

PAPER DETAILS

TITLE: General Multivariate Iyengar Type Inequalities

AUTHORS: George ANASTASSIOU

PAGES: 64-80

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/682690>

General Multivariate Iyengar Type Inequalities

GEORGE A. ANASTASSIOU

ABSTRACT. Here we give a variety of general multivariate Iyengar type inequalities for not necessarily radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer well-known univariate Iyengar type inequalities and univariate author's related results into general multivariate Iyengar inequalities.

Keywords: Iyengar inequality, Polar coordinates, Not necessarily radial function, Shell, Ball.

2010 Mathematics Subject Classification: 26D10, 26D15.

1. BACKGROUND

In the year 1938, Iyengar [5] proved the following interesting inequality:

Theorem 1.1. *Let f be a differentiable function on $[a, b]$ and $|f'(x)| \leq M_1$. Then*

$$(1.1) \quad \left| \int_a^b f(x) dx - \frac{1}{2} (b-a) (f(a) + f(b)) \right| \leq \frac{M_1 (b-a)^2}{4} - \frac{(f(b) - f(a))^2}{4M_1}.$$

In 2001, X.-L. Cheng [4] proved that

Theorem 1.2. *Let $f \in C^2([a, b])$ and $|f''(x)| \leq M_2$. Then*

$$(1.2) \quad \left| \int_a^b f(x) dx - \frac{1}{2} (b-a) (f(a) + f(b)) + \frac{1}{8} (b-a)^2 (f'(b) - f'(a)) \right| \leq \frac{M_2}{24} (b-a)^3 - \frac{(b-a)}{16M_2} \Delta_1^2,$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b) - f(a))}{(b-a)} + f'(b).$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1.1):

Theorem 1.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that for all $x \in [a, b]$ with $M > m$ we have $m \leq f'(x) \leq M$. Then*

$$\begin{aligned} & \left| \int_a^b f(x) dx - \frac{1}{2} (b-a) (f(a) + f(b)) \right| \\ & \leq \frac{(f(b) - f(a) - m(b-a))(M(b-a) - f(b) + f(a))}{2(M-m)}. \end{aligned}$$

In [7], Qi proved the following:

Received: 22 January 2019; Accepted: 27 March 2019; Published Online: 29 March 2019

*Corresponding author: G. A. Anastassiou; ganastss@memphis.edu

DOI: 10.33205/cma.543560

Theorem 1.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable function such that for all $x \in [a, b]$ with $M > 0$ we have $|f''(x)| \leq M$. Then

$$\left| \int_a^b f(x) dx - \frac{(f(a) + f(b))}{2} (b-a) + \frac{(1+Q^2)}{8} (f'(b) - f'(a)) (b-a)^2 \right| \leq \frac{M(b-a)^3}{24} (1-3Q^2),$$

where

$$Q^2 = \frac{\left(f'(a) + f'(b) - 2 \left(\frac{f(b)-f(a)}{b-a} \right) \right)^2}{M^2 (b-a)^2 - (f'(b) - f'(a))^2}.$$

In 2005, Zheng Liu, [6], proved the following:

Theorem 1.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$ and for all $x \in [a, b]$ with $M > m$ we have

$$m \leq \frac{f'(x) - f'(a)}{x-a} \leq M \text{ and } m \leq \frac{f'(b) - f'(x)}{b-x} \leq M.$$

Then

$$\left| \int_a^b f(x) dx - \frac{(f(a) + f(b))}{2} (b-a) + \left(\frac{1+P^2}{8} \right) (f'(b) - f'(a)) (b-a)^2 - \left(\frac{1+3P^2}{48} \right) (m+M) (b-a)^3 \right| \leq \frac{(M-m)(b-a)^3}{48} (1-3P^2),$$

where

$$P^2 = \frac{\left(f'(a) + f'(b) - 2 \left(\frac{f(b)-f(a)}{b-a} \right) \right)^2}{\left(\frac{M-m}{2} \right)^2 (b-a)^2 - (f'(b) - f'(a) - \left(\frac{m+M}{2} \right) (b-a))^2}.$$

Next we list some author's related results, (here $L_\infty([a, b])$ is the normed space of essentially bounded functions over $[a, b]$):

