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# On the Bézier Variant of the Srivastava-Gupta Operators

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ABSTRACT. In the present paper, we introduce the Bézier variant of the Srivastava-Gupta operators, which preserve constant as well as linear functions. Our study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness, respectively the rate of convergence for differentiable functions whose derivatives are of bounded variation.

**Keywords:** Srivastava-Gupta operators, Genuine operators, Hypergeometric series, Rate of convergence, Bounded variation.

**2010 Mathematics Subject Classification:** 26A15, 41A25, 41A35.

## 1. INTRODUCTION

Srivastava-Gupta [19] presented the following summation-integral type operators defined as follows:

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}f(t)dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x),$$

with the following special cases:

(1) If $c=0$ and $\phi_{n,c}(x)=e^{-nx}$, then we get $p_{n,k}(x,0)=e^{-nx}\dfrac{(nx)^k}{k!}$,

(2) $c=\mathbb{N}$ and $\phi_{n,c}(x)=(1+cx)^{-n/c}$, then we obtain $p_{n,k}(x,0)=\dfrac{(n/c)_k}{k!}\dfrac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}$,

(3) If $c=-1$ and $\phi_{n,c}(x)=(1-x)^n$, then $p_{n,k}(x,-1)=\binom{n}{k}x^k(1-x)^{n-k}$.

Gupta [12] introduced the general class of Durrmeyer type operators and studied some direct results. In [16], the authors considered the Bézier variant of the operators (1.1) and established the estimate of the rate of convergence of these operators for functions of bounded variation. Kajla and Acar [17] constructed mixed hybrid operators and established quantitative Voronovskaja type theorems, local approximation theorems and weighted approximation properties for these operators. Verma and Agrawal [23] presented the generalized form of the operators (1.1) and obtained some approximation properties for these operators. Acar et al. [3] proposed Stancu type generalization of the operators (1.1) and studied the rate of convergence for functions having derivatives of bounded variation and also discussed the simultaneous approximation for these operators. Recently, Neer et al. [18] introduced the Bézier variant of the

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operators which is proposed by Yadav [22] and obtained several approximation properties.

Gupta [11] introduced a modification of the operators (1.1) as

$$U_{n,c}(f;x) = (n+2c)\sum_{k=1}^{\infty} p_{n+c,k}(x,c)\int_0^{\infty} p_{n+3c,k-1}(t,c)f(t)dt + p_{n+c,0}(x,c)f(0). \tag{1.2}$$

It is important to note here that these operators preserve constant as well as linear functions. The $r^{th}(r\in\mathbb{N})$ order moments are given by

$$U_{n,c}(e_r,x) = \begin{cases} \dfrac{x\Gamma\left((n/c)-r+2\right)\Gamma(r+1)}{\Gamma\left((n/c)+1\right)c^{r-1}}\ {}_2F_1\left(\frac{n}{c}+2,1-r;2;-cx\right), & \text{for } c=\mathbb{N}\cup\{-1\},\\[2ex] \dfrac{(nx)r!}{n^r}\ {}_1F_1\left(1-r;2;-nx\right), & \text{for } c=0. \end{cases}$$

Srivastava and Gupta [20] got the rate of convergence for the Bézier variant of the Bleimann Butzer and Hahn operators for functions with bounded variation. In 2007, Guo et al. [15] studied Baskakov-Bézier operators and established direct, inverse and equivalence approximation theorems with the help of Ditzian-Totik modulus of smoothness. Very recently, Agrawal et al. [5] introduced mixed hybrid operators for which they got direct results and the rate of convergence for differentiable functions whose derivatives are of bounded variation. Many other interesting Bézier type operators were studied by several researchers, cf. [1,2,4,6,7,9,10,13,14, 21,24,25].

