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Different Variants of Bernstein Kantorovich Operators and Their Applications in Sciences and Engineering Field

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ABSTRACT In this article, we investigate various Bernstein-Kantorovich variants together with their approximation properties. Nowadays, these variants of Bernstein-Kantorovich operators have been a source of inspiration for researchers as it helps to approximate integral functions also which is not feasible in the case of discrete operators. Chaos theory has also been referred to as complexity theory. Using chaos theory complexity is also reduced as in approximation theory. Thus in order to reduce complexity and to have better understanding of images in sciences and engineering field, sampling Kantorovich operators of approximation theory are widely used in this regard for enhancement of images. Thus, we discuss the important applications of Kantorovich operators depicting pragmatic and theoretical aspects of approximation theory.

KEYWORDS
Bernstein
Kantorovich operators
q-calculus
Lupas-Stancu operators
Polya distribution

INTRODUCTION

The objective of this paper is to highlight the different variants of Bernstein-Kantorovich operators which are widely used for approximation of functions in L^p spaces. The advantage of using Kantorovich variants over discrete operators is that discrete operators are not suitable for approximating functions which are not continuous, therefore these operators were generalized into operators of integral type and one such technique is Kantorovich which helps to approximate integral functions and thus Kantorovich variant of various linear positive operators have been a source of inspiration for many scholars.

Approximation theory is an area of mathematical analysis which is mainly concerned with approximation of complicated quantities by simpler functions. This unique feature of approximation theory forces us to study this field alongwith the study of some ideas of functional anaysis. In (Rashid *et al.* 2022), discrete proportional fraction operators are used to contribute to the major effects of some innovative variants of reverse Minkowski and related H-older-type inequalities. Approximation theory gained popularity with the emergence of Weierstrass theorem (1885). Karl Weierstrass prsesented the first proof of his fundamental theorem on approximation by algebraic and trignometric polynomials but the complications in proof provoked many famous mathematicians to work on this fundamental theorem of approximation theory. Most commonly used proofs of Weierstrass theorem are of Fejer (1900) and Bernstein (1912) as proofs given by other famous mathematicians were not very productive and satisfactory (see e.g. (Bartle 1976; Cheney 1966; Lubinsky 1995; Pinkus 2000). S.N Bernstein (Bernšteın 1912) gave the most simplest and constructive proof of Weierstrass theorem using Bernstein operators $E_{\Theta} : \check{c}\,[0,1] \to \check{c}\,[0,1]$ defined in (Bernšteın 1912) as follows:

$$E_{\Theta}\left(\mathfrak{g}';\varkappa\right) = \sum_{\mathfrak{l}=0}^{\Theta} \begin{pmatrix} \Theta \\ \mathfrak{l} \end{pmatrix} \mathfrak{e}_{\Theta,\mathfrak{l}}\left(\varkappa\right) \mathfrak{g}'\left(\frac{\mathfrak{l}}{\Theta}\right), \tag{1}$$

where $\mathfrak{e}_{\Theta,\mathfrak{l}}\left(\varkappa\right) = \begin{pmatrix} \Theta \\ \mathfrak{l} \end{pmatrix} \varkappa^{\mathfrak{l}}\left(1-\varkappa\right)^{\Theta-\mathfrak{l}}$, $\mathfrak{g}' \in \check{c}\,[0,1]\,, \Theta \geq 1$ and $0 \leq \varkappa \leq 1$.

Bernstein operators are considered as foundational operators due to its immense contribution in the field of approximation theory. These operators occupy prominent position among all linear positive operators because of their efficent and noteable approxi-

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mation properties but its favourable properties gets overshadowed because of slow rate of convergence. Keeping in view these drawbacks, various modifications has been made to these operators using bezier basis (Agrawal *et al.* 2022). These operators cannot be used to approximate the function in integral metrics. In order to overcome this problem, Kantorovich made small modification of Bernstein operators in 1930 (see (Kantorovich 1930)). In that paper the author L.V Kantorovich, introduces modified operators known as nth Bernstein-Kantorovich operators $L_{\Theta} : L^1([0,1]) \to \check{c}[0,1]$ defined by

$$L_{\Theta}(\mathfrak{g}';\varkappa) = \sum_{\mathfrak{l}=0}^{\Theta} (\Theta+1)\left(\int_{\frac{\mathfrak{l}}{\Theta+1}}^{\frac{\mathfrak{l}+1}{\Theta+1}} \mathfrak{g}'(t)\,dt\right)\begin{pmatrix}\Theta\\ \mathfrak{l}\end{pmatrix}\varkappa^{\mathfrak{l}}(1-\varkappa)^{\Theta-\mathfrak{l}}, \tag{2}$$

where $\mathfrak{g}' \in L^1([0,1])$ and $\varkappa \in [0,1]$.

Study of Lebesgue integrable functions in L^1 space became possible due to modification of Bernstein operators by Kantorovich. Moreover, Altomare et al. (Altomare and Campiti 2011), established the approximation properties using Korovkin theorem and investigate the rate of convergence associated with these operators. The idea of Kantorovich modifications of sequence of linear positive operators inspires many other mathematicians to investigate some new operators within approximation theory. Many authors constructed and studied the Kantorovich type modification of some various operators (see e.g. (Agratini 2001; Barbosu 2004; Dogru and Ozalp 2001; Duman *et al.* 2006; Özarslan *et al.* 2008; Kac and Cheung 2001; Karaca 2022)

In this article, we try to give some important information about different variants of Bernstein-Kantorovich operators, hoping that this will act as a beneficial tool for all those reserachers that work in approximation theory and intend to apply Kantorovich technique in order to modify various linear positive operators.

LITERATURE REVIEW

In this section, we review some variants of Bernstein-Kantorovich operators which are being very popularly used in approximation theory. We refer readers to some papers such as (Acu 2015; Agrawal *et al.* 2015; Altomare *et al.* 2013; Barbosu 2004; Bardaro *et al.* 2007; de la Cal and Valle 2000; Deo *et al.* 2016; Gonska *et al.* 2011; Igoz 2012) for generalization of these operators in Kantorovich form.

