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Research Article

The algebra of thin measurable operators is directly finite

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau) = S_0(\mathcal{M},\tau) + \mathbb{C}I$ be the *-algebra of all operators $X = A + \lambda I$ with $A \in S_0(\mathcal{M},\tau)$ and $\lambda \in \mathbb{C}$. It is proved that every operator of $T(\mathcal{M},\tau)$ that is left-invertible in $T(\mathcal{M},\tau)$ is in fact invertible in $T(\mathcal{M},\tau)$. It is a generalization of Sterling Berberian theorem (1982) on the subalgebra of thin operators in $\mathcal{B}(\mathcal{H})$. For the singular value function $\mu(t;Q)$ of $Q = Q^2 \in S(\mathcal{M},\tau)$, the inclusion $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ holds for all $t > 0$. It gives the positive answer to the question posed by Daniyar Mushtari in 2010.

Keywords: Hilbert space, von Neumann algebra, semifinite trace, τ-measurable operator, τ-compact operator, singular value function, idempotent.

2020 Mathematics Subject Classification: 16E50, 46L51.

1. INTRODUCTION

In this paper, we extend the Sterling Berberian's result [2] (see also [12]) on direct finiteness of the algebra of thin operators on an infinite-dimensional Hilbert space to the Irving Segal's non-commutative integration setting [16]. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ, $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Let $S_0(\mathcal{M},\tau)$ be the *-algebra of all τ-compact operators and $T(\mathcal{M},\tau) = S_0(\mathcal{M},\tau) + \mathbb{C}I$ be the *-algebra of all operators $X = A + \lambda I$ with $A \in S_0(\mathcal{M},\tau)$ and a complex number λ. We prove that every operator of $T(\mathcal{M},\tau)$ left-invertible in $T(\mathcal{M},\tau)$ is actually invertible in $T(\mathcal{M},\tau)$ (Theorem 3.1). Assume that $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau)$. We have $AB \in T(\mathcal{M},\tau)$ if and only if $BA \in T(\mathcal{M},\tau)$ (Theorem 3.2). For the singular value function $\mu(t;Q)$ of $Q = Q^2 \in S(\mathcal{M},\tau)$, we have $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ for all $t > 0$ (Theorem 3.3). It is the positive answer to the question by Daniyar Mushtari of year 2010.

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2. PRELIMINARIES

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$, I be the unit of $\mathcal{M}$. Also $\mathcal{M}^+$ denotes the cone of positive elements in $\mathcal{M}$. A mapping $\varphi : \mathcal{M}^+ \to [0,+\infty]$ is called *a trace*, if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *normal*, if $X_i \uparrow X$ ($X_i, X \in \mathcal{M}^+$) $\Rightarrow$

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$\varphi(X) = \sup \varphi(X_i)$; *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{P}(\mathcal{M})$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(I - P) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [16], [14]. Let $\mathcal{L}^+$ and $\mathcal{L}^{\rm h}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)^{\rm h}$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$, then $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$. The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(s; X) = \inf\{\|XP\| : \ P \in \mathcal{P}(\mathcal{M}) \text{ and } \tau(I - P) \leq s\}.$$

Lemma 2.1. (see [10]) *We have* $\mu(s + t; XY) \leq \mu(s; X)\mu(t; Y)$ *for all* $X, Y \in S(\mathcal{M}, \tau)$ *and* $s, t > 0$.

The sets $U(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \ (\|XP\| \leq \varepsilon \text{ and } \tau(I - P) \leq \delta \text{ for some } P \in \mathcal{P}(\mathcal{M}))\}$, where $\varepsilon > 0, \ \delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $S(\mathcal{M}, \tau)$, called *the measure topology* [14]. Equipped with this topology, $S(\mathcal{M}, \tau)$ is a complete metrizable topological *-algebra in which $\mathcal{M}$ is dense. We will write $X_n \xrightarrow{\tau} X$ if a sequence $\{X_n\}_{n=1}^\infty$ converges to $X \in S(\mathcal{M}, \tau)$ in the measure topology on $S(\mathcal{M}, \tau)$.