For $\theta\geq 1$, we present the Bézier variant of the operators $U_{n,c}f$ defined by

$$U_{n,c}^{(\theta)}(f;x) = (n+2c)\sum_{k=1}^{\infty} Q_{n,k}^{(\theta)}(x,c)\int_0^{\infty} p_{n+3c,k-1}(t,c)f(t)dt + Q_{n,0}^{(\theta)}(x,c)f(0), \tag{1.3}$$

where $Q_{n,k}^{(\theta)}(x,c) = (J_{n,k}(x,c))^{\theta} - (J_{n,k+1}(x,c))^{\theta}$, with $J_{n,k}(x,c) = \sum_{j=k}^{\infty} p_{n+c,j}(x,c)$. For $\theta=1$, the operators $U_{n,c}^{(\theta)}f$ reduce to the operators $U_{n,c}f$.
Alternatively we may rewrite the operators (1.3) as

$$U_{n,c}^{(\theta)}(f;x) = \int_0^{\infty} P_{n,\theta,c}(x,t)f(t)dt, \quad x\in[0,\infty), \tag{1.4}$$

where

$$P_{n,\theta,c}(x,t) = (n+2c)\sum_{k=1}^{\infty} Q_{n,k}^{(\theta)}(x,c)p_{n+3c,k-1}(t,c) + Q_{n,0}^{(\theta)}(x,c)\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

The aim of this paper is to introduce the Bézier variant (1.3) of the Srivastava-Gupta operators, which preserve linear functions. Our further study focuses on a direct approximation theorem in terms of the Ditzian-Totik modulus of smoothness, respectively the rate of convergence for differential functions whose derivatives are of bounded variation on every finite subinterval of $(0,\infty)$, for the presented operators (1.3).

## 2. Auxiliary results

Throughout this paper, $C$ denotes a positive constant independent of $n$ and $x$, not necessarily the same at each occurrence. For these new operators (1.3) we establish some auxiliary results. The monomials $e_k(x)=x^k$, for $k\in\mathbb{N}_0$ called test functions play an important role in uniform approximation by linear positive operators.

**Lemma 2.1.** *For any $n\in\mathbb{N}$, the images of test functions by Gupta operators (1.2) are given by*

$$U_{n,c}(e_0;x)=1,\quad U_{n,c}(e_1;x)=x,\quad U_{n,c}(e_2;x)=x^2+\frac{2x(1+cx)}{n}.$$

Consequently,

$$U_{n,c}\left((t-x)^2;x\right)=\frac{2x(1+cx)}{n}. \tag{2.5}$$

**Lemma 2.2.** *Let $f$ be a real-valued function continuous and bounded on $[0,\infty)$, with $\|f\|=\sup_{x\in[0,+\infty)}|f(x)|$, then $|U_{n,c}(f)|\le\|f\|$.*

**Lemma 2.3.** *Let $f$ be a real-valued function continuous and bounded on $[0,\infty)$ and $\theta\ge 1$, then $|U^{(\theta)}_{n,c}(f)|\le\theta\|f\|$.*

*Proof.* Applying the well known property $|a^\alpha-b^\alpha|\le\alpha|a-b|$, with $0\le a,b\le 1$, $\alpha\ge 1$ and the definition of $Q^{(\theta)}_{n,k}(x,c)$, we have

$$0<(J_{n,k}(x,c))^\theta-(J_{n,k+1}(x,c))^\theta\le\theta(J_{n,k}(x,c)-J_{n,k+1}(x,c))=\theta p_{n+c,k}(x). \tag{2.6}$$

Hence, from the definition of $U^{(\theta)}_{n,c}(f)$ operators and Lemma 2.2, we get

$$|U^{(\theta)}_{n,c}(f)|\le\theta|U_{n,c}(f)|\le\theta\|f\|.$$

□

**Remark 2.1.** *We have*

$$\begin{aligned}U^{(\theta)}_{n,c}(f;x)(e_0;x)&=\sum_{k=0}^{\infty}Q^{(\theta)}_{n,k}(x,c)=[J_{n,0}(x,c)]^\theta\\&=\left[\sum_{j=0}^{\infty}p_{n+c,j}(x)\right]^\theta=1.\end{aligned}$$