Let us recall some notations. Throughout this paper, $\check{c}[a,b]$ is the space of all continuous real valued functions on $[a,b]$, $L^p_{[0,1]}$ is the class of all p power integrable functions on the interval $[0,1]$. For basic definitions and results regarding Banach spaces with the proof as well as applications, one may refer (Karaca 2022).

Kantorovich's idea has been applied also to Bernstein operators involving q-calculus. For definition and notations from q-calculus, we refer readers to (Andrews *et al.* 1999; Kac and Cheung 2001). In (Lupas 1987) Lupas introduced modified form of Bernstein operators using q-calculus and explored its approximation and shape-preserving properties. Subsequently, philips in (Phillips 2003) has done q-generalization of Bernstein operators known as Bernstein operators in q-calculus (q-Bernstein operators) defined, for every positive integer Θ and $\mathfrak{g}' \in [0,1]$, by

$$B_{\Theta}(\mathfrak{g}';q;\varkappa) = \sum_{\mathfrak{l}=0}^{\Theta} \mathfrak{g}'\left(\frac{[\mathfrak{l}]}{[\Theta]}\right)\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}\varkappa^{\mathfrak{l}}\prod_{s=0}^{\Theta-\mathfrak{l}-1}(1-q^s\varkappa). \tag{3}$$

and also studied various results including the theorem confirming uniform convergence (korovkin theorem), order of convergence and asymptotic expansion of these operators given by voronovskaya theorem. See (Ostrovska 2016) for similarities and distinctions of operators given by Lupas and Philips.

In (Dalmanoğlu 2007) modification of Bernstein operators in q-calculus using Kantorovich technique is instigated and its approximation properties are satisfied, while in (Radu 2008) Bernstein-Kantorovich operators in q-calculus are extended and their statistical convergence propertis are prepensed.

In (Dalmanoğlu 2007; Radu 2008), the another modification of the Bernstein operators using Kantorovich technique is elucidate in q-calculus, for every $\Theta \in \mathbb{N}, \varkappa \in [0,1]$ and $0 < q < 1$, by

$$K_{\Theta}(\mathfrak{g}';q;\varkappa) = [\Theta+1]\sum_{\mathfrak{l}=0}^{\Theta}\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}\left(\frac{\varkappa}{q}\right)^{\mathfrak{l}}\prod_{s=0}^{\Theta-\mathfrak{l}-1}(1-q^s\varkappa) \times \int_{\frac{[\mathfrak{l}]}{[\Theta+1]}}^{\frac{[\mathfrak{l}+1]}{[\Theta+1]}} \mathfrak{g}'(t)\,d_qt. \tag{4}$$

Subsequently, the study of operators has been intensified by Dalmanog et al. in (Dalmanog *et al.* 2010), where they reconceive the Kantorovich variant of Bernstein operators in q-calculus using the definition of q-integral of Riemann type (see (Marinković *et al.* 2008)) into the operator instead of general q-integral as:

$$B^*_{\Theta}(\mathfrak{g}';q;\varkappa) = [\Theta+1]\sum_{\mathfrak{l}=0}^{\Theta} q^{-\mathfrak{l}}\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}\prod_{s=0}^{\Theta-\mathfrak{l}-1}(1-q^s\varkappa)\int_{\frac{[\mathfrak{l}]}{[\Theta+1]}}^{\frac{[\mathfrak{l}+1]}{[\Theta+1]}} \mathfrak{g}'(t)\,d^{\mathfrak{r}}_qt. \tag{5}$$

The need of redefining the operators arises because the study of statistical convergence of operators $K_{\Theta}(\mathfrak{g}';q;\varkappa)$ to the function $\mathfrak{g}'$ is problematical with the usage of classical q-integral.

Stancu operators are instead object of a modification for various researchers. In (Gadjiev and Ghorbanalizadeh 2010) Gadjiev et al. introduced Bernstein-Stancu type polynomials with shifted knots:

$$S_{\Theta,\alpha,\beta}(\mathfrak{g}';\varkappa) = \left(\frac{\Theta+\beta_2}{\Theta}\right)^{\Theta}\sum_{\mathfrak{r}=0}^{\Theta}\mathfrak{g}'\left(\frac{\mathfrak{r}+\alpha_1}{\Theta+\beta_1}\right)\begin{pmatrix}\Theta\\ \mathfrak{r}\end{pmatrix} \left(\varkappa-\frac{\alpha_2}{\Theta+\beta_2}\right)^{\mathfrak{r}}\left(\frac{\Theta+\alpha_2}{\Theta+\beta_2}-\varkappa\right)^{\Theta-\mathfrak{r}}, \tag{6}$$

where $\frac{\alpha_2}{\Theta+\beta_2} \le \varkappa \le \frac{\Theta+\alpha_2}{\Theta+\beta_2}$ and $\alpha_{\mathfrak{l}}, \beta_{\mathfrak{l}}\,(\mathfrak{l}=1,2)$ are such that $0 \le \alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2$.

Motivated by above operators, *İçöz* in (İçöz 2012) designate a generalized form of Bernstein-Stancu operators using Kantorovich technique under same assumptions as:

$$S^*_{\Theta,\alpha,\beta}(\mathfrak{g}';\varkappa) = (\Theta+\beta_1+1)\left(\frac{\Theta+\beta_2}{\Theta}\right)^{\Theta} \sum_{\mathfrak{r}=0}^{\Theta}\begin{pmatrix}\Theta\\ \mathfrak{r}\end{pmatrix}\left(\varkappa-\frac{\alpha_2}{\Theta+\beta_2}\right)^{\mathfrak{r}}\left(\frac{\Theta+\alpha_2}{\Theta+\beta_2}-\varkappa\right)^{\Theta-\mathfrak{r}}\int_{\frac{\mathfrak{r}+\alpha_1}{\Theta+\beta_1+1}}^{\frac{\mathfrak{r}+\alpha_1+1}{\Theta+\beta_1+1}}\mathfrak{g}'(s)\,ds. \tag{7}$$

To show the extend of research in direction of q-calculus, we mention the work done by Muraru in (Muraru 2011). She bring forward the q analogue of Bernstein-Schurer operators that are given by

$$E_{\Theta,p}\left(\mathfrak{g}';q,\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta+p}p_{\Theta,\mathfrak{l}}\left(q,\varkappa\right)\mathfrak{g}'\left(\frac{[\mathfrak{l}]_q}{[\Theta]_q}\right), \tag{8}$$

where $p_{\Theta,\mathfrak{l}}\left(q,\varkappa\right)=\begin{bmatrix}\Theta+p\\ \mathfrak{l}\end{bmatrix}\varkappa^{\mathfrak{l}}\left(1-\varkappa\right)_q^{\Theta+p-\mathfrak{l}},\varkappa\in[0,1]$ and prepense various results involving necessary and sufficient condition for convergence of operators given by the Korovkin and Voronovskaja theorem concerning asymptotic convergence of linear positive operators.