The set of τ-compact operators $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \lim\limits_{t\to\infty} \mu(t; X) = 0\}$ is an ideal in $S(\mathcal{M}, \tau)$. For any closed and densely defined linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, the *null projection* $\mathrm{n}(X) = \mathrm{n}(|X|)$ is the projection onto its kernel $\mathrm{Ker}(X)$, the *range projection* $\mathrm{r}(X)$ is the projection onto the closure of its range $\mathrm{Ran}(X)$ and the *support projection* $\mathrm{supp}(X)$ of X is defined by $\mathrm{supp}(X) = I - \mathrm{n}(X)$.

The two-sided ideal $\mathcal{F}(\mathcal{M}, \tau)$ in $\mathcal{M}$ consisting of all elements of τ-finite range is defined by

$$\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : \ \tau(\mathrm{r}(X)) < \infty\} = \{X \in \mathcal{M} : \ \tau(\mathrm{supp}(X)) < \infty\}.$$

Equivalently, $\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : \ \mu(t; X) = 0 \text{ for some } t > 0\}$. Clearly, $S_0(\mathcal{M}, \tau)$ is the closure of $\mathcal{F}(\mathcal{M}, \tau)$ with respect to the measure topology [9].

3. Main results

Throughout the sequel, let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra, with some distinguished faithful normal semifinite trace τ.

Lemma 3.2. *We have* $|X| \in T(\mathcal{M}, \tau)$ *for every* $X \in T(\mathcal{M}, \tau)$.

Proof. The ideal $\mathcal{F}(\mathcal{M}, \tau)$ is a C^*-subalgebra in $\mathcal{M}$. Hence $F(\mathcal{M}, \tau) = \mathcal{F}(\mathcal{M}, \tau) + \mathbb{C}I$ is an unital C^*-subalgebra in $\mathcal{M}$ and if $X \in F(\mathcal{M}, \tau)$, then $|X| \in F(\mathcal{M}, \tau)$. Assume that $X \in T(\mathcal{M}, \tau)$, i.e., $X = A + \lambda I$ with $A \in S_0(\mathcal{M}, \tau)$ and $\lambda \in \mathbb{C}$. Since $\mathcal{F}(\mathcal{M}, \tau)$ is t_τ-dense in $S_0(\mathcal{M}, \tau)$, there exists a sequence $\{A_n\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{M}, \tau)$ such that $A_n \xrightarrow{\tau} A$ as $n \to \infty$. Then the sequence $X_n = A_n + \lambda I$, $n \in \mathbb{N}$, lies in $F(\mathcal{M}, \tau)$ and t_τ-converges to the operator X as $n \to \infty$. According to the results given above, $|X_n| = B_n + |\lambda| I$ with some $B_n \in F(\mathcal{M}, \tau)^{\rm h}$, $n \in \mathbb{N}$. Since $X_n \xrightarrow{\tau} X$ as $n \to \infty$, we have $X_n^* \xrightarrow{\tau} X^*$ as $n \to \infty$ by t_τ-continuity of the involution in $S(\mathcal{M}, \tau)$. Then via joint t_τ-continuity of the multiplication in $S(\mathcal{M}, \tau)$, we have $X_n^* X_n \xrightarrow{\tau} X^* X$ as $n \to \infty$. Therefore, we obtain $|X_n| \xrightarrow{\tau} |X|$ as $n \to \infty$ by t_τ-continuity of the real function $f(t) = \sqrt{t}$, $t \geq 0$ [18]. Thus the sequence $\{B_n\}_{n=1}^\infty$ t_τ-converges to a some operator $B \in S_0(\mathcal{M}, \tau)^{\rm h}$ and $|X| = B + |\lambda| I$. $\square$

Lemma 3.3. (see [4, Corollary 2.4]) *If $X \in T(\mathcal{M},\tau)$ and $XX^* \leq X^*X$, then $XX^* = X^*X$.*