In order to present our further results, we recall from [8] the definitions of the Ditizian-Totik modulus of smoothness. Let $\varphi(x)=\sqrt{x(1+cx)}$, then

$$\omega_\varphi(f,t)=\sup_{0<h\le t}\ \sup_{x\pm h\varphi(x)/2\ge 0}\left\{\left|f\left(x+\frac{h\varphi(x)}{2}\right)-f\left(x-\frac{h\varphi(x)}{2}\right)\right|\right\},$$

and the appropriate Peetre's $K$-functional is defined by

$$\overline{K}_\varphi(f,t)=\inf_{g\in V_\varphi}\{\|f-g\|+t\|\varphi g'\|\},\quad t>0,$$

where $V_\varphi = \{g \in C[0,+\infty) \,|\, g \in AC_{\mathrm{loc}}[0,+\infty), \|\varphi g'\| < \infty\}$. According to Th. 3.1.2, [8], it is well known that $\overline{K}_\varphi(f,t) \sim \omega_\varphi(f,t)$, which means that there exists a constant $M>0$, such that

$$M^{-1}\omega_\varphi(f,t) \le \overline{K}_\varphi(f,t) \le M\omega_\varphi(f,t). \tag{2.7}$$

## 3. Direct Theorem

Now we are able to prove the following direct approximation theorem in terms of Ditzian-Totik modulus of smoothness.

**Theorem 3.1.** *Let $f \in C_B[0,\infty)$ and $\theta \ge 1$, then for any $x \in [0,\infty)$, we have*

$$\left|U_{n,c}^{(\theta)}(f;x) - f(x)\right| \le C\omega_\varphi\left(f, \frac{\varphi(x)}{\sqrt{n}}\right), \tag{3.8}$$

*where $C$ is an absolute constant.*

*Proof.* By the definition of $\overline{K}_\varphi(f,t)$ and the relation (2.7), for fixed $n,x$, we can choose $g = g_{n,x} \in V_\varphi$ such that

$$||f-g|| + \frac{1}{\sqrt{n}}||\varphi g'|| + \frac{1}{n}||g'|| \le \omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.9}$$

Using Remark 2.1, we can write

$$\begin{aligned} |\ U_{n,c}^{(\theta)}(f) - f\ | &\le |\ U_{n,c}^{(\theta)}(f-g;x)\ | + |f-g| + |\ U_{n,c}^{(\theta)}(g;x) - g(x)\ | \\ &\le C||f-g|| + |\ U_{n,c}^{(\theta)}(g;x) - g(x)\ |\ . \end{aligned} \tag{3.10}$$

We only need to estimate the second term in the above relation. We will have to split the estimate into two domains, i.e. $x \in F_n^c = [0,1/n]$ and $x \in F_n = (1/n,\infty)$.
Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we get

$$\left|U_{n,c}^{(\theta)}(g;x) - g(x)\right| = \left|U_{n,c}^{(\theta)}\left(\int_x^t g'(u)du; x\right)\right|. \tag{3.11}$$

If $x \in F_n = (1/n,\infty)$, then $U_{n,c}^{(\theta)}\left((t-x)^2;x\right) \sim \frac{2\theta}{n}\varphi^2(x)$. We have

$$\left|\int_x^t g'(u)du\right| \le ||\varphi g'||\left|\int_x^t \frac{1}{\varphi(u)}du\right|. \tag{3.12}$$

For any $x,t \in (0,\infty)$, we find that

$$\begin{aligned} \left|\int_x^t \frac{1}{\varphi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1+cu)}}du\right| \\ &\le \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{(1+cu)}}\right)du\right| \\ &\le 2\left(\sqrt{t} - \sqrt{x} + \frac{\sqrt{(1+ct)} - \sqrt{(1+cx)}}{c}\right) \\ &= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{(1+ct)}+\sqrt{(1+cx)}}\right) \\ &< 2|t-x|\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{(1+cx)}}\right) \\ &\le \frac{2(c+1)}{\sqrt{c(c-1)}}\frac{|t-x|}{\varphi(x)}. \end{aligned} \tag{3.13}$$

