in X^N$ is a finite sequence $\big\{(z_0^1,z_0^2,\dots,z_0^N),(z_1^1,z_1^2,\dots,z_1^N),\ \dots,(z_k^1,z_k^2,\dots,z_k^N)\big\}\subset X^N$ satisfying the following conditions:*

(i) $(z_0^1,z_0^2,\dots,z_0^N)=(\nu_1,\nu_2,\dots,\nu_N)$ and $(z_k^1,z_k^2,\dots,z_k^N)=(\rho_1,\rho_2,\dots,\rho_N)$;
(ii) $\big((z_i^1,z_i^2,\dots,z_i^N),(z_{i+1}^1,z_{i+1}^2,\dots,z_{i+1}^N)\big)\in\Re^N$ for all $i=0,1,2,...,k-1$.

Clearly, a path of length k involves $k+1$ elements of X^N, although they are not necessarily distinct. Moreover, let $\gamma\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N),\Re^N\big)$ be the class of all paths in $\Re^N$ from $(\nu_1,\nu_2,\dots,\nu_N)$ to $(\rho_1,\rho_2,\dots,\rho_N)$.

Now, we introduce the notion of generalized $\mathtt{F}_{\Re^N}$-contraction mapping and $\mathtt{F}_{\Re^N}$-graph contraction mapping for $N \geq 2$.

Definition 47. *Let (X, d) be a metric space endowed with a binary relation $\Re$ and A^N is a non-empty subset of $X^N(M; \Re^N)$. A mapping $M : X^N \to X$ is called a generalized $\mathtt{F}_{\Re^N}$-contraction with respect to A^N, if for each $(\nu_1, \nu_2, \ldots, \nu_N), (\rho_1, \rho_2, \ldots, \rho_N) \in A^N$ with $\big((\nu_1, \nu_2, ..., \nu_N), (\rho_1, \rho_2, ..., \rho_N)\big) \in \Re^N$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d\big(M(\nu_1, \nu_2, ..., \nu_N), M(\rho_1, \rho_2, ..., \rho_N)\big) > 0 \implies$$

$$\tau + \mathtt{F}\begin{pmatrix} d\big(M(\nu_1, \nu_2, ..., \nu_N), M(\rho_1, \rho_2, ..., \rho_N)\big)+ \\ d\big(M(\nu_2, \nu_3, ..., \nu_1), M(\rho_2, \rho_3, ..., \rho_1)\big)+ \\ . \\ . \\ . \\ d\big(M(\nu_N, \nu_1, ..., \nu_{N-1}), M(\rho_N, \rho_1, ..., \rho_{N-1})\big) \end{pmatrix} \leq \mathtt{F}\Big(\sum_{i=1}^{N} d(\nu_i, \rho_i)\Big).$$

Definition 48. *Let (X, d) be a metric space endowed with a binary relation $\Re$ and $X^N(M; \Re^N)$ be a non-empty subset of X. A mapping $M : X^N \to X$ is called a $\mathtt{F}_{\Re^N}$-graph contraction, if for each $(\nu_1, \nu_2, \ldots, \nu_N), (\rho_1, \rho_2, \ldots, \rho_N) \in X^N(M; \Re^N)$ with $\big((\nu_1, \nu_2, \ldots, \nu_N), (\rho_1, \rho_2, \ldots, \rho_N)\big) \in \Re^N$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that $d\big(M(\nu_1, \nu_2, \ldots, \nu_N), M(\rho_1, \rho_2, \ldots, \rho_N)\big) > 0 \implies$*

$$\tau + \mathtt{F}\begin{pmatrix} d\big(M(\nu_1, \nu_2, ..., \nu_N), M(\rho_1, \rho_2, ..., \rho_N)\big)+ \\ d\big(M(\nu_2, \nu_3, ..., \nu_1), M(\rho_2, \rho_3, ..., \rho_1)\big)+ \\ . \\ . \\ . \\ d\big(M(\nu_N, \nu_1, ..., \nu_{N-1}), M(\rho_N, \rho_1, ..., \rho_{N-1})\big) \end{pmatrix} \leq \mathtt{F}\Big(\sum_{i=1}^{N} d(\nu_i, \rho_i)\Big). \quad (10)$$

Now using Theorem 24, we will prove a multidimensional result which conforms the existence of fixed points of N-order.