Lemma 3.4. *The idempotents of $T(\mathcal{M},\tau)$ are the operators P, $I-P$, where P runs over the idempotent operators of $S_0(\mathcal{M},\tau)$.*

Proof. Assume that $X = A + \lambda I \in T(\mathcal{M},\tau)$ and $X^2 = X$. Then $A^2 + 2\lambda A + \lambda^2 I = A + \lambda I$, i.e., $\lambda \in \{0,1\}$. If $\lambda = 0$, then $A^2 = A$ and $A \in S_0(\mathcal{M},\tau)$ is an idempotent operator. Then $I - A \in T(\mathcal{M},\tau)$ and is also an idempotent. If $\lambda = 1$, then $A^2 = -A = (-A)^2$ and $-A \in S_0(\mathcal{M},\tau)$ is an idempotent operator. Then $I - (-A) \in T(\mathcal{M},\tau)$ and is also an idempotent. □

Consider $F_0(\mathcal{M},\tau) = \{A \in S_0(\mathcal{M},\tau) : \ \tau(\mathrm{r}(A)) < +\infty\}$ and $\mathcal{A}(\mathcal{M},\tau) = F_0(\mathcal{M},\tau) + \mathbb{C}I$. Then $\mathcal{A}(\mathcal{M},\tau)$ is a *-subalgebra of $T(\mathcal{M},\tau)$.

Lemma 3.5. *$\mathcal{A}(\mathcal{M},\tau)$ contains every idempotent of $T(\mathcal{M},\tau)$.*

Proof. Let Q be an idempotent operator of $S(\mathcal{M},\tau)$. Then

$$(Q + Q^* - I)^2 = I + (Q - Q^*)(Q - Q^*)^*$$

and by [6, Theorem 2.21] there exists a unique "range" projection $Q^\sharp \in \mathcal{P}(\mathcal{M})$, defined by the formula $Q^\sharp = Q(Q + Q^* - I)^{-1}$ with $(Q + Q^* - I)^{-1} \in \mathcal{M}$ and subject to the condition $Q^\sharp \cdot S(\mathcal{M},\tau) = Q \cdot S(\mathcal{M},\tau)$. By [6, Theorem 2.23], there exists a unique decomposition $Q = P + Z$, where $P = Q^\sharp \in \mathcal{P}(\mathcal{M})$ and $Z \in S(\mathcal{M},\tau)$ is a nilpotent so that $Z^2 = 0$ and $ZP = 0$, $PZ = Z$. Thus $QP = P$ and $PQ = Q$. Assume that $Q \in S_0(\mathcal{M},\tau)$. Since $QP = P$, we have $P \in S_0(\mathcal{M},\tau)$. Since the singular function $\mu(t;P) = \chi_{(0,\tau(P)]}(t)$ for all $t > 0$, we conclude that $P \in \mathcal{F}(\mathcal{M},\tau)$. Then by equality $PQ = Q$, we have $Q \in F_0(\mathcal{M},\tau)$ and apply Lemma 3.4. □

Lemma 3.6. *$F_0(\mathcal{M},\tau)$ is a regular ring.*

Proof. We show that for every operator $A \in F_0(\mathcal{M},\tau)$ the equation $AXA = A$ possesses a solution in $F_0(\mathcal{M},\tau)$. For $A \in F_0(\mathcal{M},\tau)$, the range projection $\mathrm{r}(A)$ and the support projection $\mathrm{supp}(A)$ lie in $\mathcal{F}(\mathcal{M},\tau)$. Consider the projection $P = \mathrm{r}(A) \bigvee \mathrm{supp}(A)$ in $\mathcal{F}(\mathcal{M},\tau)$ and the reduced von Neumann algebra $\mathcal{M}_P = P\mathcal{M}P$, the reduced faithful normal finite trace τ_P with $\tau_P(X) = \tau(PXP)$, $X \in \mathcal{M}_P^+$. The algebra $\mathcal{M}_P$ is finite, therefore $S(\mathcal{M}_P,\tau_P)$ is a regular ring by [15, Theorem 4.3]. Since $A \in S(\mathcal{M}_P,\tau_P)$, the equation $AXA = A$ admits a solution in $S(\mathcal{M}_P,\tau_P) \subset F_0(\mathcal{M},\tau)$. □