Combining (3.11)-(3.13) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |U_{n,c}^{(\theta)}(g;x)-g(x)| &< \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)U_{n,c}^{(\theta)}(|t-x|;x) \\ &\leq \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)\left(U_{n,c}^{(\theta)}((t-x)^2;x)\right)^{1/2} \\ &\leq \frac{2(c+1)}{\sqrt{c(c-1)}}||\varphi g'||\varphi^{-1}(x)\left(\theta\, U_{n,c}((t-x)^2;x)\right)^{1/2}. \end{aligned}$$

Now applying the relation (2.5), we get

$$|U_{n,c}^{(\theta)}(g;x)-g(x)| < C\frac{||\varphi g'||}{\sqrt{n}}. \tag{3.14}$$

For $x\in F_n^c=[0,1/n]$, $U_{n,c}^{(\theta)}\left((t-x)^2;x\right)\sim \frac{2\theta}{n^2}$ and

$$\left|\int_x^t g'(u)du\right| \leq ||g'||\,|t-x|.$$

Therefore, using Cauchy-Schwarz inequality we have

$$|U_{n,c}^{(\theta)}(g;x)-g(x)| \leq ||g'||U_{n,c}^{(\theta)}(|t-x|;x) \leq C||g'||\frac{\sqrt{2\theta}}{\sqrt{n}} < \frac{C}{n}||g'||. \tag{3.15}$$

From (3.14) and (3.15), we have

$$|U_{n,c}^{(\theta)}(g;x)-g(x)| < C\left(\frac{||\varphi g'||}{\sqrt{n}}+\frac{1}{n}||g'||\right). \tag{3.16}$$

Using $\overline{K_\varphi}(f,t)\sim\omega_\varphi(f,t)$ and (3.9), (3.10), (3.16), we get the desired relation (3.8). This completes the proof of the theorem.

□

## 4. Rate of Convergence

Let $f\in DBV_\gamma(0,\infty)$, $\gamma\geq 0$, be the class of differentiable functions defined on $(0,\infty)$, whose derivatives $f'$ are of bounded variation on every finite subinterval of $(0,\infty)$ and $|f(t)|\leq Mt^\gamma$, for all $t>0$ and some $M>0$. The functions $f\in DBV_\gamma(0,\infty)$, could be represented as

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g$ is a function of bounded variation on each finite subinterval of $(0,\infty)$.

**Lemma 4.4.** *Let $x\in(0,\infty)$, then for $\theta\geq 1$ and sufficiently large $n$, we have*

*i)* $\zeta_{n,\theta,c}(x,y)=\int_0^y P_{n,\theta,c}(x,t)dt \leq \frac{\theta\rho}{n}\frac{\varphi^2(x)}{(x-y)^2}, \quad 0\leq y<x,$

*ii)* $1-\zeta_{n,\theta,c}(x,z)=\int_z^\infty P_{n,\theta,c}(x,t)dt \leq \frac{\theta\rho}{n}\frac{\varphi^2(x)}{(z-x)^2}, \quad x<z<\infty,$

*where* $\rho\geq 2$.

*Proof.*

$i)$ Using Lemma 2.3 and (2.5), we get

$$\begin{aligned}\zeta_{n,\theta,c}(x,y) = \int_0^y P_{n,\theta,c}(x,t)dt &\leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 P_{n,\theta,c}(x,t)dt \\ &\leq U_{n,c}^{(\theta)}((t-x)^2;x)\,(x-y)^{-2} \leq \theta U_{n,c}((t-x)^2;x)(x-y)^{-2} \\ &\leq \frac{\theta\rho}{n}\frac{\varphi^2(x)}{(x-y)^2}, \quad 0 \leq y < x.\end{aligned}$$

$ii)$ The second relation can be proved analogously. □

**Theorem 4.2.** *Let $f \in DBV_\gamma(0,\infty)$, $\theta \geq 1$ and $\bigvee_a^b(f_x')$ be the total variation of $f_x'$ on $[a,b] \subset (0,\infty)$. Then, for every $x \in (0,\infty)$ and sufficiently large $n$, we have*

$$\begin{aligned}\left|U_{n,c}^{(\theta)}(f;x) - f(x)\right| \leq{}& \frac{\sqrt{\theta}}{\theta+1}\left|f'(x+)+\theta f'(x-)\right|\sqrt{\frac{\rho}{n}}\varphi(x) + \sqrt{\frac{\rho}{n}}\,\varphi(x)\frac{\theta^{3/2}}{\theta+1}\left|f'(x+)-f'(x-)\right| \\ &+\frac{\theta\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}(f_x') \\ &+\frac{\theta\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x}^{x+x/k}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}(f_x'),\end{aligned}$$