Theorem 1.6. ([3]) Let $n \in \mathbb{N}$, $f \in AC^n([a, b])$ (i.e. $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions). We assume that $f^{(n)} \in L_\infty([a, b])$. Then

(i)

$$(1.3) \quad \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!} \left[(t-a)^{n+1} + (b-t)^{n+1} \right],$$

for all $t \in [a, b]$,

(ii) at $t = \frac{a+b}{2}$, the right hand side of (1.3) is minimized, and we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n},$$

(iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$ for all $k = 0, 1, \dots, n-1$, then we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])} (b-a)^{n+1}}{(n+1)! 2^n}$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$(1.4) \quad \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right],$$

(v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (1.4) we get:

$$(1.5) \quad \left| \int_a^b f(x) dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right]$$

for $j = 0, 1, 2, \dots, N \in \mathbb{N}$,

(vi) when $N = 2$ and $j = 1$, (1.5) turns to

$$(1.6) \quad \left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])} (b-a)^{n+1}}{(n+1)! 2^n},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (1.6) that

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{[a,b],\infty} \frac{(b-a)^2}{4},$$

a similar to Iyengar inequality (1.1).

We mention here $L_1([a,b])$ is the normed space of integrable functions over $[a,b]$.

Theorem 1.7. ([3]) Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$. Then

(i)

$$(1.7) \quad \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} [(t-a)^n + (b-t)^n],$$

for all $t \in [a,b]$,

(ii) at $t = \frac{a+b}{2}$, the right hand side of (1.7) is minimized, and we get:

$$\left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])} (b-a)^n}{n! 2^{n-1}},$$

(iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$(1.8) \quad \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n [j^n + (N-j)^n],$$

(v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (1.8) we get:

$$(1.9) \quad \left| \int_a^b f(x) dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n [j^n + (N-j)^n],$$

for $j = 0, 1, 2, \dots, N \in \mathbb{N}$,

(vi) when $N = 2$ and $j = 1$, (1.9) turns to

$$(1.10) \quad \left| \int_a^b f(x) dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (1.10) that

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{L_1([a,b])} (b-a).$$

We mention here $L_q([a, b])$ is the normed space of functions f such that $|f|^q$ is integrable over $[a, b]$

Theorem 1.8. ([3]) Let $f \in AC^n([a, b])$, $n \in \mathbb{N}$; $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $f^{(n)} \in L_q([a, b])$. Then

(i)

$$(1.11) \quad \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right|$$

$$\leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) \left(p(n-1) + 1 \right)^{\frac{1}{p}}} \left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right],$$

for all $t \in [a, b]$,

(ii) at $t = \frac{a+b}{2}$, the right hand side of (1.11) is minimized, and we get:

$$\begin{aligned} & \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \\ & \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \end{aligned}$$

(iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_a^b f(x) dx \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, N \in \mathbb{N}$, it holds

$$\begin{aligned} (1.12) \quad & \left| \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \\ & \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \end{aligned}$$

(v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \dots, n-1$, from (1.12) we get:

$$\begin{aligned} (1.13) \quad & \left| \int_a^b f(x) dx - \left(\frac{b-a}{N}\right) [j f(a) + (N-j) f(b)] \right| \\ & \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \end{aligned}$$

for $j = 0, 1, 2, \dots, N \in \mathbb{N}$,

(vi) when $N = 2$ and $j = 1$, (1.13) turns to

$$(1.14) \quad \left| \int_a^b f(x) dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (1.14) that

$$\left| \int_a^b f(x) dx - \left(\frac{b-a}{2}\right) (f(a) + f(b)) \right| \leq \frac{\|f'\|_{L_q([a,b])}}{\left(1 + \frac{1}{p}\right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}.$$

We need

Remark 1.1. We define the ball $B(0, R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1} .

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure on the ball, that is the volume of $B(0, R)$, which exactly is $\text{Vol}(B(0, R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma(\frac{N}{2} + 1)}$.

Following [8, pp. 149-150, exercise 6], and [9, pp. 87-88, Theorem 5.2.2] we can write for $F : \overline{B(0, R)} \rightarrow \mathbb{R}$ a Lebesgue integrable function that

$$(1.15) \quad \int_{B(0,R)} F(x) dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega) r^{N-1} dr \right) d\omega,$$

and we use this formula a lot.