Idempotents P, Q of a ring $\mathcal{R}$ are said to be *equivalent* (in $\mathcal{R}$), written $P \sim Q$, if there exist elements $X, Y \in \mathcal{R}$ such that $XY = P$ and $YX = Q$ (replacing X, Y by PXQ, QYP, one can suppose that $X \in P\mathcal{R}Q$, $Y \in Q\mathcal{R}P$ [13, p. 22]). Projections (=self-adjoint idempotents) P, Q of a ring with involutions are said to be *-*equivalent* if there exists an element X such that $XX^* = P$ and $X^*X = Q$.

Theorem 3.1. *If $X, Y \in T(\mathcal{M},\tau)$ such that $XY = I$, then $YX = I$.*

Proof. In the terms of ring theory, we assert that the ring $T(\mathcal{M},\tau)$ is "directly finite" [11, p. 49]. Since $F_0(\mathcal{M},\tau)$ (by Lemma 3.6) and $\mathcal{A}(\mathcal{M},\tau)/F_0(\mathcal{M},\tau) \cong \mathbb{C}$ are both regular rings, $\mathcal{A}(\mathcal{M},\tau)$ is a regular ring [11, p. 2, Lemma 1.3]; since, moreover, the involution of $\mathcal{A}(\mathcal{M},\tau)$ is proper ($AA^* = 0$ implies $A = 0$), the algebra $\mathcal{A}(\mathcal{M},\tau)$ is *-regular in the sense of von Neumann [1, p. 229].

If X, Y are elements of $T(\mathcal{M},\tau)$ such that $XY = I$, then $P = YX$ is an idempotent of $T(\mathcal{M},\tau)$ such that $P \sim I$ in $T(\mathcal{M},\tau)$. By Lemma 3.5, we have $P \in \mathcal{A}(\mathcal{M},\tau)$; since $\mathcal{A}(\mathcal{M},\tau)$ is *-regular, there exists a projection $Q \in \mathcal{A}(\mathcal{M},\tau)$ such that $Q \cdot \mathcal{A}(\mathcal{M},\tau) = P \cdot \mathcal{A}(\mathcal{M},\tau)$ [1, p. 229, Proposition 3]. Then $P \sim Q$ in $\mathcal{A}(\mathcal{M},\tau)$ [13, p. 21, Theorem 14], a fortiori $P \sim Q$ in $T(\mathcal{M},\tau)$; already $P \sim I$ in $T(\mathcal{M},\tau)$, so $Q \sim I$ in $T(\mathcal{M},\tau)$ by transitivity. Since $T(\mathcal{M},\tau)$

satisfies the "square root" axiom (SR) and contains square roots of its positive elements (see Lemma 3.2 and [13, p. 90]), it follows that the projections P, I are *-equivalent in $T(\mathcal{M},\tau)$ [13, p. 35, Theorem 27], say $X \in T(\mathcal{M},\tau)$ with $XX^* = P$, $X^*X = I$. By Lemma 3.3, $P = I$; then $Q \cdot \mathcal{A}(\mathcal{M},\tau) = P \cdot \mathcal{A}(\mathcal{M},\tau) = \mathcal{A}(\mathcal{M},\tau)$ shows that $P = I$, that is, $YX = I$. □

Theorem 3.1 can obviously be reformulated as follows: if $A, B \in S_0(\mathcal{M},\tau)$ and $A+B+AB = 0$, then $AB = BA$. On invertibility in $S(\mathcal{M},\tau)$, see [17], [7] and [8].