*where $\rho \geq 2$ and the auxiliary function $f_x'$ is defined by*

$$f_x'(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t \leq 1. \end{cases}$$

*Proof.* Since $\int_0^\infty P_{n,\theta,c}(x,t)dt = U_{n,c}^{(\theta)}(e_0;x) = 1$, we can write

$$\begin{aligned}U_{n,c}^{(\theta)}(f;x) - f(x) &= \int_0^\infty (f(t)-f(x))\,P_{n,\theta,c}(x,t)dt \\ &= \int_0^\infty \left(\int_x^t f'(u)du\right)P_{n,\theta,c}(x,t)dt. \qquad (4.17)\end{aligned}$$

Using definition of the function $f_x'$, for any $f \in DBV_\gamma(0,\infty)$, it follows

$$\begin{aligned}f'(t) = \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big) + f_x'(t) &+ \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\left(\text{sgn}(t-x)+\frac{\theta-1}{\theta+1}\right) \\ &+ \delta_x(t)\left(f'(x) - \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\right), \qquad (4.18)\end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1\ , & x = t \\ 0\ , & x \neq t. \end{cases}$$

It is clear that

$$\int_0^\infty P_{n,\theta,c}(x,t)\int_x^t \left(f'(x) - \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\right)\delta_x(u)dudt = 0.$$

Using the definition of operators (1.4), then simple computations lead us to

$$
\begin{aligned}
E_1 &= \int_0^\infty \left( \int_x^t \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)du\right) P_{n,\theta,c}(x,t)dt \\
&= \frac{1}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big| \int_0^\infty |t-x| P_{n,\theta,c}(x,t)dt \\
&\le \frac{1}{\theta+1}\Big(f'(x+)+\theta f'(x-)\Big)\Big(U_{n,c}^{(\theta)}((e_1-x)^2;x)\Big)^{1/2} \le \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x) \qquad (4.19)
\end{aligned}
$$

and

$$
\begin{aligned}
E_2 &= \int_0^\infty \left( \int_x^t \frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\Big(\mathrm{sgn}(u-x)+\frac{\theta-1}{\theta+1}\Big)du\right) P_{n,\theta,c}(x,t)dt \\
&\le \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big| \int_0^\infty |t-x|\, P_{n,\theta,c}(x,t)dt = \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big| U_{n,c}^{(\theta)}\left(|t-x|\,;x\right) \\
&\le \frac{\theta}{\theta+1}\Big|f'(x+)-f'(x-)\Big|\Big(U_{n,c}^{(\theta)}\left((e_1-x)^2;x\right)\Big)^{1/2} \le \frac{\theta^{3/2}}{\theta+1}\Big|f'(x+)-f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x). \qquad (4.20)
\end{aligned}
$$

Involving the relations (4.17)–(4.20), we obtain the following estimate

$$
\begin{aligned}
\Big|U_{n,c}^{(\theta)}(f;x)-f(x)\Big| \le |A_{n,\theta,c}(f'_x,x)+B_{n,\theta,c}(f'_x,x)| &+ \frac{\sqrt{\theta}}{\theta+1}\Big|f'(x+)+\theta f'(x-)\Big|\sqrt{\frac{\rho}{n}}\varphi(x) \\
&+ \frac{\theta^{3/2}}{\theta+1}|f'(x+)-f'(x-)|\sqrt{\frac{\rho}{n}}\,\varphi(x), \qquad (4.21)
\end{aligned}
$$

where

$$A_{n,\theta,c}(f'_x,x) = \int_0^x \left(\int_x^t f'_x(u)du\right) P_{n,\theta,c}(x,t)dt$$

and

$$B_{n,\theta,c}(f'_x,x) = \int_x^\infty \left(\int_x^t f'_x(u)du\right) P_{n,\theta,c}(x,t)dt.$$