Typically here the function $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ is not radial. A radial function f is such that there exists a function g with $f(x) = g(r)$, where $r = |x|$, $r \in [0, R]$, for all $x \in \overline{B(0, R)}$.

We need

Remark 1.2. Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. Consider that $f : \overline{A} \rightarrow \mathbb{R}$ is not radial. A radial function f is such that there exists a function g with $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, for all $x \in \overline{A}$. Here x can be written uniquely as $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, see ([8], p. 149-150 and [2], p. 421), furthermore for $F : \overline{A} \rightarrow \mathbb{R}$ a Lebesgue integrable function we have that

$$(1.16) \quad \int_A F(x) dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega.$$

Here

$$\text{Vol}(A) = \frac{\omega_N (R_2^N - R_1^N)}{N} = \frac{\pi^{\frac{N}{2}} (R_2^N - R_1^N)}{\Gamma(\frac{N}{2} + 1)}.$$

In this article we derive general multivariate Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geq 2$, for not necessarily radial functions. Our results are based on Theorems 1.1-1.8.

2. MAIN RESULTS

We present the following non-radial multivariate Iyengar type inequalities:

We start with

Theorem 2.9. Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \rightarrow \mathbb{R}$ that is not necessarily radial, and that $f \in C^1(\overline{A})$. Assume that $\left| \frac{\partial f(s\omega)}{\partial s} \right| \leq M_1$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$, where $M_1 > 0$.

Then

$$\begin{aligned} & \left| \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right| \\ & \leq \frac{M_1 \pi^{\frac{N}{2}} (R_2 - R_1)^2}{2\Gamma(\frac{N}{2})} - \frac{\int_{S^{N-1}} (f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1})^2 d\omega}{4M_1}. \end{aligned}$$

Proof. Here $f(s\omega) s^{N-1} \in C^1([R_1, R_2])$, $N \geq 2$, for all $\omega \in S^{N-1}$. By (1.1) we get

$$\begin{aligned} & \left| \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{1}{2} (R_2 - R_1) (f(R_1\omega) R_1^{N-1} + f(R_2\omega) R_2^{N-1}) \right| \\ & \leq \frac{M_1 (R_2 - R_1)^2}{4} - \frac{(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1})^2}{4M_1} =: \lambda_1(\omega), \end{aligned}$$

for all $\omega \in S^{N-1}$.

Equivalently, we have

$$-\lambda_1(\omega) \leq \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{1}{2} (R_2 - R_1) (f(R_1\omega) R_1^{N-1} + f(R_2\omega) R_2^{N-1}) \leq \lambda_1(\omega),$$

for all $\omega \in S^{N-1}$.

Hence it holds

$$\begin{aligned} & - \int_{S^{N-1}} \lambda_1(\omega) d\omega \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega \\ & - \frac{1}{2} (R_2 - R_1) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \\ & \leq \int_{S^{N-1}} \lambda_1(\omega) d\omega. \end{aligned}$$

That is (by (1.16))

$$\begin{aligned} & - \left[\frac{\pi^{\frac{N}{2}} M_1 (R_2 - R_1)^2}{2\Gamma(\frac{N}{2})} - \frac{\int_{S^{N-1}} (f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1})^2 d\omega}{4M_1} \right] \\ & \leq \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \\ & \leq \frac{\pi^{\frac{N}{2}} M_1 (R_2 - R_1)^2}{2\Gamma(\frac{N}{2})} - \frac{\int_{S^{N-1}} (f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1})^2 d\omega}{4M_1}, \end{aligned}$$

proving the claim. $\square$

We continue with

Theorem 2.10. Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \rightarrow \mathbb{R}$ that is not necessarily radial, and that $f \in C^2(\overline{A})$. Assume that $\left| \frac{\partial^2 f(s\omega)}{\partial s^2} \right| \leq M_2$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$, where $M_2 > 0$. Set

$$\begin{aligned} \Delta_1(\omega) &:= (f(s\omega) s^{N-1})'(R_1) - \frac{2(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1})}{R_2 - R_1} \\ &+ (f(s\omega) s^{N-1})'(R_2), \quad \forall \omega \in S^{N-1}. \end{aligned}$$