Kantorovich modification of the operators (8) have been introduced by Ozarsland et al. in (Özarslan and Vedi 2013) defined, for every $\mathfrak{g}'\in\mathfrak{C}\left[0,p+1\right],0<q<1$ and $p\in\Theta_0=\{0,1,2,...\}$, by

$$K_{\Theta}^{p}\left(\mathfrak{g}';q;\varkappa\right)=\sum_{\mathfrak{r}=0}^{\Theta+p}\begin{bmatrix}\Theta+p\\ \mathfrak{r}\end{bmatrix}\varkappa^{\mathfrak{r}}\prod_{s=0}^{\Theta+p-\mathfrak{r}-1}\left(1-q^s\varkappa\right)\times$$

$$\int_0^1\mathfrak{g}'\left(\frac{[\mathfrak{r}]}{[\Theta+1]}+\frac{1+(q-1)\,[\mathfrak{r}]}{[\Theta+1]}t\right)d_qt. \tag{9}$$

Ren et al. modified the operators (8) in (Ren and Zeng 2013). A new variant of Bernstein-Schurer operators in q-calculus are given by

$$\widetilde{\mathfrak{e}}_{\Theta,p}\left(\mathfrak{g}';q;\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta+p}\widetilde{p}_{\Theta,\mathfrak{l}}^{*}\left(q,\varkappa\right)\mathfrak{g}'\left(\frac{[\mathfrak{l}]_q}{[\Theta]_q}\right), \tag{10}$$

where $\widetilde{p}_{\Theta,\mathfrak{l}}^{*}\left(q,\varkappa\right)=\frac{[\Theta]_q^{\Theta+p}}{[\Theta+p]_q^{\Theta+p}}\begin{bmatrix}\Theta+p\\ \mathfrak{l}\end{bmatrix}_q\varkappa^{\mathfrak{l}}\left(\frac{[\Theta+p]_q}{[\Theta]_q}-\varkappa\right)_q^{\Theta+p-\mathfrak{l}}$ and $\varkappa\in[0,1]$.

Authors investigated compulsory Korovkin type statistical convergence theorem for uniform convergence, Voronovskaja theorem concerning asymptotic convergence, the rate of statistical convergence using various tools such as modulus of continuity and a Lipschitz function for the operators in (Ren and Zeng 2013).

Agrawal et al. in (Agrawal *et al.* 2015) present a stancu variant of operators (8) using Kantorovich technique defined, for every $\mathfrak{g}'\in C\left[0,1+p\right]$ endowed with the norm $\|\mathfrak{g}'\|=\sup_{\varkappa\in[0,1]}\left|\mathfrak{g}'\left(\varkappa\right)\right|,\alpha,\beta\in\mathbb{R}$ such that $0\leq\alpha\leq\beta$ and $0<q<1$, by

$$\kappa_{\Theta,p}^{(\alpha,\beta)}\left(\mathfrak{g}';q;\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta+p}\widetilde{p}_{\Theta,\mathfrak{l}}^{*}\left(q,\varkappa\right)\int_0^1\mathfrak{g}'\left(\frac{[\mathfrak{l}]_q+q^{\mathfrak{l}}t+\alpha}{[\Theta+1]_q+\beta}\right)d_qt. \tag{11}$$

Kantorovich (Kantorovich 1930) gave the integral modification of Bernstein operators so as to approximate integrable functions defined on $[0,1]$. In (Ozarslan and Duman 2016) Özarslan et al. introduced modified Kantorovich operators based on non-negative parameter ρ as:

$$K_{\Theta,\rho}\left(\mathfrak{g}';\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta}\mathfrak{e}_{\Theta,\mathfrak{l}}\left(\varkappa\right)\int_0^1\mathfrak{g}'\left(\frac{\mathfrak{l}+t^{\rho}}{\Theta+1}\right)dt. \tag{12}$$

and investigated the order of convergence and various approximation properties of these operators using various mathematical tools concerning smoothness of operators. In that paper, authors also showed that modified Kantorovich operators based on non-negative parameter ρ depicts faster rate of convergence to a function than that of Kantorovich operators in classical form.

Another new general approach is considered by Mursaleen et al. (Mursaleen *et al.* 2015). In that paper authors prepensed another modified form of Bernstein operators in ((p,q)-calculus) known as (p,q)-Bernstein operators defined, for every $\varkappa\in[0,1],0<p<q\leq1$, by

$$B_{\Theta,p,q}\left(\mathfrak{g}';\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta}\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}_{p,q}\varkappa^{\mathfrak{l}}\prod_{s=0}^{\Theta-\mathfrak{l}-1}\left(p^s-q^s\varkappa\right)\mathfrak{g}'\left(\frac{[\mathfrak{l}]_{p,q}}{[\Theta]_{p,q}}\right). \tag{13}$$

These operators turned into classical Bernstein operators in q-calculus for p=1.

For notations used in operators (13), we refer reader to (Mursaleen *et al.* 2015). Also details on (p,q)-calculus can be found in (Hounkonnou *et al.* 2013; Katriel and Kibler 1992; Sahai and Yadav 2007).