For a complete proof of the theorem, it remains to estimate the terms $A_{n,\theta,c}(f'_x,x)$ and $B_{n,\theta,c}(f'_x,x)$. Since $\int_a^b d_t\zeta_{n,\theta,c}(x,t) \le 1$, for all $[a,b] \subseteq (0,\infty)$, using integration by parts and applying Lemma 4.4 with $y = x-(x/\sqrt{n})$, it follows

$$
\begin{aligned}
|A_{n,\theta,c}(f'_x,x)| &= \left|\int_0^x \left(\int_x^t f'_x(u)du\right) d_t\zeta_{n,\theta,c}(x,t)\right| = \left|\int_0^x \zeta_{n,\theta,c}(x,t)f'_x(t)dt\right| \\
&\le \left(\int_0^y + \int_y^x\right) |f'_x(t)|\,|\zeta_{n,\theta,c}(x,t)|\,dt \\
&\le \theta\frac{\rho\varphi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \int_y^x \bigvee_t^x (f'_x)dt \\
&\le \theta\frac{\rho\varphi^2(x)}{n}\int_0^y \bigvee_t^x (f'_x)(x-t)^{-2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x (f'_x).
\end{aligned}
$$

Taking $u=x/(x-t)$ into account, we get

$$\theta\frac{\rho\varphi^2(x)}{n}\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x(f_x')dt=\theta\frac{\rho(1+cx)}{n}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x}(f_x')du$$
$$\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\bigvee_{x-x/u}^{x}(f_x')du\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x').$$

Hence, we reach the following estimation

$$|A_{n,\theta,c}(f_x',x)|\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_{x-x/k}^{x}(f_x')+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}(f_x'). \tag{4.22}$$

Using again the integration by parts and applying Lemma 4.4 with $z=x+x/\sqrt{n}$, it follows

$$\begin{aligned}
|B_{n,\theta,c}(f_x^{'},x)|&=\left|\int_x^\infty\left(\int_x^t f_x'(u)du\right)P_{n,\theta,c}(x,t)dt\right|\\
&=\left|\int_x^z\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))+\int_z^\infty\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))\right|\\
&=\left|\left[\left(\int_x^t f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,t))\right]_x^z-\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right.\\
&\left.+\int_z^\infty\left(\int_x^t f_x'(u)du\right)d_t(1-\zeta_{n,\theta,c}(x,t))\right|\\
&=\left|\left(\int_x^z f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,z))-\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right.\\
&\left.+\left[\left(\int_x^t f_x'(u)du\right)(1-\zeta_{n,\theta,c}(x,t))\right]_z^\infty-\int_z^\infty f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right|\\
&=\left|\int_x^z f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt+\int_z^\infty f_x'(t)(1-\zeta_{n,\theta,c}(x,t))dt\right|\\
&<\theta\frac{\rho\varphi^2(x)}{n}\int_z^\infty\bigvee_x^t(f')_x(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f_x')dt\\
&\leq\theta\frac{\rho\varphi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f_x')(t-x)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f_x'). \qquad (4.23)
\end{aligned}$$

Taking $u=x/(t-x)$ into account, we get

$$\theta\frac{\rho\varphi^2(x)}{n}\int_{x+x/\sqrt{n}}^\infty\bigvee_x^t(f_x')(t-x)^{-2}dt=\theta\frac{\rho\varphi^2(x)}{xn}\int_0^{\sqrt{n}}\bigvee_x^{x+x/u}(f_x')du$$
$$\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\int_k^{k+1}\bigvee_x^{x+x/u}(f_x')du\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+x/k}(f_x'). \tag{4.24}$$

Using the relations (4.23)-(4.24), we get the following estimation

$$|B_{n,\theta,c}(f_x^{'},x)|\leq\theta\frac{\rho(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}\ ]}\bigvee_x^{x+x/k}(f_x')+\frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}(f_x'). \tag{4.25}$$

The relations (4.21), (4.22) and (4.25) lead us to the desired result. □

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