Then

$$\begin{aligned}
& \left| \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right. \\
& + \left. \frac{(R_2 - R_1)^2}{8} \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_2) d\omega - \int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_1) d\omega \right] \right| \\
& \leq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{M_2}{12} (R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M_2} \int_{S^{N-1}} \Delta_1^2(\omega) d\omega.
\end{aligned}$$

Proof. Here $f(s\omega) s^{N-1} \in C^2([R_1, R_2])$, $N \geq 2$, for all $\omega \in S^{N-1}$. By (1.2) we get

$$\begin{aligned}
& \left| \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{1}{2} (R_2 - R_1) (f(R_1 \omega) R_1^{N-1} + f(R_2 \omega) R_2^{N-1}) \right. \\
& + \left. \frac{1}{8} (R_2 - R_1)^2 \left((f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1) \right) \right| \\
& \leq \frac{M_2}{24} (R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M_2} \Delta_1^2(\omega) =: \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.

Equivalently, we have

$$\begin{aligned}
- \lambda_2(\omega) & \leq \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{(R_2 - R_1)}{2} (f(R_1 \omega) R_1^{N-1} + f(R_2 \omega) R_2^{N-1}) \\
& + \frac{1}{8} (R_2 - R_1)^2 \left((f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1) \right) \leq \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.

Hence it holds

$$\begin{aligned}
- \int_{S^{N-1}} \lambda_2(\omega) d\omega & \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega \\
& - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \\
& + \frac{(R_2 - R_1)^2}{8} \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_2) d\omega - \int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_1) d\omega \right] \\
& \leq \int_{S^{N-1}} \lambda_2(\omega) d\omega.
\end{aligned}$$

That is (by (1.16))

$$\begin{aligned}
& - \left[\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{M_2}{12} (R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M_2} \int_{S^{N-1}} \Delta_1^2(\omega) d\omega \right] \\
& \leq \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \\
& + \frac{(R_2 - R_1)^2}{8} \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_2) d\omega - \int_{S^{N-1}} (f(s\omega) s^{N-1})'(R_1) d\omega \right] \\
& \leq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{M_2}{12} (R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M_2} \int_{S^{N-1}} \Delta_1^2(\omega) d\omega,
\end{aligned}$$

proving the claim. $\square$

We give

Theorem 2.11. *Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \rightarrow \mathbb{R}$ that is not necessarily radial, and that $f \in C^1(\overline{A})$. Let $M > m$ and assume that $m \leq \frac{\partial f(s\omega)}{\partial s} \leq M$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$.*

Then

$$\begin{aligned} & \left| \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right| \\ & \leq \frac{1}{2(M - m)} \int_{S^{N-1}} [(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1} - m(R_2 - R_1)) \\ & \quad \times (M(R_2 - R_1) - f(R_2\omega) R_2^{N-1} + f(R_1\omega) R_1^{N-1})] d\omega. \end{aligned}$$

Proof. Similar to the proof of Theorem 2.9 by using Theorem 1.3 and (1.16). $\square$

We give

Theorem 2.12. *Let the spherical shell $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. Consider $f : \overline{A} \rightarrow \mathbb{R}$ that is not necessarily radial, and that $f \in C^2(\overline{A})$. Assume that $\left| \frac{\partial^2 f(s\omega)}{\partial s^2} \right| \leq M_3$, for all $s \in [R_1, R_2]$, and for all $\omega \in S^{N-1}$, where $M_3 > 0$.*

Set

$$\begin{aligned} & Q_1^2(\omega) \\ & := \frac{\left[(f(s\omega) s^{N-1})'(R_1) + (f(s\omega) s^{N-1})'(R_2) - 2 \left(\frac{f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}}{R_2 - R_1} \right) \right]^2}{\left[M_3^2 (R_2 - R_1)^2 - ((f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1))^2 \right]}, \end{aligned}$$

for all $\omega \in S^{N-1}$.