Theorem 49. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose that $M : X^N \to X$ be a mapping and there exists a non-empty subset A^N of $X^N(M; \Re^N)$ such that the following conditions hold:*

(a) $M(A^N) \subseteq A^N$;
(b) M is $(\Re^N, k)$-continuous mapping;
(c) M is a generalized $\mathtt{F}_{\Re^N}$-contraction with respect to A^N;
(d) there exists $\mathrm{Y}^N \subseteq A^N$ such that $M(A^N) \subseteq \mathrm{Y}^N \subseteq A^N$ and (Y^N, D^N) is $\Re^N$-complete.

Then M has a fixed point of N-order.

Proof. Let A^N be a non-empty subset of $X^N(M,\Re^N)$ and $(\nu_0^1,\nu_0^2,\dots,\nu_0^N)\in A^N$. Then by the virtue of subset A^N, we have

$$\left(\begin{array}{r}(\nu_0^1,\nu_0^2,\dots,\nu_0^N),(M(\nu_0^1,\nu_0^2,\dots,\nu_0^N),M(\nu_0^2,\nu_0^3,\dots,\nu_0^1),\\ \dots,M(\nu_0^N,\nu_0^1,\dots,\nu_0^{N-1}))\end{array}\right)\in\Re^N.$$

If $(\nu_0^1,\nu_0^2,\dots,\nu_0^N)=\left(\begin{array}{r}M(\nu_0^1,\nu_0^2,\dots,\nu_0^N),M(\nu_0^2,\nu_0^3,\dots,\nu_0^N,\nu_0^1),\\ \dots,M(\nu_0^N,\nu_0^1,\dots,\nu_0^{N-1})\end{array}\right)$, then proof is complete. So in view of assumption (a), there exists $(\nu_1^1,\nu_1^2,\dots,\nu_1^N)$ in A^N such that

$$(\nu_1^1,\nu_1^2,\dots,\nu_1^N)=\left(\begin{array}{r}M(\nu_0^1,\nu_0^2,\dots,\nu_0^N),M(\nu_0^2,\nu_0^3,\dots,\nu_0^N,\nu_0^1),\\ \dots,M(\nu_0^N,\nu_0^1,\dots,\nu_0^{N-1})\end{array}\right).$$

Again, since $(\nu_1^1,\nu_1^2,\dots,\nu_1^N)\in A^N$ so

$$\left(\begin{array}{r}(\nu_1^1,\nu_1^2,\dots,\nu_1^N),(M(\nu_1^1,\nu_1^2,\dots,\nu_1^N),M(\nu_1^2,\nu_1^3,\dots,\nu_1^N,\nu_1^1),\\ \dots,M(\nu_1^N,\nu_1^1,\dots,\nu_1^{N-1}))\end{array}\right)\in\Re^N.$$

If $(\nu_1^1,\nu_1^2,\dots,\nu_1^N)=\left(\begin{array}{r}M(\nu_1^1,\nu_1^2,\dots,\nu_1^N),M(\nu_1^2,\nu_1^3,\dots,\nu_1^N,\nu_1^1),\\ \dots,M(\nu_1^N,\nu_1^1,\dots,\nu_1^{N-1})\end{array}\right)$, then the proof is complete. Otherwise we will continue this process again and again and obtain a $\Re^N$-preserving sequence of points $\{(\nu_n^1,\nu_n^2,\dots,\nu_n^N)\}$ in A^N such that

$$(\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)=\left(\begin{array}{r}M(\nu_n^1,\nu_n^2,\dots,\nu_n^N),M(\nu_n^2,\nu_n^3,\dots,\nu_n^N,\nu_n^1),\\ \dots,M(\nu_n^N,\nu_n^1,\dots,\nu_n^{N-1})\end{array}\right)$$

and

$$\big((\nu_n^1,\nu_n^2,\dots,\nu_n^N),(\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)\big)\in\Re^N,\ \text{for all } n\in\mathbb{N}.$$

