CONSTRUCTIVE MATHEMATICAL ANALYSIS
6 (2023), No. 2, pp. 102-106
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

On an interpolation sequence for a weighted Bergman space on a Hilbert unit ball

MOHAMMED EL AÏDI*

ABSTRACT. The purpose is to provide a generalization of Carleson's Theorem on interpolating sequences when dealing with a sequence in the open unit ball of a Hilbert space. Precisely, we interpolate a sequence by a function belonging to a weighted Bergman space of infinite order on a unit Hilbert ball and we furnish explicitly the upper bound corresponding to the interpolation constant.

Keywords: Analytic functions, interpolation sequences, weighted Bergman spaces, pseudohyberbolic distance, Fréchet differentiable functions.

2020 Mathematics Subject Classification: 30A99, 30H05, 32A36, 28E99, 46A04.

1. INTRODUCTION

Let us recall a known result that it has been shown that a sequence $\Gamma = (a_k)_{k\in\mathbb{N}}$ is interpolated by a function in $B^{\infty}_{\varphi^c}(\mathbb{D}^n)$, the set of holomorphic functions f on the complex unit ball $\mathbb{D}^n$ such that φf is bounded, where φ is strictly positive continuous function on $[0, 1)$ satisfying a few meaningful assumptions and the power c is a strictly positive constant [3]. Precisely, it has been shown the following theorem.

Theorem 1.1 ([3]). *Let $\Gamma = (a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}^n$ and $\prod_{j\in\mathbb{N}\setminus\{k\}} |\psi_{a_j}(a_k)| \geq \varphi(|a_k|)$ for all $k \in \mathbb{N}$. Then Γ is interpolated by a function in $B^{\infty}_{\varphi^c}(\mathbb{D}^n)$. Furthermore, an upper bound of the interpolation constant is given explicitly and it is independent of n and φ.*

$|\psi_{a_j}(a_k)|$ is the pseudohyperbolic distance between a_j and a_k such that $\psi_{a_j}(\cdot)$ is the $\mathbb{D}^n$-valued Möbius map on $\mathbb{D}^n$. Apropos of the proof of Theorem 1.1, concisely the author sets up an interpolating function belonging in $B^{\infty}_{\varphi^c}(\mathbb{D}^n)$ given in terms of a series of functions.

The goal of the present article is to show that Theorem 1.1 remains true when we swap $\mathbb{D}^n$ by a unit Hilbert ball, so we give a positive response on a question raised in Remark 3.1 in [3]. Therefore, let $\mathbb{B}_H = \{x \in H : \|x\|_H < 1\}$ be the open unit ball in $H = (H, \langle\cdot,\cdot\rangle_H; \|\cdot\|_H)$, an infinite dimensional complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle_H$ and the norm $\|\cdot\|_H$. E.g., $H = L_2(X, \mu)$, the space of square-integrable measurable functions on X with respect to the measure μ such that $\langle f, g\rangle_H = \int_X f(x)\overline{g(x)}d\mu(x)$ and $\|f\|_H = \left(\int_X |f(x)|^2 d\mu(x)\right)^{\frac{1}{2}}$.

Instead to use a holomorphic function, we employ a complex-valued analytic function on $\mathbb{B}_H$, i.e., a Fréchet differentiable function at all points in $\mathbb{B}_H$.

Received: 20.01.2023; Accepted: 03.06.2023; Published Online: 04.06.2023
*Corresponding author: Mohammed El Aïdi; melaidi@unal.edu.co
DOI: 10.33205/cma.1240126

2. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREM

Let $\mathcal{A}(\mathbb{B}_H)$ be the space of analytic functions on $\mathbb{B}_H$, ϕ be a strictly positive continuous function on $[0,1)$, where its inverse is logarithmically convex. Let $L^\infty_\phi(\mathbb{B}_H) = \left(L^\infty_\phi(\mathbb{B}_H), \|\cdot\|_{\infty,\phi}\right)$ be the space of complex-valued measurable functions f on $\mathbb{B}_H$ such that $\phi(\|x\|_H).f(x)$ is bounded for all $x \in \mathbb{B}_H$ and $||f||_{\infty,\phi} = \sup_{x\in\mathbb{B}_H} \phi(||x||_H)|f(x)| < \infty$.