Theorem 3.2. *Assume that $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau)$. Then $AB \in T(\mathcal{M},\tau)$ if and only if $BA \in T(\mathcal{M},\tau)$.*

Proof. "⇒". If $B \in S_0(\mathcal{M},\tau)$, then $BA \in S_0(\mathcal{M},\tau) \subset T(\mathcal{M},\tau)$. Assume that $B \notin S_0(\mathcal{M},\tau)$. Then $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M},\tau)$. Hence,

$$AB = \lambda A + AK = \mu I + K_1 \tag{3.1}$$

for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M},\tau)$.

Case 1: $\mu = 0$. Then we have $A \in S_0(\mathcal{M},\tau)$ by (3.1); hence $BA \in S_0(\mathcal{M},\tau) \subset T(\mathcal{M},\tau)$.

Case 2: $\mu \neq 0$. Then by (3.1), we have $\lambda A = \mu I + K_2$ with $K_2 = K_1 - AK \in S_0(\mathcal{M},\tau)$. Therefore, $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2$ and

$$BA = (\lambda I + K)\Big(\frac{\mu}{\lambda} I + \frac{1}{\lambda} K_2\Big) = I + K_3$$

with $K_3 = K_1 - AK + \frac{\mu}{\lambda} K + \frac{1}{\lambda} KK_1 - \frac{1}{\lambda} KAK \in S_0(\mathcal{M},\tau)$. Thus $BA \in T(\mathcal{M},\tau)$.

"⇐". We know that $X \in T(\mathcal{M},\tau)$ if and only if $X^* \in T(\mathcal{M},\tau)$, and apply the proof given above to the pair $\{A^*, B^*\}$. □

Corollary 3.1. *If $A \in S(\mathcal{M},\tau)$ and $B \in T(\mathcal{M},\tau) \setminus S_0(\mathcal{M},\tau)$ then the following conditions are equivalent:*

(i) $AB \in T(\mathcal{M},\tau)$;
(ii) $BA \in T(\mathcal{M},\tau)$;
(iii) $A \in T(\mathcal{M},\tau)$.

Proof. "(i)⇒(iii)". Let $B = \lambda I + K$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M},\tau)$. Then $AB = \lambda A + AK = \mu I + K_1$ for some $\mu \in \mathbb{C}$ and $K_1 \in S_0(\mathcal{M},\tau)$. Thus $\lambda A = \mu I + K_1 - AK$ and $A = \frac{\mu}{\lambda} I + \frac{1}{\lambda} K_1 - \frac{1}{\lambda} AK \in T(\mathcal{M},\tau)$. □

Theorem 3.3. *If $Q \in S(\mathcal{M},\tau)$ is such that $Q^2 = Q$, then $\mu(t;Q) \in \{0\} \bigcup [1,+\infty)$ for all $t > 0$. For the symmetry $U = 2Q - I$, we have $\mu(t;U) \geq 1$ for all $t > 0$.*

Proof. For $Q = Q^2 \notin S_0(\mathcal{M},\tau)$, we have $\mu(t;Q) \geq 1$ for all $t > 0$, see [5, Lemma 3.8]. Let $Q = Q^2 \in S_0(\mathcal{M},\tau)$ and P be "the range" projection of the idempotent Q, see the proof of Lemma 3.5. Since $QP = P$ and $P \in \mathcal{P}(\mathcal{M}) \bigcap \mathcal{F}(\mathcal{M},\tau)$, by Lemma 2.1 we have

$$1 = \mu(s+t;P) = \chi_{(0,\tau(P)]}(s+t) = \mu(s+t;QP) \leq \mu(s;P)\mu(t;Q) = \mu(t;Q)$$

for all $s, t > 0$ with $s + t \leq \tau(P)$. By tending s to $0+$, we obtain $\mu(t;Q) \geq 1$ for all $0 < t < \tau(P)$. By the right continuity of the function $\mu(t;\cdot)$, we have $\mu(\tau(P);Q) \geq 1$. If $t > \tau(P)$ then $\mu(t;P) = 0$; by the equality $PQ = Q$ and by Lemma 2.1, we obtain

$$0 \leq \mu(t;Q) = \mu(t;PQ) \leq \mu(t-\varepsilon;P)\mu(\varepsilon;Q) = 0$$

for all $\varepsilon > 0$ with $t - \varepsilon > \tau(P)$.