Then

$$\begin{aligned} & \left| \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right| \\ & + \frac{(R_2 - R_1)^2}{8} \int_{S^{N-1}} (1 + Q_1^2(\omega)) \left((f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1) \right) d\omega \\ & \leq \frac{M_3 (R_2 - R_1)^3}{24} \int_{S^{N-1}} (1 - 3Q_1^2(\omega)) d\omega. \end{aligned}$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.4 and (1.16). $\square$

We continue with

Theorem 2.13. *Here all as in Theorem 2.9, and let $M_1 > m_1$. Assume that*

$$m_1 \leq \frac{(f(s\omega) s^{N-1})'(x) - (f(s\omega) s^{N-1})'(R_1)}{x - R_1} \leq M_1,$$

and

$$m_1 \leq \frac{(f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(x)}{R_2 - x} \leq M_1,$$

for all $x \in [R_1, R_2]$, for all $\omega \in S^{N-1}$.

Set

$$P_1^2(\omega) = \frac{\left[(f(s\omega) s^{N-1})'(R_1) + (f(s\omega) s^{N-1})'(R_2) - 2 \left(\frac{f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}}{R_2 - R_1} \right) \right]^2}{\left(\frac{M_1 - m_1}{2} \right)^2 (R_2 - R_1)^2 - \left[(f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1) - \left(\frac{m_1 + M_1}{2} \right) (R_2 - R_1) \right]^2},$$

for all $\omega \in S^{N-1}$.

Then

$$\begin{aligned} & \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right. \\ & + \frac{(R_2 - R_1)^2}{8} \int_{S^{N-1}} (1 + P_1^2(\omega)) \left((f(s\omega) s^{N-1})'(R_2) - (f(s\omega) s^{N-1})'(R_1) \right) d\omega \\ & \left. - \frac{(R_2 - R_1)^3}{48} (m_1 + M_1) \int_{S^{N-1}} (1 + 3P_1^2(\omega)) d\omega \right| \\ & \leq \frac{(M_1 - m_1)(R_2 - R_1)^3}{48} \int_{S^{N-1}} (1 - 3P_1^2(\omega)) d\omega. \end{aligned}$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.5 and (1.16). $\square$

We present

Theorem 2.14. Consider $f : \bar{A} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ (i.e. $(f(s\omega) s^{N-1})^{(n-1)} \in AC([R_1, R_2])$ absolutely continuous functions), for all $\omega \in S^{N-1}$, $N \geq 2$. We assume that $(f(s\omega) s^{N-1})^{(n)} \in L_\infty([R_1, R_2])$, for all $\omega \in S^{N-1}$. There exists $K_1 > 0$ such that $\left\| (f(s\omega) s^{N-1})^{(n)} \right\|_{L_\infty([R_1, R_2])} \leq K_1$, where $s \in [R_1, R_2]$, for all $\omega \in S^{N-1}$.

Then

(i)

$$\begin{aligned} (2.17) \quad & \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) (t - R_1)^{k+1} \right. \right. \\ & + \left. \left. (-1)^k \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) (R_2 - t)^{k+1} \right] \right| \\ & \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{K_1}{(n+1)!} \left[(t - R_1)^{n+1} + (R_2 - t)^{n+1} \right], \end{aligned}$$

for all $t \in [R_1, R_2]$,

(ii) at $t = \frac{R_1 + R_2}{2}$, the right hand side of (2.17) is minimized, and we get:

$$\begin{aligned} & \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right. \right. \\ & + \left. \left. (-1)^k \int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right] \right| \\ & \leq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{K_1}{(n+1)!} \frac{(R_2 - R_1)^{n+1}}{2^{n-1}}, \end{aligned}$$

(iii) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k (f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_A f(y) dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{K_1}{(n+1)!} \frac{(R_2 - R_1)^{n+1}}{2^{n-1}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.18) \quad \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{k+1} \left[j^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) \right. \right. \\ \left. \left. + (-1)^k (\bar{N} - j)^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) \right] \right| \\ \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{K_1}{(n+1)!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{n+1} [j^{n+1} + (\bar{N} - j)^{n+1}],$$