The weighted Bergman space of infinite order on $\mathbb{B}_H$ is defined by

$$B^\infty_\phi(\mathbb{B}_H) = \left\{f \text{ complex measurable functions on } \mathbb{B}_H : f \in \mathcal{A}(\mathbb{B}_H) \cap L^\infty_\phi(\mathbb{B}_H)\right\}.$$

The space $B^\infty_\phi(\mathbb{B}_H)$ is endowed with the induced norm $\|\cdot\|_{\infty,\phi}$. We suppose that the continuous function ϕ is not identically equal to one which implies that $B^\infty_\phi(\mathbb{B}_H)$ contains strictly $H^\infty(\mathbb{B}_H)$, the Hardy space of order infinity on $\mathbb{B}_H$. We recall that interpolating a sequence by a function in $H^\infty(\mathbb{B}_H)$ has been conducted in [8].

Let $(l^\infty_\phi, ||\cdot||_{l^\infty_\phi})$ be the weighted space of bounded sequences with respect to the sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{B}_H$ and which is defined by

$$l^\infty_\phi = \left\{v = (v_k)_{k\in\mathbb{N}} \in \mathbb{C} \text{ such that } (\phi(\|x_k\|_H)|v_k|)_{k\in\mathbb{N}} \in l^\infty\right\}$$

such that $\|v\|_{l^\infty_\phi} = \sup_{k\in\mathbb{N}} (\phi(\|x_k\|_H)|v_k|)$. In the sequel, we need the following definition of an interpolation sequence.

Definition 2.1. *Let c be a positive constant, we say that $\Gamma = (x_k)_{k\in\mathbb{N}}$ is an interpolation sequence for $B^\infty_{\phi^c}(\mathbb{B}_H)$ if for every complex-valued sequence $v = (v_k)_{k\in\mathbb{N}} \in l^\infty_{\phi^{c-4}}$, there is $f \in B^\infty_{\phi^c}(\mathbb{B}_H)$ such that $f(a_k) = v_k$. The associated interpolation constant is the smaller constant M such that $||f||_{\infty,\phi} \leq M||v||_{l^\infty_{\phi^{c-4}}}$.*

The pseudohyperbolic distance between two points x, y belonging to $\mathbb{B}_H$ is defined by $\|\Phi_y(x)\|_H$ such that $\Phi_y(x)$ is the Möbius transformation on $\mathbb{B}_H$ defined by $\Phi_y(x) = (s_yQ_y + P_y)m_y(x)$ such that m_y is the $\mathbb{B}_H$-valued analytic map on $\mathbb{B}_H$ and defined as $m_y(x) = \frac{y-x}{1-\langle y,x\rangle_H}$, $P_y(x) = \frac{\langle y,x\rangle_H}{\|y\|^2_H} y$, $Q_y(x) = x - P_y(x)$, and $s_y = \sqrt{1-\|y\|^2_H}$. It is known (see Page 99 in [5]) that

$$\|\Phi_y(x)\|^2_H = 1 - \frac{(1-\|x\|^2_H)(1-\|y\|^2_H)}{|1-\langle x,y\rangle_H|^2}. \tag{2.1}$$

Our main result states

Theorem 2.2. *Let $\Gamma = (x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}_H$ such that $\prod_{j\in\mathbb{N}\setminus\{k\}} \|\Phi_{x_j}(x_k)\|_H \geq \phi(\|x_k\|_H)$ for all $k \in \mathbb{N}$ such that ϕ be a strictly positive continuous function on $[0,1)$ such that its inverse is logarithmically convex. Then Γ is interpolated by a function belonging to $B^\infty_{\phi^c}(\mathbb{B}_H)$ and an upper bound of the associated interpolation constant is provided explicitly and does not rely on the weight function ϕ.*

As we observe that the announcement of the main result is almost the same as the one stated in Theorem 1.1, where $\mathbb{D}_n$ is substituted by $\mathbb{B}_n$ and the complex modulus is substituted by $\|\cdot\|_H$. The novelty of the proof of Theorem 2.2 is that we use the pseudohyperbolic distance between two points in $\mathbb{B}_H$ and essentially Equality (2.1).

In the following section, we furnish the proof of Theorem 2.2 in two parts and we employ the techniques used in [2, 3, 6, 7]. The first part is on building an interpolation function, see Subsection 3.1, and the second one focuses on the interpolation constant, see Subsection 3.2.