Since M is $(\Re^N,k)$-continuous, we get $\mathcal{S}_M^N$ is also $(\Re^N,k)$-continuous. From the generalized $\mathtt{F}_{\Re^N}$-contractive condition of M, we deduce that $\mathcal{S}_M^N$ is also a generalized $\mathtt{F}_{\Re^N}$-contraction. Applying Theorem 24, there exists $\hat{\mathrm{Z}}=(\nu_1^*,\nu_2^*,\dots,\nu_N^*)\in X^N$ such that $\mathcal{S}_M^N(\hat{\mathrm{Z}})=\hat{\mathrm{Z}}$, i.e., $(\nu_1^*,\nu_2^*,\dots,\nu_N^*)$ is a fixed point of $\mathcal{S}_M^N$. Using Lemma 42, we have $(\nu_1^*,\nu_2^*,\dots,\nu_N^*)$ is a fixed point of N-order of M. This completes the proof. □

If we take $\tau=\log\frac{1}{\varrho}$ and $\mathtt{F}=\log\nu$ in Theorem 49 then we get the following corollary as a direct consequence of Theorem 49.

Corollary 50. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose that $M:X^N\to X$ be a mapping and there exists a non-empty subset A^N of $X^N(M;\Re^N)$ such that the following conditions hold:*

(a) $M(A^N)\subseteq A^N$,
(b) M is $(\Re^N,k)$-continuous mapping,

(c) there exists $\varrho \in [0,1)$ such that

$$\sum_{i=1}^{N} d\left(\begin{array}{c} M(\nu_i, \nu_{i+1}, ..., \nu_N, \nu_1, ..., \nu_{i-1}), \\ M(\rho_i, \rho_{i+1}, ..., \rho_N, \rho_1, ..., \rho_{i-1}) \end{array} \right) \leq \varrho \sum_{i=1}^{N} d(\nu_i, \rho_i),$$

for each $(\nu_1, \nu_2, ..., \nu_N), (\rho_1, \rho_2, ..., \rho_N) \in A^N$ such that $\big((\nu_1, \nu_2, ..., \nu_N), (\rho_1, \rho_2, ..., \rho_N)\big) \in \Re^N$, then M has a fixed point of N-order.

(d) there exists $\mathrm{Y}^N \subseteq A^N$ and $M(A^N) \subseteq \mathrm{Y}^N \subseteq A^N$, so that (Y^N, D^N) is $\Re^N$-complete.

Using similar technique as in the proof of Theorem 49, we obtain the following multidimensional result for the existence of fixed points of N-order.

Theorem 51. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose $X^N(M; \Re^N)$ be a non-empty and $M : X^N \to X$ be a mapping such that the following conditions hold:*

(a) $\Re$ is M_G^N-d-closed;
(b) M is $(\Re^N, k)$-continuous;
(c) M is $\mathtt{F}_{\Re^N}$-graph contraction on X^N;
(d) there exists $\mathrm{Y}^N \subseteq X^N(M; \Re^N)$ such that $M(X^N(M; \Re^N)) \subseteq \mathrm{Y}^N \subseteq X^N(M; \Re^N)$ and (Y^N, D^N) is $\Re^N$-complete,

then M has a fixed point of N-order.

Proof. Suppose $X^N(M; \Re^N)$ be a non-empty set and $(\nu_0^1, \nu_0^2, \ldots, \nu_0^N) \in X^N(M; \Re^N)$. Then, we have

$$\big((\nu_0^1, \nu_0^2, \ldots, \nu_0^N), (M(\nu_0^1, \nu_0^2, \ldots, \nu_0^N), \ldots, M(\nu_0^N, \nu_0^1, \ldots, \nu_0^{N-1}))\big) \in \Re^N.$$