(v) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k (f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for $k = 1, \dots, n-1$, from (2.18) we get:

$$(2.19) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{\bar{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega) d\omega \right) \right. \right. \\ \left. \left. + (\bar{N} - j) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \\ \times \frac{K_1}{(n+1)!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{n+1} [j^{n+1} + (\bar{N} - j)^{n+1}],$$

for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$,

(vi) when $\bar{N} = 2$ and $j = 1$, (2.19) turns to

$$(2.20) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right| \\ \leq \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{K_1}{(n+1)!} \frac{(R_2 - R_1)^{n+1}}{2^{n-1}},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (2.20) that

$$\left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right| \\ \leq \frac{\pi^{\frac{N}{2}} K_1}{2\Gamma(\frac{N}{2})} (R_2 - R_1)^2.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.6 along with (1.16). $\square$

We continue with

Theorem 2.15. Consider $f : \bar{A} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ (i.e. $(f(s\omega) s^{N-1})^{(n-1)} \in AC([R_1, R_2])$ absolutely continuous functions), for all $\omega \in S^{N-1}$, $N \geq 2$. Here there exists $K_2 > 0$ such that $\left\| (f(s\omega) s^{N-1})^{(n)} \right\|_{L_1([R_1, R_2])} \leq K_2$, where $s \in [R_1, R_2]$, for all $\omega \in S^{N-1}$.

Then

(i)

$$(2.21) \quad \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) (t - R_1)^{k+1} \right. \right. \\ \left. \left. + (-1)^k \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) (R_2 - t)^{k+1} \right] \right| \\ \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} [(t - R_1)^n + (R_2 - t)^n],$$

for all $t \in [R_1, R_2]$,(ii) at $t = \frac{R_1+R_2}{2}$, the right hand side of (2.21) is minimized, and we get:

$$\left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right. \right. \\ \left. \left. + (-1)^k \int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right] \right| \\ \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \frac{(R_2 - R_1)^n}{2^{n-2}},$$

(iii) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k (f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_A f(y) dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \frac{(R_2 - R_1)^n}{2^{n-2}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.22) \quad \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{k+1} \left[j^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) \right. \right. \\ \left. \left. + (-1)^k (\bar{N} - j)^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) \right] \right| \\ \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^n [j^n + (\bar{N} - j)^n],$$

(v) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k (f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for $k = 1, \dots, n-1$, from (2.22) we get:

$$(2.23) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{\bar{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1 \omega) d\omega \right) \right. \right. \\ \left. \left. + (\bar{N} - j) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \\ \times \frac{K_2}{n!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^n [j^n + (\bar{N} - j)^n],$$

for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$,

(vi) when $\overline{N} = 2$ and $j = 1$, (2.23) turns to

$$(2.24) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right| \\ \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \frac{(R_2 - R_1)^n}{2^{n-2}},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (2.24) that

$$\left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1 \omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2 \omega) d\omega \right) \right| \\ \leq \frac{2\pi^{\frac{N}{2}} K_2}{\Gamma\left(\frac{N}{2}\right)} (R_2 - R_1).$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.7 along with (1.16). $\square$

We continue with

Theorem 2.16. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} > 1$. Consider $f : \overline{A} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC^n([R_1, R_2])$ (i.e. $(f(s\omega) s^{N-1})^{(n-1)} \in AC([R_1, R_2])$ absolutely continuous functions), for all $\omega \in S^{N-1}$, $N \geq 2$. We assume that $(f(s\omega) s^{N-1})^{(n)} \in L_q([R_1, R_2])$, for all $\omega \in S^{N-1}$. There exists $K_3 > 0$ such that $\left\| (f(s\omega) s^{N-1})^{(n)} \right\|_{L_q([R_1, R_2])} \leq K_3$, where $s \in [R_1, R_2]$, for all $\omega \in S^{N-1}$.

Then

(i)

$$(2.25) \quad \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) (t - R_1)^{k+1} \right. \right. \\ \left. \left. + (-1)^k \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) (R_2 - t)^{k+1} \right] \right| \\ \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \left[(t - R_1)^{n+\frac{1}{p}} + (R_2 - t)^{n+\frac{1}{p}} \right],$$

for all $t \in [R_1, R_2]$,

(ii) at $t = \frac{R_1 + R_2}{2}$, the right hand side of (2.25) is minimized, and we get:

$$\left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2 - R_1)^{k+1}}{2^{k+1}} \right. \\ \left. \times \left[\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega + (-1)^k \int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right] \right| \\ \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