Afterward, Mursaleen et al. (Mursaleen *et al.* 2016) constructed of Bernstein-Kantorovich operators in ((p,q)-calculus) as:

$$K_{\Theta}^{(p,q)}\left(\mathfrak{g}';\varkappa\right)=\frac{[\Theta]_{p,q}}{p^{\frac{\Theta(\Theta-1)}{2}}}\sum_{\mathfrak{l}=0}^{\Theta}\frac{b_{\Theta,\mathfrak{l}}^{(p,q)}\left(\varkappa\right)}{p^{\Theta-\mathfrak{l}}q^{\mathfrak{l}}}\int_{\frac{[\mathfrak{l}]_{p,q}}{p^{\mathfrak{l}-\Theta-1}[\Theta]_{p,q}}}^{\frac{[\mathfrak{l}+1]_{p,q}}{p^{\mathfrak{l}-\Theta}[\Theta]_{p,q}}}\mathfrak{g}'\left(t\right)d_pt, \tag{14}$$

where $\varkappa\in[0,1],b_{\Theta,\mathfrak{l}}^{(p,q)}\left(\varkappa\right)=\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}_{p,q}\left(\varkappa\right)_{p,q}^{\mathfrak{l}}\left(1-\varkappa\right)_{p,q}^{\Theta-\mathfrak{l}}$ and $\left(\varkappa\right)_{p,q}^{\mathfrak{l}}=\varkappa\left(px\right)\left(p^2\varkappa\right).......\left(p^{\mathfrak{l}-1}\varkappa\right)=p^{\frac{\mathfrak{l}(\mathfrak{l}-1)}{2}}\varkappa^{\mathfrak{l}}$.

Moreover, authors study the local approximation property of $K_{\Theta}^{(p,q)}\left(\mathfrak{g}';\varkappa\right)$ and obtain faster rate of convergence and better error estimates of operators as compared to Bernstein-Kantorovich operators in q-calculus.

Realising the essentials of Bernstein-Stancu, Mursaleen el at. in (Mursaleen *et al.* 2017) introduce another variant of Bernstein-Stancu in q-calculus using Kantorovich technique as follows:

$$K_{\Theta,q}^{(\alpha,\beta)}=\left(\frac{[\Theta]+\beta_2}{[\Theta]}\right)^{\Theta}\sum_{\mathfrak{l}=0}^{\Theta}\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}\left(\varkappa-\frac{\alpha_2}{[\Theta]+\beta_2}\right)_q^{\mathfrak{l}}$$

$$\times\left(\frac{[\Theta]+\alpha_2}{[\Theta]+\beta_2}-\varkappa\right)_q^{\Theta-\mathfrak{l}}\int_0^1\mathfrak{g}'\left(\frac{[\mathfrak{l}]\,q^{\mathfrak{l}}t+\alpha_1}{[\Theta+1]+\beta_1}\right)d_qt. \tag{15}$$

New Kantorovich-type operators based on P̀olya-Eggenberger distribution (Eggenberger and Pólya 1923) are introduced by Kajla et al. in (Kajla and Araci 2017) as:

$$K_{\Theta,\rho}^{[\alpha]}\left(\mathfrak{g}';\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta}p_{\Theta,\mathfrak{l}}^{[\alpha]}\left(\varkappa\right)\int_0^1\mathfrak{g}'\left(\frac{\mathfrak{l}+t^{\rho}}{\Theta+1}\right)dt, \tag{16}$$

where $\rho>0$ and $p_{\Theta,\mathfrak{l}}^{[\alpha]}\left(\varkappa\right)=\binom{\Theta}{\mathfrak{l}}\frac{1}{1^{[\Theta,-\alpha]}}\chi^{\mathfrak{l},-\alpha}\left(1-\chi\right)^{[\Theta-\mathfrak{l},-\alpha]}$.

In (Acu *et al.* 2018) Acu et al. study a new type of Bernstein operators depending on the parameter $\lambda\in[-1,1]$ proposed by Cai et al. (Cai *et al.* 2018) as follows:

$$B_{\Theta,\lambda}\left(\mathfrak{g}';\varkappa\right)=\sum_{\mathfrak{l}=0}^{\Theta}\widetilde{\mathfrak{e}}_{\Theta,\mathfrak{l}}\left(\lambda;\varkappa\right)\mathfrak{g}'\left(\frac{\mathfrak{l}}{\Theta}\right), \tag{17}$$

where $\widetilde{\mathfrak{e}}_{\Theta,0}\left(\lambda;\varkappa\right),\mathfrak{l}=0,1,2,3.....$, are defined as:

$$\widetilde{\mathfrak{e}}_{\Theta,0}\left(\lambda;\varkappa\right)=\mathfrak{e}_{\Theta,0}\left(\varkappa\right)-\frac{\lambda}{\Theta+1}\mathfrak{e}_{\Theta+1,1}\left(\varkappa\right),$$

$$\widetilde{\mathfrak{e}}_{\Theta,\mathfrak{l}}(\lambda;\varkappa)=\mathfrak{e}_{\Theta,\mathfrak{l}}(\varkappa)+\lambda\left(\frac{\Theta-2k+1}{\Theta^2-1}\mathfrak{e}_{\Theta+1,\mathfrak{l}}(\varkappa)-\frac{\Theta-2k-1}{\Theta^2-1}\mathfrak{e}_{\Theta+1,\mathfrak{l}+1}(\varkappa)\right)$$

$$\widetilde{\mathfrak{e}}_{\Theta,\Theta}(\lambda;\varkappa)=\mathfrak{e}_{\Theta,\Theta}(\varkappa)-\frac{\lambda}{\Theta+1}\mathfrak{e}_{\Theta+1,\Theta}(\varkappa).$$

Moreover, Ana et al. considered a Kantorovich modification of λ-Bernstein operators, namely

$$K_{\Theta,\lambda}(\mathfrak{g}';\varkappa)=(\Theta+1)\sum_{\mathfrak{l}=0}^{\Theta}\widetilde{\mathfrak{e}}_{\Theta,\mathfrak{l}}(\varkappa)\int_{\frac{\mathfrak{l}}{\Theta+1}}^{\frac{\mathfrak{l}+1}{\Theta+1}}\mathfrak{g}'(t)\,dt. \tag{18}$$

In particular, in (Acu *et al.* 2018) (see Example 4) authors obtained better error estimate for λ-Bernstein operators as compared to classical Kantorovich operators.