Now in view of assumption (a) and from $\mathtt{F}_{\Re^N}$-graph contraction condition (10), we have

$$\big((\nu_1^1, \nu_1^2, \ldots, \nu_1^N), (\nu_2^1, \nu_2^2, \ldots, \nu_2^N)\big) =$$
$$\left(\begin{array}{c} \left(M(\nu_0^1, \nu_0^2, \ldots, \nu_0^N), M(\nu_0^2, \nu_0^3, \ldots, \nu_0^N, \nu_0^1), \ldots, M(\nu_0^N, \nu_0^1, \ldots, \nu_0^{N-1}) \right), \\ \left(M(\nu_1^1, \nu_1^2, \ldots, \nu_1^N), M(\nu_1^2, \nu_1^3, \ldots, \nu_1^N, \nu_1^1), \ldots, M(\nu_1^N, \nu_1^1, \ldots, \nu_1^{N-1} \right) \end{array} \right).$$

Continuing this process again and again, we get a $\Re^N$-preserving Cauchy sequence of points $(\nu_n^1, \nu_n^2, \ldots, \nu_n^N)$ in X^N such that

$$(\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N) = \left(\begin{array}{c} M(\nu_n^1, \nu_n^2, \ldots, \nu_n^N), M(\nu_n^2, \nu_n^3, \ldots, \nu_n^N, \nu_n^1), \\ \ldots, M(\nu_n^N, \nu_n^1, \ldots, \nu_n^{N-1}) \end{array} \right)$$

and

$$\big((\nu_n^1, \nu_n^2, \ldots, \nu_n^N), (\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N)\big) \in \Re^N, \text{ for all } n \in \mathbb{N}.$$

If we take $(\nu_n^1, \nu_n^2, \ldots, \nu_n^N) = (\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N)$ for some $n \in \mathbb{N}$, then $\{(\nu_n^1, \nu_n^2, \ldots, \nu_n^N)\}$ is called a fixed point of M. Therefore we assume $(\nu_n^1, \nu_n^2, \ldots, \nu_n^N) \neq (\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N)$ for all $n \in \mathbb{N}$. Now proceeding the proof of Theorem 49 we get the conclusion. □

Corollary 52. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose $X^N(M;\Re^N)$ be a non-empty set and $M : X^N \to X$ be a mapping such that the following conditions hold:*

(a) $\Re$ is M_G^N-d-closed,
(b) M is $(\Re^N, k)$-continuous,
(c) there exists $\varrho \in [0,1)$ such that

$$\sum_{i=1}^{N} d\begin{pmatrix} M(\nu_i,\nu_{i+1},...,\nu_N,\nu_1,...,\nu_{i-1}), \\ M(\rho_i,\rho_{i+1},...,\rho_N,\rho_1,...,\rho_{i-1}) \end{pmatrix} \le \varrho \sum_{i=1}^{N} d(\nu_i,\rho_i),$$

for all $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big) \in G^N(M;\Re^N)$,
(d) there exists $\mathrm{Y}^N \subseteq X^N(M;\Re^N)$ such that $M(X^N(M;\Re^N)) \subseteq \mathrm{Y}^N \subseteq X^N(M;\Re^N)$ and (Y^N, D^N) is $\Re^N$-complete.

Then M has a fixed point of N-th order.

5. Application to nonlinear matrix equations

In this section, we follow the following notations:

- $\mathcal{X}_n$ denotes the set of all $n \times n$ Complex matrices;
- $\mathcal{H}_n \subset \mathcal{X}_n$ is the set of all $n \times n$ Hermitian matrices;
- $\mathcal{P}_n \subset \mathcal{H}_n$ is the set of all $n \times n$ positive definite matrices;
- $\mathcal{H}_n^+ \subset \mathcal{H}_n$ is the set of all $n \times n$ positive semidefinite matrices.

and for $U, V \in \mathcal{X}_n$, we denote the following notations:

- $U \succ 0 \Longleftrightarrow U \in \mathcal{P}_n$;
- $U \succeq 0 \Longleftrightarrow U \in \mathcal{H}_n^+$;
- $U - V \succ 0 \Longleftrightarrow U \succ V$;
- $U - V \succeq 0 \Longleftrightarrow U \succeq V$.