3. Proof of the main Theorem

3.1. **On an appropriate interpolating function.** Let us consider the following series of functions on $\mathbb{B}_H$

$$G(x) = \sum_{k=1}^{\infty} v_k G_k(x) \text{ for } x \in \mathbb{B}_H, \tag{3.2}$$

where $(v_k)_{k\in\mathbb{N}} \in l^{\infty}_{\phi^{c-4}}$ such that each G_k is an analytic function on $\mathbb{B}_H$ defined as

$$G_k(x) = \left(\frac{1-\|x_k\|_H^2}{1-\langle x_k, x\rangle_H}\right)^4 \mathcal{W}(x_k, x)\mathcal{V}(x_k, x) \prod_{j\in\mathbb{N}\setminus\{k\}} \frac{\langle \Phi_{x_j}(x_k), \Phi_{x_j}(x)\rangle_H}{\|\Phi_{x_j}(x_k)\|_H^2},$$

where $x \in \mathbb{B}_H$, $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}_H$, $\mathcal{W}(x_k,\cdot)$ and $\mathcal{V}(x_k,\cdot)$ are two analytic functions on $\mathbb{B}_H$. Precisely,

$$\mathcal{W}(x_k, x) = \exp\left[-\sum_{m\in\mathbb{N}} (\mathfrak{f}(x) - \mathfrak{f}(x_k)) \frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{1-|\langle x_m, x_k\rangle_H|^2}\right]$$

with $\mathfrak{f}(x) = \frac{1+\langle x_m, x\rangle_H}{1-\langle x_m, x\rangle_H}$ which is well defined due the fact by using Cauchy-Schwarz inequality, we have $1-\langle x_m, x\rangle_H > 0$ and $\mathcal{V}(x_k, x) = \exp(\partial u(\widetilde{\psi}(x_k)).(\widetilde{\psi}(x) - \widetilde{\psi}(x_k)))$, where $\partial u(\widetilde{\psi}(x_k)).(\widetilde{\psi}(x) - \widetilde{\psi}(x_k))$ is the inner product in $\mathbb{C}^n$ between $\partial u(\widetilde{\psi}(x_k))$ and $\widetilde{\psi}(x) - \widetilde{\psi}(x_k)$ where u is a real-valued convex function on $\mathbb{C}^n$ and $\widetilde{\psi}$ is a $\mathbb{C}^n$-valued surjective map on $\mathbb{B}_H$. Consequently, from the definitions of $\mathcal{W}$ and $\mathcal{V}$, we have $G_k(x_k) = 1$ and for $j \neq k$ we have $G_k(x_j) = 0$ this due to the fact that $\Phi_{x_j}(x_j) = 0$, see (2.1). Whence, the sequence $(a_k)_{k\in\mathbb{N}}$ is interpolated by G and in the next subsection, we prove that $G \in B^{\infty}_{\phi^c}(\mathbb{B}_H)$ and provide explicitly an upper bound associated to the interpolation constant.

3.2. **On the interpolation constant.** By using the hypothesis of Theorem 2.2, that is, for each $k \in \mathbb{N}$, $\prod_{j\in\mathbb{N}\setminus\{k\}} \|\Phi_{x_j}(x_k)\|_H$ is bigger than $\phi(\|x_k\|_H)$, we have

$$|G_k(x)| \leq \left(\frac{1-\|x_k\|_H^2}{1-\langle x_k, x\rangle_H}\right)^4 |\mathcal{W}(x_k, x)||\mathcal{V}(x_k, x)|\phi^{-2}(\|x_k\|_H). \tag{3.3}$$

Let us look an upper bound for $|\mathcal{W}(x_k, x)|$. So, since that we work in a complex Hilbert space, we have $\Re\mathfrak{f}(x) = \frac{1-|\langle x_m, x\rangle_H|^2}{|1-\langle x_m, x\rangle_H|^2}$. Whence, we have

$$\begin{aligned} |\mathcal{W}(x_k, x)| &= \exp\left[-\sum_{m\in\mathbb{N}} \frac{1-|\langle x_m, x\rangle_H|^2}{|1-\langle x_m, x\rangle_H\rangle^2} \frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{1-|\langle x_m, x_k\rangle_H|^2}\right] \\ &\times \exp\left[\sum_{m\in\mathbb{N}} \frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{|1-\langle x_m, x_k\rangle_H|^2}\right]. \end{aligned} \tag{3.4}$$