In (Chen *et al.* 2017) Chen et al. studied another modification of Bernstein operators, depending on a non-negative real parameter α (α-Bernstein operators) defined, for every $\mathfrak{g}'\in\mathfrak{C}[0,1],\Theta\geq 2$ and $0\leq\varkappa\leq 1$, by

$$T_{\Theta}^{\alpha}(\mathfrak{g}';\varkappa)=\sum_{\mathfrak{l}=0}^{\Theta}p_{\Theta,\mathfrak{l}}^{(\alpha)}(\varkappa)\,\mathfrak{g}'\left(\frac{\mathfrak{l}}{\Theta}\right), \tag{19}$$

where

$$p_{\Theta,\mathfrak{l}}^{(\alpha)}(\varkappa)=\left[\binom{\Theta-2}{\mathfrak{l}}(1-\alpha)\varkappa+\binom{\Theta-2}{\mathfrak{l}-2}(1-\alpha)(1-\varkappa)+\binom{\Theta}{\mathfrak{l}}\alpha\varkappa(1-\varkappa)\right]\varkappa^{\mathfrak{l}-1}(1-\varkappa)^{\Theta-\mathfrak{l}-1}$$

The authors also proved the degree of convergence, Voronovskaja theorem concerning asymptotic formula and shape preserving properties for operator(19).

In (Mohiuddine *et al.* 2017) Mohiuddine et al. proposed and investigated the kantorovich type modification of operators (19) defined, for for every $\mathfrak{g}'\in\mathfrak{C}[0,1],\Theta\geq 2$ and $0\leq\varkappa\leq 1$, by

$$\widehat{\mathfrak{l}}_{\Theta,\alpha}(\mathfrak{g}';\varkappa)=(\Theta+1)\sum_{s=0}^{\Theta}p_{\Theta,s}^{(\alpha)}\int_{\frac{s}{\Theta+1}}^{\frac{s+1}{\Theta+1}}\mathfrak{g}'(t)\,dt. \tag{20}$$

Araci et al. also proved various approximation properties with the help of Bohman-Korovkin's principle which is necessary and sufficient criteria for uniform convergence and used various tools such as the modulus of smoothness and Lipschitz type function to study the approximation rate of operators. They also derived Voronovskaja type asymptotic convergence theorem and Korovkin type A-statistical approximation theorem of these operators.

A short time ago, the Kantorovich modification of the operators (19) by Araci et al. is defined in (Araci *et al.* 2019), for every $\mathfrak{g}'\in\mathfrak{C}[0,1],\alpha>0$ and $\rho>0$, by

$$\kappa_{m,\rho}^{\alpha,a}(\mathfrak{g}';\varkappa)=\sum_{\mathfrak{l}=0}^{\Theta}p_{\Theta,\mathfrak{l}}^{(\alpha)}(\varkappa)\int_0^1 f\left(\frac{\mathfrak{l}+at^{\rho}}{\Theta+a}\right)dt, \tag{21}$$

where $\varkappa\in[0,1]$ and $p_{\Theta,\mathfrak{l}}^{(\alpha)}$ is defined above.

Afterward, α, q-Bernstein operators by Cai et al. in (Cai and Xu 2018) are presented as:

$$\mathfrak{T}_{\Theta,q,\alpha}(\mathfrak{g}';\varkappa)=\sum_{\mathfrak{k}=0}^{\Theta}p_{\Theta,q,\mathfrak{k}}^{(\alpha)}f\left(\frac{[\mathfrak{k}]_q}{[\Theta]_q}\right), \tag{22}$$

where $\alpha\in[0,1],q\in(0,1),\varkappa\in[0,1],\mathfrak{g}'\in\mathfrak{C}[0,1]$ and

$$p_{1,q,0}^{(\alpha)}(\varkappa)=1-\varkappa,\ p_{1,q,1}^{(\alpha)}(\varkappa)=\varkappa,$$

$$p_{\Theta,q,\mathfrak{k}}^{(\alpha)}(\varkappa)=\left(\begin{bmatrix}\Theta-2\\ \mathfrak{l}\end{bmatrix}_q(1-\alpha)\varkappa+\begin{bmatrix}\Theta-2\\ \mathfrak{l}-2\end{bmatrix}_q(1-\alpha)q^{\Theta-\mathfrak{k}-2}\left(1-q^{\Theta-\mathfrak{k}-1}\varkappa\right)+\begin{bmatrix}\Theta\\ \mathfrak{l}\end{bmatrix}_q\times\alpha\varkappa\left(1-q^{\Theta-\mathfrak{k}-1}\varkappa\right)\right)\varkappa^{\mathfrak{l}-1}(1-\varkappa)_q^{\Theta-\mathfrak{l}-1},\Theta\geq 2$$

Subsequently, Cai et al. establish a more general approach to Kantorovich operators known as bivariate α, q-Bernstein-Kantorovich operators in (Cai *et al.* 2019) as:

$$\kappa_{\mathfrak{m}_1,\mathfrak{m}_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(\mathfrak{g}';\varkappa;\mathfrak{s})=[\Theta_1+1]_{q_1}[\Theta_2+1]_{q_2}\times\sum_{\mathfrak{k}_1=0}^{\Theta_1}\sum_{\mathfrak{k}_2=0}^{\Theta_2}p_{\Theta_1,q_1,\mathfrak{k}_1}^{(\alpha_1)}(\varkappa)\,p_{\Theta_2,q_2,\mathfrak{k}_2}^{(\alpha_2)}(\mathfrak{s})\,q_1^{-\mathfrak{k}_1}q_1^{-\mathfrak{k}_2}\times\int_{\frac{[\mathfrak{k}_1]_{q_1}}{[\Theta_1+1]_{q_1}}}^{\frac{[\mathfrak{k}_1+1]_{q_1}}{[\Theta_1+1]_{q_1}}}\int_{\frac{[\mathfrak{k}_2]_{q_2}}{[\Theta_2+1]_{q_2}}}^{\frac{[\mathfrak{k}_2+1]_{q_2}}{[\Theta_2+1]_{q_2}}}\mathfrak{g}'(t;u)\,dq_1td q_2u,$$

where $\varkappa,\mathfrak{s}\in[0,1],\mathfrak{g}'\in\mathfrak{C}([0,1]\times[0,1]),0<q_1<q_2<1$ and $\alpha_1,\alpha_2\in[0,1]$.