Let B^* is the conjugate transpose of B and $\lambda^+(B^*B)$ is the largest eigenvalue of B^*B. We use the symbol $\|.\|$ for the spectral norm of B and defined by $\|B\| = \sqrt{\lambda^+(B^*B)}$.

The symbol $\|.\|_{tr}$ is used for the metric induced by trace norm and it is defined by $\|B\|_{tr} = \sum_{j=1}^{n} s_j(B)$, where $s_j(B), j = 1,2,...,n$, are the singular values of $B \in \mathcal{X}_n$. Hence, $(\mathcal{H}_n, \|.\|_{tr})$ forms a complete metric space. See ([8], [9], [18]) for more details. Moreover, the binary relation $\preceq$ on $\mathcal{H}_n$ defined by:

$$U \preceq V \Longleftrightarrow V \succeq U$$

for all $U, V \in \mathcal{H}_n$.

In this section, we apply Theorem 24 to establish a solution of the nonlinear matrix equation.

$$U = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(U) A_i \tag{11}$$

where A_i is an any $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is continuous order preserving mapping (i.e., if $U, V \in \mathcal{H}_n$ with $U \preceq V$ implies that $\mathcal{G}(U) \preceq \mathcal{G}(V)$) with $\mathcal{G}(0) = 0$.

Now we state the following lemmas which are very useful in this sequel:

Lemma 53. *If $U, V \in \mathcal{H}_n^+$ such that $U \succeq 0$ and $V \succeq 0$, Then*

$$0 \leq tr(UV) \leq \|U\| tr(V).$$

Lemma 54. *If $U \in \mathcal{H}_n$ and $U \prec I$, then $\|U\| < 1$.*

Theorem 55. *Consider the matrix equation* (11) *and suppose that there is a positive numbers k and τ such that*

(i) For every $U, V \in \mathcal{H}_n^+$ with $U \preceq V$ and $\sum_{i=1}^n A_i^* \mathcal{G}(U) A_i \neq \sum_{i=1}^n A_i^* \mathcal{G}(V) A_i$, we have

$$|tr(\mathcal{G}(V) - \mathcal{G}(U))| \leq \frac{|tr(V - U)|}{k(1 + \tau\sqrt{tr(V - U)})^2}; \tag{12}$$

(ii) $\sum_{i=1}^m A_i A_i^* \prec kI_n$ and $\sum_{i=1}^m A_i^* \mathcal{G}(U) A_i \succ 0$.

Then the matrix equation (11) has a solution. Moreover, the iteration

$$U_n = Q + \sum_{i=1}^n A_i^* \mathcal{G}(U_{n-1}) A_i \tag{13}$$

where $U_0 \in \mathcal{H}_n$ such that $U_0 \preceq Q + \sum_{i=1}^n A_i^* \mathcal{G}(U_0) A_i$, converges in the sense of trace norm $\|.\|_{tr}$, to the solution of the nonlinear matrix equation (11).

Proof. We define a mapping $M : \mathcal{H}_n \to \mathcal{H}_n$ by

$$M(U) = Q + \sum_{i=1}^n A_i^* \mathcal{G}(U) A_i$$

for all $U \in \mathcal{H}_n$ and a set by

$$\mathcal{H}_n^+(M, \preceq) = \{A \in \mathcal{H}^+ : A \preceq M(A) \text{ or } M(A) - A \succeq 0\}.$$