Let us show that the terms $\exp\left[\sum_{m\in\mathbb{N}} \frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{|1-\langle x_m, x_k\rangle_H|^2}\right]$ is upper bounded by $\exp(1)\phi^{-2}(\|x_k\|_H)$. For $x > 0$, we have $1 - x \leq \exp(-x)$, thus by employing, successively, this inequality with $y_{m,k} = \frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{|1-\langle x_m, x_k\rangle_H|^2} > 0$, the square of the pseudohyperbolic distance equality (2.1), and the assumption of Theorem 2.2, we obtain

$$\begin{aligned}\exp\left[-\sum_{m\in\mathbb{N}} y_{m,k}\right] &= \prod_{m\in\mathbb{N}} \exp(-y_{m,k}) \\ &= \exp(-1) \prod_{m\in\mathbb{N}\setminus\{k\}} \exp(-y_{m,k}) \\ &\geq \exp(-1) \prod_{m\in\mathbb{N}\setminus\{k\}} \|\Phi_{x_m}(x_k)\|_H^2 \geq \exp(-1)\phi^2(\|x_k\|_H).\end{aligned}$$

Hence, Equality (3.4) implies

$$|\mathcal{W}(x_k,x)| \leq \frac{\exp(1)}{\phi^2(\|x_k\|_H)} \exp\left[-\sum_{m\in\mathbb{N}} A_{m,k}(x)\right] \tag{3.5}$$

such that $A_{m,k}(x) = \frac{1-|\langle x_m,x\rangle_H|^2}{|1-\langle x_m,x\rangle_H|^2}\frac{(1-\|x_m\|_H^2)(1-\|x_k\|_H^2)}{1-|\langle x_m,x_k\rangle_H|^2}$.

Let us reorder the sequence $(x_k)_{k\in\mathbb{N}}$, for obtaining an increasing sequence $(\|x_k\|_H)_{k\in\mathbb{N}}$, then by using the fact that $\frac{1-|\langle x_m,x\rangle_H|^2}{1-|\langle x_m,x_k\rangle_H|^2} \geq \frac{1-\|x_m\|_H^2}{8(1-|\langle x_k,x\rangle_H|^2)}$ whenever $\|x_m\|_H \geq \|x_k\|_H$, for the proof see Lemmas 3.8 and 3.9 in [8], and Inequality (3.5) becomes

$$|\mathcal{W}(x_k,x)| \leq \frac{\exp(1)}{\phi^2(\|x_k\|_H)} \exp\left[-\frac{\mathfrak{X}\mathfrak{T}_k}{8}\right] \tag{3.6}$$

such that $\mathfrak{X} = \frac{1-\|x_k\|_H^2}{1-|\langle x_k,x\rangle_H|^2}$ and $\mathfrak{T}_k = \sum_{m\geq k}\left(\frac{1-\|x_m\|_H^2}{|1-\langle x_m,x\rangle_H|}\right)^2$.

Let $b_m(x) = \left(\frac{1-\|x_m\|_H^2}{|1-\langle x_m,x\rangle_H|}\right)^2$, then thanks to the triangle inequality, we have $b_k(x) \leq 4\mathfrak{X}^2$, and we observe that the function $g_{\mathfrak{X}}(\tau) = \mathfrak{X}^2\exp\left(-\frac{\mathfrak{X}\tau}{8}\right)$ for $\tau > 0$, is at most equal $h(\tau) = \min\left(1, \frac{256}{\exp(2)\tau^2}\right)$. Accordingly, Inequality (3.6) becomes

$$\begin{aligned}b_k(x)\,|\mathcal{W}(x_k,x)| &\leq \frac{4\exp(1)\mathfrak{X}^2}{\phi^2(\|x_k\|_H)}\exp\left(-\frac{\mathfrak{X}\mathfrak{T}_k}{8}\right) \\ &\leq \frac{4\exp(1)}{\phi^2(\|x_k\|_H)} h\left(\mathfrak{T}_k\right).\end{aligned} \tag{3.7}$$

Now, from the definition of $\mathcal{V}$ and the use the properties of the convex function u, we have $|\mathcal{V}(x_k,x)| \leq \exp(u(\widetilde{\psi}(x)) - u(\widetilde{\psi}(x_k)))$. Furthermore, since that the inverse of ϕ is logarithmically convex, let us choose $u(\widetilde{\psi}(x)) = -c\log(\phi(\|x\|_H))$ and we have

$$|\mathcal{V}(x_k,x)| \leq \phi^c(\|x_k\|_H)\phi^{-c}(\|x\|_H). \tag{3.8}$$

We recall that G_k satisfies

$$|G_k(x)| \leq \left(\frac{1-\|x_k\|_H^2}{1-\langle x_k,x\rangle_H}\right)^4 |\mathcal{W}(x_k,x)||\mathcal{V}(x_k,x)|\phi^{-2}(\|x_k\|_H). \tag{3.9}$$