APPLICATIONS

Approximation Theory is rigorous branch of study, developed in different directions by mathematicians. It's centrality in the development of many area of mathematics and its diverse application in Sciences and Engineering field makes it attractive field of study and research. Approximation Theory has two aspects:

- Pragmatic: Which is concerned largely with computational practicalities.
- Theoretical: Which is more concerned with applications to theoretical issues.

Sampling Kantorovich operators are concerned more with pragmatic aspects. In (Bardaro *et al.* 2007) the authors introduced the sampling Kantorovich operators and studied their convergence in the general setting of orlicz spaces in the one-dimensional space. Afterward, the results to multivariate setting have been extended in (Costarelli and Vinti 2011), to the nonlinear case in (Costarelli and Vinti 2013; Vinti and Zampogni 2009; Bardaro and Mantellini 2012) in a more general context. Results regarding the order of approximation of these operators are shown in (Costarelli *et al.* 2014b). In (Cluni *et al.* 2013) the authors obtain application to civil engineering by using multivariate sampling Kantorovich operators $(S_\omega)_{\omega>0}$, defined by:

$$(S_\omega\mathfrak{g}')(\underline{\varkappa})=\sum_{\mathfrak{l}\in\mathbb{Z}^{\Theta}}\chi(\omega\underline{\varkappa}-t_{\underline{\mathfrak{l}}})\left[\frac{\omega^{\Theta}}{A_{\underline{\mathfrak{l}}}}\int_{R_{\underline{\mathfrak{l}}}^{\omega}}\mathfrak{g}'(\underline{u})\,d\underline{u}\right], \tag{23}$$

where ($\underline{\varkappa}\in\mathbb{R}^{\Theta}$) and $\mathfrak{g}':\mathbb{R}^{\Theta}\to\mathbb{R}$ is locally integrable function.

For notations and more details about operators, we refer reader to (Cluni *et al.* 2013). In that paper, authors realized the importance of sampling Kantorovich operators to seismic engineering by demonstrating that structural analysis using sampling Kantorovich operators produce clear understanding of masonry texture, the geometry of buildings and possible structural damage, which further helps to estimate seismic risk of structure. The models for the reproduction of the behaviour of structures under seismic action and comparison of behaviour of the building using various models are obtained in (Costarelli *et al.* 2014a). In (Cheney 1966) a real world case study in terms of structural analysis is analyzed. For better understanding readers can refer Figure in (Cluni *et al.* 2013)

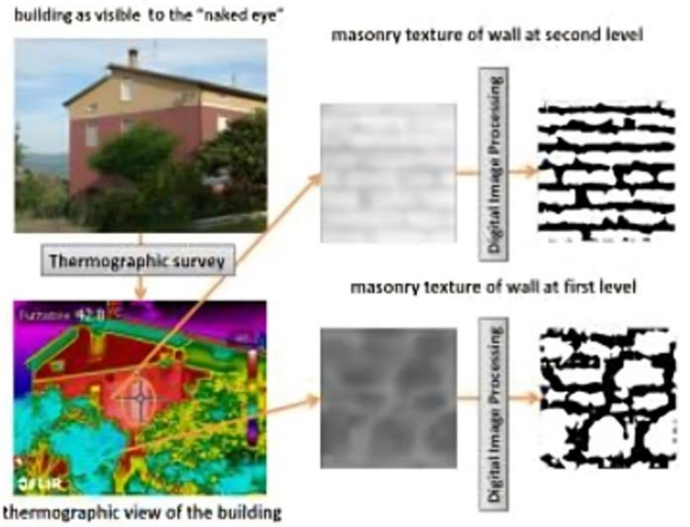
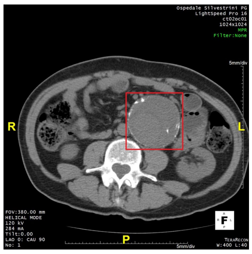


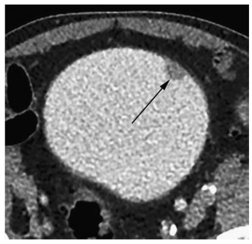
Figure 1 Image depicting importance of Sampling Kantorovich operators in structural analysis of building.

Sampling Kantorovich operators are very useful in sampling and signal theories. Cluni et al. in (Costarelli *et al.* 2014a) also recognized that the use of sampling Kantorovich operators in approximating discontinuous signals is predominant as it reduces "time-jitter" errors. Angeloni et al. in (Angeloni *et al.* 2005) developed that sampling Kantorovich operators represents an approximate version of classical sampling series, based on Whittaker-Kotelnikov-Shannon sampling theorem.

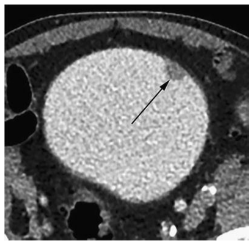
In (Costarelli and Vinti 2014), Costarelli et al. presented an application of sampling Kantorovich operators to digital image processing (D.I.P) and also studied the various results regarding the convergence of sampling Kantorovich operators. In that paper, the various usage of the D.I.P technique are discussed from mathematical and medical point of view. Moreover, some new applications are obtained by considering biomedical images. A concrete example is showed in (Costarelli and Vinti 2014) and deduce that enhancement of images using sampling Kantorovich operators are very useful from medical point of view as it allows doctor to perform a better diagnosis. For better understanding one can refer images below (Costarelli *et al.* 2014a)



(a) CT image without contrast medium. In the red square is depicted the aorta artery



(b) Enhanced by the sampling Kantorovich operators $S_{20}I$ based upon a bivariate Jackson type kernel



(c) ROI of the CT image without contrast medium of (a), depicted the aorta artery

Figure 2 Processing a portion of a CT (computer tomography) image depicting the aorta artery using Sampling Kantorovich operators.