(iii) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k(f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for all $k = 0, 1, \dots, n-1$, we obtain

$$\left| \int_A f(y) dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.26) \quad \left| \int_A f(y) dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{k+1} \left[j^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_1) d\omega \right) + (-1)^k (\bar{N} - j)^{k+1} \left(\int_{S^{N-1}} (f(s\omega) s^{N-1})^{(k)}(R_2) d\omega \right) \right] \right|$$

$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (\bar{N} - j)^{n+\frac{1}{p}} \right],$$

(v) if $(f(s\omega) s^{N-1})^{(k)}(R_1) = (f(s\omega) s^{N-1})^{(k)}(R_2) = 0$, for all $\omega \in S^{N-1}$, (i.e. $\frac{\partial^k(f(s\omega) s^{N-1})}{\partial s^k}$ vanish on $\partial B(0, R_1)$ and $\partial B(0, R_2)$) for $k = 1, \dots, n-1$, from (2.26) we get:

$$(2.27) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{\bar{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1\omega) d\omega \right) + (\bar{N} - j) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2\omega) d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

$$\times \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \left(\frac{R_2 - R_1}{\bar{N}} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (\bar{N} - j)^{n+\frac{1}{p}} \right],$$

for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$,

(vi) when $\bar{N} = 2$ and $j = 1$, (2.27) turns to

$$(2.28) \quad \left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right|$$

$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)! \left(n + \frac{1}{p}\right) (p(n-1) + 1)^{\frac{1}{p}}} \frac{(R_2 - R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

(vii) when $n = 1$ (without any boundary conditions), we get from (2.28) that

$$\left| \int_A f(y) dy - \left(\frac{R_2 - R_1}{2} \right) \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right) \right|$$

$$\leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} K_3}{\Gamma\left(\frac{N}{2}\right) \left(1 + \frac{1}{p}\right)} (R_2 - R_1)^{1+\frac{1}{p}}.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.8 along with (1.16). $\square$

We continue with results on the ball. We present

Theorem 2.17. Consider $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC([0, R])$, for all $\omega \in S^{N-1}$, $N \geq 2$. We further assume

that $\frac{\partial f(s\omega)s^{N-1}}{\partial s} \in L_\infty([0, R])$, for all $\omega \in S^{N-1}$. Suppose there exists $M_1 > 0$ such that $\left\| \frac{\partial f(s\omega)s^{N-1}}{\partial s} \right\|_{\infty, (s \in [0, R])} \leq M_1$, for all $\omega \in S^{N-1}$.

Then

(i)

$$(2.29) \quad \left| \int_{B(0, R)} f(y) dy - \left(\int_{S^{N-1}} f(R\omega) d\omega \right) R^{N-1} (R-t) \right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)} [t^2 + (R-t)^2],$$

for all $t \in [0, R]$,

(ii) at $t = \frac{R}{2}$, the right hand side of (2.29) is minimized, and we get:

$$\left| \int_{B(0, R)} f(y) dy - \left(\int_{S^{N-1}} f(R\omega) d\omega \right) \frac{R^N}{2} \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega) = 0$, for all $\omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanishes on $\partial B(0, R)$), we obtain

$$\left| \int_{B(0, R)} f(y) dy \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.30) \quad \left| \int_{B(0, R)} f(y) dy - \frac{R^N}{\bar{N}} (\bar{N} - j) \int_{S^{N-1}} f(R\omega) d\omega \right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\bar{N}} \right)^2 [j^2 + (\bar{N} - j)^2],$$

(v) when $\bar{N} = 2$ and $j = 1$, (2.30) turns to

$$\left| \int_{B(0, R)} f(y) dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega) d\omega \right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.14, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

We continue with

Theorem 2.18. Consider $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega)s^{N-1} \in AC([0, R])$, for all $\omega \in S^{N-1}$, $N \geq 2$. Suppose there exists $M_2 > 0$ such that $\left\| \frac{\partial f(s\omega)s^{N-1}}{\partial s} \right\|_{L_1([0, R])} \leq M_2$, for all $\omega \in S^{N-1}$.