Let $Q \in S(\mathcal{M},\tau)$ be such that $Q^2 = Q$. For the symmetry $U = 2Q - I$, we have $U^2 = I$ and by Lemma 2.1 obtain

$$1 = \mu(2t;I) = \mu(2t;U^2) \leq \mu(t;U)\mu(t;U) = \mu(t;U)^2$$

for all $t > 0$. □

Note that for $Q \in \mathcal{M}$ such that $Q^2 = Q$ the relation $\mu(t; Q) \in \{0\} \bigcup [1, \|Q\|]$ for all $t > 0$ was obtained by another way in [3, item 1) of Lemma 3.8]. Theorem 3.3 gives the positive answer to the question by Daniyar Mushtari of year 2010.

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REFERENCES

[1] S. K. Berberian: *Baer *-rings. Die Grundlehren der mathematischen Wissenschaften, Band 195*, Springer-Verlag, New York-Berlin (1972).
[2] S. K. Berberian: *The algebra of thin operators is directly finite*, Publ. Sec. Mat. Univ. Autònoma Barcelona, **26** (2) (1982), 5-7.
[3] A. M. Bikchentaev: *Local convergence in measure on semifinite von Neumann algebras*, Proc. Steklov Inst. Math., **255** (2006), 35-48.
[4] A. M. Bikchentaev: *On normal τ-measurable operators affiliated with semifinite von Neumann algebras*, Math. Notes, **96** (3-4) (2014), 332-341.
[5] A. M. Bikchentaev: *Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra*, Math. Notes, **98** (3-4) (2015), 382-391.
[6] A. M. Bikchentaev: *On idempotent τ-measurable operators affiliated to a von Neumann algebra*, Math. Notes, **100** (3-4) (2016), 515-525.
[7] A. M. Bikchentaev: *On τ-essentially invertibility of τ-measurable operators*, Internat. J. Theoret. Phys., **60** (2) (2021), 567-575.
[8] A. M. Bikchentaev: *Essentially invertible measurable operators affiliated to a semifinite von Neumann algebra and commutators*, Sib. Math. J., **63** (2) (2022), 224-232.
[9] P. Dodds, B. de Pagter: *Normed Köthe spaces: A non-commutative viewpoint*, Indag. Math. (N.S.), **25** (2) (2014), 206-249.
[10] T. Fack, H. Kosaki: *Generalized s-numbers of τ-measurable operators*, Pacific J. Math., **123** (2) (1986), 269-300.
[11] K. R. Goodearl: *von Neumann regular rings, Monographs and Studies in Mathematics, vol. 4. Pitman (Advanced Publishing Program)*, Boston, Mass.-London (1979).
[12] I. Halperin: *On a theorem of Sterling Berberian*, C. R. Math. Rep. Acad. Sci. Canada, **3** (1) (1981), 33-35.
[13] I. Kaplansky: *Rings of operators*, W.A. Benjamin, Inc., New York-Amsterdam (1968).
[14] E. Nelson: *Notes on non-commutative integration*, J. Funct. Anal., **15** (2) (1974), 103-116.
[15] K. Saitô: *On the algebra of measurable operators for a general AW^*-algebra. II*, Tohoku Math. J. (2), **23** (3) (1971), 525-534.
[16] I. E. Segal: *A non-commutative extension of abstract integration*, Ann. Math., **57** (3) (1953), 401-457.
[17] I. D. Tembo: *Invertibility in the algebra of τ-measurable operators, in: Operator algebras, operator theory and applications, Oper. Theory Adv. Appl., vol. 195*, Birkhäuser Verlag, Basel (2010), 245-256.
[18] O. E. Tikhonov: *Continuity of operator functions in topologies connected with a trace on a von Neumann algebra*, Soviet Math. (Iz. VUZ), **31** (1) (1987), 110-114.

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