In (Karaca *et al.* 2019) Karaca et al. main contribution was to provide a unique method for assessing key stroke subtypes' features using a mobile phone connected to a cloud system depicting huge progress in innovative healthcare technologies that rendered healthcare data bigger.

Angeloni et al. (Angeloni *et al.* 2020) examined the convergence properties of a family of multidimensional sampling Kantorovich type operators. Besides that, quantitative estimates, order of approximation and Voronovskaja type asymptotic convergence theorem have been established. Very recently, Bawa et al. (Bawa *et al.* 2022) elucidate the approximation properties of a Kantorovich-Lupaş-Stancu operators based on Pòlya distribution.

CONCLUSION

We conclude that approximation theory is an intensive research area that is widely used in sciences and engineering field. In this paper, we have discussed different variants of Bernstein Kantorovich operators showing their pragmatic and theoretical applicatons in different areas of research. The importance of approximation theory in numerous scientific fields makes it one of the most active research areas. The theory is relevant to both engineering and mathematical fields, such as constructive approximation of functions, partial and integral equation solutions, machine learning and image processing.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

LITERATURE CITED

Acu, A. M., 2015 Stancu–Schurer–Kantorovich operators based on q-integers. Applied Mathematics and Computation **259**: 896–907.

Acu, A.-M., N. Manav, and D. F. Sofonea, 2018 Approximation properties of λ-Kantorovich operators. Journal of inequalities and applications **2018**: 1–12.

Agratini, O., 2001 An approximation process of Kantorovich type. Miskolc Mathematical Notes **2**: 3–10.

Agrawal, P., N. Bhardwaj, and P. Bawa, 2022 Bézier variant of modified α-bernstein operators. Rendiconti del Circolo Matematico di Palermo Series 2 **71**: 807–827.

Agrawal, P., M. Goyal, and A. Kajla, 2015 q-Bernstein-Schurer-Kantorovich type operators. Bollettino dell'Unione Matematica Italiana **8**: 169–180.

Altomare, F. and M. Campiti, 2011 *Korovkin-type approximation theory and its applications*, volume 17. Walter de Gruyter.

Altomare, F., M. C. Montano, and V. Leonessa, 2013 On a generalization of Szász–Mirakjan–Kantorovich operators. Results in Mathematics **63**: 837–863.

Andrews, G. E., R. Askey, and R. Roy, 1999 *Special functions*, volume 71. Cambridge university press.

Angeloni, L., D. Costarelli, and G. Vinti, 2020 Approximation properties of mixed sampling-Kantorovich operators. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **115**: 1–14.

Angeloni, L., G. Vinti, *et al.*, 2005 Rate of approximation for nonlinear integral operators with application to signal processing. Differential and Integral Equations **18**: 855–890.

Araci, S., A. Kajla, and P. Agarwal, 2019 A Kantorovich variant of a generalized Bernstein operators .

Barbosu, D., 2004 Kantorovich-Stancu type operators. J. Inequal. Pure Appl. Math **5**: 6.

Bardaro, C. and I. Mantellini, 2012 On convergence properties for a class of Kantorovich discrete operators. Numerical functional analysis and optimization **33**: 374–396.

Bardaro, C., G. Vinti, P. Butzer, and R. Stens, 2007 Kantrovich-type generalized sampling series in the setting of orlicz spaces. Sampling Theory in Signal and Image Processing **6**: 29.

Bartle, R. G., 1976 The element of real analysis, john willy & sons. Inc., New York .

Bawa, P., N. Bhardwaj, and P. Agrawal, 2022 Quantitative voronovskaya type theorems and gbs operators of kantorovich variant of lupaş-stancu operators based on pólya distribution. Mathematical Foundations of Computing **5**: 269–293.

Bernšteın, S., 1912 Démonstration du théoreme de Weierstrass fondée sur le calcul des probabilities. Comm. Soc. Math. Kharkov **13**: 1–2.

Cai, Q.-B., W.-T. Cheng, and B. Çekim, 2019 Bivariate α, q-Bernstein–Kantorovich operators and gbs operators of bivariate α, q-bernstein–kantorovich type. Mathematics **7**: 1161.

Cai, Q.-B., B.-Y. Lian, and G. Zhou, 2018 Approximation properties of λ-Bernstein operators. Journal of Inequalities and Applications **2018**: 1–11.

Cai, Q.-B. and X.-W. Xu, 2018 Shape-preserving properties of a new family of generalized Bernstein operators. Journal of inequalities and applications **2018**: 1–14.

Chen, X., J. Tan, Z. Liu, and J. Xie, 2017 Approximation of functions by a new family of generalized Bernstein operators. Journal of Mathematical Analysis and Applications **450**: 244–261.

Cheney, E. W., 1966 Introduction to approximation theory .

Cluni, F., D. Costarelli, A. M. Minotti, and G. Vinti, 2013 Multivariate sampling Kantorovich operators: approximation and applications to civil engineering. EURASIP, Proc. SampTA pp. 400–403.

Costarelli, D., F. Cluni, A. M. Minotti, and G. Vinti, 2014a Applications of sampling Kantorovich operators to thermographic images for seismic engineering. arXiv preprint arXiv:1411.2584 .

Costarelli, D. and G. Vinti, 2011 Approximation by multivariate generalized sampling Kantorovich operators in the setting of orlicz spaces. Bollettino dell'Unione Matematica Italiana **4**: 445–468.

Costarelli, D. and G. Vinti, 2013 Approximation by nonlinear multivariate sampling Kantorovich type operators and applications to image processing. Numerical Functional Analysis and Optimization **34**: 819–844.

Costarelli, D. and G. Vinti, 2014 Sampling Kantorovich operators and their applications to approximation problems and to digital image processing. In *Proceedings of 8th international conference on applied mathematics, simulation, modelling (ASM'14), Florence, Italy November*, pp. 22–24.

Costarelli, D., G. Vinti, *et al.*, 2014b Order of approximation for sampling Kantorovich operators. Journal of Integral Equations and Applications **26**: 345–367.