Whence, by using Inequalities (3.7)-(3.9) we obtain

$$\phi^c(\|x\|_H)\phi^{4-c}(\|x_k\|_H)|G_k(x)| \leq 4\exp(1)b_k(x)h\left(\mathfrak{T}_k\right). \tag{3.10}$$

The function $h(\tau)$ decreases on $[\mathfrak{T}_{k+1},\mathfrak{T}_k]$, then by using Inequality (3.10), we have

$$\phi^c(\|x\|_H)\phi^{4-c}(\|x_k\|_H)|G_k(x)| \leq 4\exp(1)\int_{\mathfrak{T}_{k+1}}^{\mathfrak{T}_k} h(\tau)d\tau. \tag{3.11}$$

Therefore, by using the definition of $h(\tau)$ and Inequality (3.11), we have

$$\begin{aligned}\sum_{k\in\mathbb{N}} \phi^c(\|x\|_H)\phi^{4-c}(\|x_k\|_H)|G_k(x)| &\leq 4\exp(1)\sum_{k\in\mathbb{N}}\int_{\mathfrak{T}_{k+1}}^{\mathfrak{T}_k} h(\tau)d\tau \\ &\leq 4\exp(1)\int_0^{\infty} h(\tau)d\tau \\ &= 47.0886. \end{aligned} \tag{3.12}$$

We recall that $G(x) = \sum_{k=1}^{\infty} v_k G_k(x)$, then from (3.12), we have

$$\begin{aligned}|G(x)| \leq \sum_{k=1}^{\infty} |v_k||G_k(x)| &\leq \|v\|_{l^{\infty}_{\phi^{c-4}}} \sum_{k=1}^{\infty} \phi^{4-c}(|x_k|)|G_k(x)| \\ &\leq 47.0886\|v\|_{l^{\infty}_{\phi^{c-4}}}\phi^{-c}(\|x\|_H).\end{aligned}$$

Thus, $\|G\|_{\infty,\phi} = \sup_{x\in\mathbb{B}_H}\phi^c(\|x\|_H)|G(x)| \leq 47.0886\|v\|_{l^{\infty}_{\phi^{c-4}}} < \infty$, i.e., $G \in B^{\infty}_{\phi^c}(\mathbb{B}_H)$, consequently the sequence Γ is interpolated by the function G, furthermore an upper bound of the interpolation constant is equal to 47.0886. The proof of Theorem 2.2 is complete.

On an extension

We are asking whether it possible to state an analogue result of Theorem 2.2, for a proper subspace of a suitable weighted Bergman space of infinite order on $\mathbb{B}_H$ and containing a proper subspace of $H^{\infty}(\mathbb{B}_H)$. E.g., interpolating sequences for a proper space of $H^{\infty}(\mathbb{D})$ has been conducted by Dyakonov [1]. Also, we are asking whether our result remains true for a function belonging to a Bloch-type space on $\mathbb{B}_H$, see, e.g., [9].

References

[1] K. M. Dyakonov: *A free interpolation problem for a subspace of H^{∞}*, Bull Lond Math Soc., **50** (2018), 477–486.
[2] B. Berndtsson: *Interpolating sequences for H^{∞} in the ball*, Nederl Akad Wetensch Indag Math., **47** (1) (1985), 1–10.
[3] M. El Aïdi: *On the interpolation constant for weighted Bergman spaces of infinite order*, Complex Var. Elliptic Equ., **64** (6) (2019), 1043–1049.
[4] P. Galindo, A. Miralles: *Interpolating sequences for bounded analytic functions*, Proc Amer Math Soc., **135** (10) (2007), 3225–3231.
[5] K. Goebel, S. Reich: *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, Inc., New York and Basel, 1984.
[6] P. Jones: *L^{∞}-estimates for the δ problem in a half-plane*, Acta Math., **150** (1983) 137–152.
[7] X. Massaneda: *Interpolation by holomorphic functions in the unit ball with polynomial growth* Ann Fac Sci Toulouse Math., **6** (2) (1997), 277–296.
[8] A. Miralles: *Interpolating sequences for $H^{\infty}(B_H)$*, Quaest Math., **39** (6) (2016), 785–795.
[9] T. T. Quang: *Banach-valued Bloch-type functions on the unit ball of a Hilbert space and weak spaces of Bloch-type*, Constr. Math. Anal., **6** (1) (2023), 6–21.

Mohammed El Aïdi
Universidad Nacional De Colombia, Sede Bogotá
Facultad De Ciencias
Departamento De Matemáticas
Edificio 404, Bogotá , D.C. Colombia
ORCID: 0000-0002-3032-0879
E-mail address: melaidi@unal.edu.co