Then

(i)

$$(2.31) \quad \left| \int_{B(0, R)} f(y) dy - \left(\int_{S^{N-1}} f(R\omega) d\omega \right) R^{N-1} (R-t) \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)},$$

for all $t \in [0, R]$,

(ii) if $f(R\omega) = 0$, for all $\omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanishes on $\partial B(0, R)$) from (2.31), we obtain

$$\left| \int_{B(0, R)} f(y) dy \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iii) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.32) \quad \left| \int_{B(0,R)} f(y) dy - \frac{R^N}{\bar{N}} (\bar{N} - j) \int_{S^{N-1}} f(R\omega) d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)},$$

(iv) when $\bar{N} = 2$ and $j = 1$, (2.32) turns to

$$\left| \int_{B(0,R)} f(y) dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega) d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.15, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

We continue with

Theorem 2.19. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Consider $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC([0, R])$, for all $\omega \in S^{N-1}$, $N \geq 2$. We further assume that $\frac{\partial f(s\omega) s^{N-1}}{\partial s} \in L_q([0, R])$, for all $\omega \in S^{N-1}$. Suppose there exists $M_3 > 0$ such that $\left\| \frac{\partial f(s\omega) s^{N-1}}{\partial s} \right\|_{L_q([0, R])} \leq M_3$, for all $\omega \in S^{N-1}$.

Then

$$(2.33) \quad \begin{aligned} & (i) \quad \left| \int_{B(0,R)} f(y) dy - \left(\int_{S^{N-1}} f(R\omega) d\omega \right) R^{N-1} (R-t) \right| \\ & \leq \frac{2\pi^{\frac{N}{2}} M_3}{\Gamma\left(\frac{N}{2}\right) \left(1 + \frac{1}{p}\right)} \left[t^{1+\frac{1}{p}} + (R-t)^{1+\frac{1}{p}} \right], \end{aligned}$$

for all $t \in [0, R]$,

(ii) at $t = \frac{R}{2}$, the right hand side of (2.33) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y) dy - \left(\int_{S^{N-1}} f(R\omega) d\omega \right) \frac{R^N}{2} \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega) = 0$, for all $\omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanishes on $\partial B(0, R)$), we obtain

$$\left| \int_{B(0,R)} f(y) dy \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, \dots, \bar{N} \in \mathbb{N}$, it holds

$$(2.34) \quad \begin{aligned} & \left| \int_{B(0,R)} f(y) dy - \frac{R^N}{\bar{N}} (\bar{N} - j) \int_{S^{N-1}} f(R\omega) d\omega \right| \\ & \leq \frac{2\pi^{\frac{N}{2}} M_3}{\left(1 + \frac{1}{p}\right) \Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\bar{N}} \right)^{1+\frac{1}{p}} \left[j^{1+\frac{1}{p}} + (\bar{N} - j)^{1+\frac{1}{p}} \right], \end{aligned}$$

(v) when $\bar{N} = 2$ and $j = 1$, (2.34) turns to

$$\left| \int_{B(0,R)} f(y) dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega) d\omega \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\left(1 + \frac{1}{p}\right) \Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.16, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

REFERENCES

- [1] R. P. Agarwal, S. S. Dragomir: *An application of Hayashi's inequality for differentiable functions*, Computers Math. Applic., 6 (1996), 95-99.
- [2] G. A. Anastassiou: *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.
- [3] G. A. Anastassiou: *General Iyengar type inequalities*, submitted, 2018.
- [4] Xiao-Liang Cheng: *The Iyengar-type inequality*, Applied Math. Letters 14 (2001), 975-978.
- [5] K. S. K. Iyengar: *Note on an inequality*, Math. Student 6, (1938), 75-76.
- [6] Zheng Liu: *Note on Iyengar's inequality*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 16 (2005), 29-35.
- [7] F. Qi: *Further generalizations of inequalities for an integral*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 8 (1997), 79-83.
- [8] W. Rudin: *Real and Complex Analysis*, International Student edition, Mc Graw Hill, London, New York, 1970.
- [9] D. Stroock: *A Concise Introduction to the Theory of Integration*, Third Edition, Birkhäuser, Boston, Basel, Berlin, 1999.

DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF MEMPHIS,
MEMPHIS, TN 38152,
U.S.A.
E-mail address: ganastss@memphis.edu