Dalmanog, Ö., O. Dog, *et al.*, 2010 On statistical approximation properties of Kantorovich type q-Bernstein operators. Mathematical and Computer Modelling **52**: 760–771.

Dalmanoğlu, Ö., 2007 Approximation by Kantorovich type q-Bernstein operators .

de la Cal, J. and A. M. Valle, 2000 A generalization of Bernstein–Kantorovič operators. Journal of mathematical analysis and applications **252**: 750–766.

Deo, N., M. Dhamija, and D. Miclăuş, 2016 Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution. Applied Mathematics and Computation **273**: 281–289.

Dogru, O. and N. Ozalp, 2001 Approximation by Kantorovich type generalization of meyer-konig and zeller operators. Glasnik matematički **36**: 311–318.

Duman, O., M. Özarslan, and O. Doğru, 2006 On integral type generalizations of positive linear operators. Studia Mathematica **174**: 1–12.

Eggenberger, F. and G. Pólya, 1923 Über die statistik verketteter vorgänge. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik **3**: 279–289.

Gadjiev, A. and A. Ghorbanalizadeh, 2010 Approximation properties of a new type Bernstein–Stancu polynomials of one and two variables. Applied mathematics and computation **216**: 890–901.

Gonska, H., M. Heilmann, and I. Raşa, 2011 Kantorovich operators of order k. Numerical functional analysis and optimization **32**: 717–738.

Hounkonnou, M. N., J. Désiré, and B. Kyemba, 2013 R (p, q)-calculus: differentiation and integration. SUT J. Math **49**: 145–167.

İçöz, G., 2012 A Kantorovich variant of a new type Bernstein–Stancu polynomials. Applied Mathematics and Computation **218**: 8552–8560.

Igoz, G., 2012 A kantorovich variant of a new type bernstein-stancu polynomails. AppI Math Comput **218**: 8552–8560.

Kac, V. and P. Cheung, 2001 *Quantum calculus*. Springer Science & Business Media.

Kajla, A. and S. Araci, 2017 Blending type approximation by Stancu-Kantorovich operators based on pólya-eggenberger distribution. Open Physics **15**: 335–343.

Kantorovich, L., 1930 Sur certains développements suivant les polynômes de la forme de s. Bernstein, I, II, CR Acad. URSS **563**: 568.

Karaca, Y., 2022 Global attractivity, asymptotic stability and blow-up points for nonlinear functional-integral equations'solutions and applications in banach space bc (r+) with computational complexity. Fractals **30**: 2240188.

Karaca, Y., M. Moonis, Y.-D. Zhang, and C. Gezgez, 2019 Mobile cloud computing based stroke healthcare system. International Journal of Information Management **45**: 250–261.

Katriel, J. and M. Kibler, 1992 Normal ordering for deformed boson operators and operator-valued deformed stirling numbers. Journal of Physics A: Mathematical and General **25**: 2683.

Lubinsky, D., 1995 Weierstrass'theorem in the twentieth century: A selection. Quaestiones Mathematicae **18**: 91–130.

Lupas, A., 1987 A q-analogue of the Bernstein operator. In *Seminar on numerical and statistical calculus, University of Cluj-Napoca*, volume 9.

Marinković, S., P. Rajković, and M. Stanković, 2008 The inequalities for some types of q-integrals. Computers & Mathematics with Applications **56**: 2490–2498.

Mohiuddine, S., T. Acar, and A. Alotaibi, 2017 Construction of a new family of Bernstein-Kantorovich operators. Mathematical Methods in the Applied Sciences **40**: 7749–7759.

Muraru, C.-V., 2011 Note on q-Bernstein-Schurer operators. Stud. Univ. Babes-Bolyai Math **56**: 489–495.

Mursaleen, M., K. J. Ansari, and A. Khan, 2015 On (p, q)-analogue of Bernstein operators. Applied Mathematics and Computation **266**: 874–882.

Mursaleen, M., K. J. Ansari, and A. Khan, 2016 Some approximation results for Bernstein-Kantorovich operators based on (p, q)-calculus. UPB Sci. Bull., Ser. A **78**: 129–142.

Mursaleen, M., K. J. Ansari, and A. Khan, 2017 Approximation by kantorovich type q-Bernstein-Stancu operators. Complex Analysis and Operator Theory **11**: 85–107.

Ostrovska, S., 2016 The q-versions of the Bernstein operator: from mere analogies to further developments. Results in Mathematics **69**: 275–295.

Ozarslan, M. A. and O. Duman, 2016 Smoothness properties of modified Bernstein-Kantorovich operators .

Özarslan, M. A., O. Duman, and H. Srivastava, 2008 Statistical approximation results for Kantorovich-type operators involving some special polynomials. Mathematical and computer modelling **48**: 388–401.

Özarslan, M. A. and T. Vedi, 2013 q-Bernstein-Schurer-Kantorovich operators. Journal of Inequalities and Applications **2013**: 444.

Phillips, G. M., 2003 Bernstein polynomials. In *Interpolation and Approximation by Polynomials*, pp. 247–290, Springer.

Pinkus, A., 2000 Weierstrass and approximation theory. Journal of Approximation Theory **107**: 1–66.

Radu, C., 2008 Statistical approximation properties of Kantorovich operators based on q-integers, creat. math. Inform **17**: 75–84.

Rashid, S., S. Sultana, Y. Karaca, A. Khalid, and Y.-M. Chu, 2022 Some further extensions considering discrete proportional fractional operators. Fractals **30**: 2240026.

Ren, M.-Y. and X.-M. Zeng, 2013 On statistical approximation properties of modified q-Bernstein-Schurer operators. Bulletin of the Korean Mathematical Society **50**: 1145–1156.

Sahai, V. and S. Yadav, 2007 Representations of two parameter quantum algebras and p, q-special functions. Journal of mathematical analysis and applications **335**: 268–279.

Vinti, G. and L. Zampogni, 2009 Approximation by means of nonlinear Kantorovich sampling type operators in orlicz spaces. Journal of Approximation Theory **161**: 511–528.

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