Then M is well defined mapping, $\mathcal{H}_n^+(M, \preceq)$ is a non-empty set as $Q \in \mathcal{H}^+$ and $M(Q) - Q = \sum_{i=1}^n A_i^* \mathcal{G}(Q) A_i \succeq 0$. It is easy to verify that for every positive semidefinite matrix B, $M(B)$ is also positive semidefinite matrix and $\mathcal{H}_n^+(M, \preceq)$ is $\preceq$-complete. Now, we will prove that the set $\mathcal{H}_n^+(M, \preceq)$ is invariant under the mapping M, that is $M(\mathcal{H}_n^+(M, \preceq)) \subseteq \mathcal{H}_n^+(M, \preceq)$. For this, it is sufficient to prove that $M(B) \in \mathcal{H}_n^+(M, \preceq)$ for every $B \in \mathcal{H}_n^+(M, \preceq)$. Let $B \in \mathcal{H}_n^+(M, \preceq)$ then $M(B) - B \succeq 0$ and

$$M(M(B)) - M(B) = \sum_{i=1}^n A_i^* \big(\mathcal{G}(M(B)) - \mathcal{G}(B)\big) A_i \succeq 0, \tag{14}$$

that is $M(B) \preceq M(M(B))$, which implies $M(B) \in \mathcal{H}_n^+(M, \preceq)$.

Next, we will show that M is a generalized $\mathtt{F}_{\preceq}$-contraction mapping with respect to $\mathcal{H}_n^+(M,\preceq)$. For this, let $\tau>0$ be any real number and $\mathtt{F}:\mathbb{R}^+\to\mathbb{R}$ be mapping defined as

$$\mathtt{F}(\varrho)=-\frac{1}{\sqrt{\varrho}} \text{ for all } \varrho\in\mathbb{R}^+.$$

Then from (12), for each $U,V\in\mathcal{H}_n^+(M,\preceq)$ with $U\preceq V$ and $\mathcal{G}(U)\preceq\mathcal{G}(V)$, we have

$$\begin{aligned}
\|M(V)-M(U)\|_{tr} &= tr(M(V)-M(U))\\
&= tr\Big(\sum_{i=1}^{m}A_i^*(\mathcal{G}(V)-\mathcal{G}(U))A_i\Big)\\
&= \sum_{i=1}^{m}tr(A_i^*(\mathcal{G}(V)-\mathcal{G}(U)A_i)\\
&= \sum_{i=1}^{m}tr(A_iA_i^*(\mathcal{G}(V)-\mathcal{G}(U)))\\
&= tr\Big(\big(\sum_{i=1}^{m}A_iA_i^*\big)(\mathcal{G}(V)-\mathcal{G}(U))\Big)\\
&\le \Big(\|\sum_{i=1}^{m}A_iA_i^*\|\Big)\|\mathcal{G}(V)-\mathcal{G}(U)\|_{tr}\\
&\le \frac{\|\sum_{i=1}^{m}A_iA_i^*\|}{k}\left(\frac{\|V-U\|_{tr}}{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}\right)\\
&< \left(\frac{\|V-U\|_{tr}}{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}\right)
\end{aligned}$$

and so

$$\frac{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}{\|V-U\|_{tr}}\le\frac{1}{\|M(V)-M(U)\|_{tr}}.$$

This implies that

$$\left(\tau+\frac{1}{\sqrt{\|V-U\|_{tr}}}\right)^2\le\frac{1}{\|M(V)-M(U)\|_{tr}}$$

or

$$\tau+\frac{1}{\sqrt{\|V-U\|_{tr}}}\le\frac{1}{\sqrt{\|M(V)-M(U)\|_{tr}}}.$$

This yields that

$$\tau-\frac{1}{\sqrt{\|M(V)-M(U)\|_{tr}}}\le-\frac{1}{\sqrt{\|V-U\|_{tr}}}.$$

Hence

$$\tau + \mathtt{F}(\|M(V) - M(U)\|_{tr}) \leq \mathtt{F}(\|V - U\|_{tr}),$$

which shows that M is a generalized $\mathtt{F}_{\preceq}$-contraction with respect to $\mathcal{H}_n^+(M, \preceq)$. Since all the assumptions of Theorem 24 are satisfied therefore there exists $Z \in \mathcal{H}_n$ such that $M(Z) = Z$, i.e., the matrix equation (11) has a solution. □

Authors Contribution Statement The authors contributed equally and they read and approved the final copy of the manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgment The authors are very grateful to the anonymous referees for their careful reading and valuable comments, suggestions towards the improvement of this manuscript.

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