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TASKS AND META-TASKS TO PROMOTE PRODUCTIVE MATHEMATICAL DISCOURSE IN COLLABORATIVE DIGITAL ENVIRONMENTS

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ABSTRACT: Rich tasks can be vehicles for productive mathematical discussions. How to support such discourse in collaborative digital environments is the focus of our theorization and empirical examination of task design that emerges from a larger research project. We present the theoretical foundations of our task design principles that developed through an iterative research design for a project that involves secondary teachers in online courses to learn discursively dynamic geometry by collaborating on construction and problem-solving tasks in a cyberlearning environment. In this study, we discuss a task and the collaborative work of a team of teachers to illustrate relationships between the task design, productive mathematical discourse, and the development of new mathematics knowledge for the teachers. Implications of this work suggest further investigations into interactions between characteristics of task design and learners mathematical activity.

Key words: Collaboration, dynamic geometry, mathematical discourse, task design, technology

INTRODUCTION

Mathematical tasks shape significantly what learners learn and structure their classroom discourse (Hiebert & Wearne, 1993). Such discussions when productive involve essential mathematical actions and ideas such as representations, procedures, relations, patterns, invariants, conjectures, counterexamples, and justifications and proofs about objects and relations among them. Nowadays, these mathematical objects and relations can be conveniently and powerfully represented in digital environments such as computers, tablets, and smartphones. Most of these environments contain functionality for collaboration. However, in such collaborative, digital environments, the design of tasks that promote productive mathematical discussions still requires continued theorization and empirical examination (Margolinas, 2013). To theorize and investigate features of tasks that promote mathematical discussions, we are guided by this question: What features of tasks support productive discourse in collaborative, digital environments? Knowing these features will inform the design of rich tasks that promote mathematical discussions so that engaged and attentive learners build mathematical ideas and convincing forms of argumentation and justification in digital and virtual environments.

In virtual collaborative environments, the resources available to teachers to orchestrate collaboration and discourse among learners are different from those in traditional presential classroom environments. The salient difference is that in presential classroom environments the teacher is physically present, whereas in a virtual learning environment the teacher is artificially present; that is, the teacher exists largely as an artifact of digital tools. Consequently, the design of the tasks that are to be objects of learners' activities in virtual environments need to be constructed in ways that support particular learning goals such as productive mathematical discourse.

We share Sierpinska's (2004) consideration that "the design, analysis, and empirical testing of mathematical tasks, whether for purposes of research or teaching, is one of the most important responsibilities of mathematics education" (p. 10). In this paper, we focus on the design of tasks that embody particular intentionalities of an educational designer who aims to promote and support productive discourse in collaborative, digital environments. Our work employs a specific virtual environment that supports synchronous collaborative discourse and provides tools for mathematics discussions and for creating graphical and semiotic objects for doing mathematics. The environment, Virtual Math Teams (VMT), has been the focus of years of development

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by a team led by Gerry Stahl, Drexel University, and Stephen Weimar, The Math Forum @ Drexel University, and the target of much research (see, for example, Stahl, 2008; Stahl, 2009). Recently, research has been conducted on an updated VMT with a multiuser version of a dynamic geometry environment, GeoGebra, (Grisi-Dicker, Powell, Silverman, & Fetter, 2012; Powell, Grisi-Dicker, & Alqahtani, 2013; Stahl, 2013, 2015). Our tasks are designed for this new environment—VMTwG. Though the environment and its functionalities are not the specific focus of this paper, we will later describe some of its important features to provide context for understanding our design of tasks. Our focus here is to describe how we address challenges involved in designing tasks to orchestrate productive mathematical discourse in an online synchronous and collaborative environment. We first describe the theoretical foundation that guides our design of tasks to promote potentially productive mathematical discourse among small groups of learners working in VMTwG. Afterward, we describe our task-design methodology and follow with an example of a task along with the mathematical insights a small team of teachers developed discursively as they engaged with the task. We conclude with implications and suggestions areas for further research.

THEORETICAL PERSPECTIVE

The theoretical foundation of our perspective on task design for collaborative digital environments to promote productive mathematical discourse rests on a dialogic notion of mathematics (Gattegno, 1987), a view of mathematics curriculum (Hewitt, 1999), what we call epistemic tools (Ray, 2013), the co-active infrastructure of dynamic mathematics environments (Hegedus & Moreno-Armella, 2010), and a sociocultural theory both of task and activity (Christiansen & Walther, 1986) and of instrument-mediated activity (Rabardel & Beguin, 2005).

Our notion of productive mathematical discourse rests on a particular view of what constitutes mathematics. From a psychological perspective, Gattegno (1987) posits that doing mathematics is based on dialog and perception:

No one doubts that mathematics stands by itself, is the clearest of the dialogues of the mind with itself. Mathematics is created by mathematicians conversing first with themselves and with one another. Still, because these dialogues could blend with other dialogues which refer to perceptions of reality taken to exist outside Man…Based on the awareness that relations can be perceived as easily as objects, the dynamics linking different kinds of relationships were extracted by the minds of mathematicians and considered *per se*. (pp. 13-14)

Mathematics results when a mathematician or any interlocutor talks to herself and to others about specific perceived objects, relations among objects, and dynamics involved with those relations (or relations of relations). For dialogue about these relations and dynamics to become something that can be reflected upon, it is important that they not be ephemeral but rather have residence in a material (physical or semiotic) record or inscription. On the one hand, through moment-to-moment discursive interactions, interlocutors can create inscriptions and, during communicative actions, achieve shared meanings of them. On the other hand, inscriptions can represent encoded meanings that—based on previous discursive interactions—interlocutors can grasp as they decode the inscriptions. Thus, inscriptive meanings and the specific perceived content of experience are dialectically related and mutually constitutive through discourse.

Voiced and inscribed mathematical meanings arise through discursive interactions, discussions. Pirie and Schwarzenberger (1988) define a mathematical discussion involving learners this way: "It is purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction" (p. 461). From a sociocultural perspective, we understand "purposeful talk" as goal-directed discourse and "on a mathematical subject" as about mathematical objects, relations and dynamics of relations. In the setting of VMT with interlocutors—teams of pupils, students, or teachers—collaborating and usually without the contemporaneous presence of a teacher, the discursive contributions and interactions genuinely emanate from the interlocutors. As such, we define productive mathematical discourse to be goal-directed discursive exchanges about mathematical objects, relations, and dynamics of relations, including questioning, affirming, reasoning, justifying, and generalizing.

Through discourse, interlocutors among themselves construct or from others become aware of mathematical content. As Hewitt (1999) posits, mathematical content intended for learners to engage can be parsed into two essential categories. The first category pertains to content that is arbitrary in the sense that it refers to semiotic conventions such as names, labels, and notations. These conventions are historical and cultural, examples of which are the Cartesian axes, coordinates, names of coordinates, and notational rules. These conventions could have been otherwise and hence are arbitrary. Moreover, they cannot be constructed or appropriated through attentive noticing or awareness but rather must be known through memorization and association.

The second essential category concerns mathematical content that is necessary. These are ideas or properties that can be derived by attending to and noticing relations among objects as well as dynamics linking relations. For instance, when two planar, congruent circles have exactly two points of intersection, then an isosceles triangle can always be formed by choosing its vertices to be the circles' centers and one intersection point. This conclusion, once known can be considered a cultural tool, is derivable, could not be otherwise, and therefore necessary. Relations among objects, dynamics of relations, and properties that can be worked out are necessary mathematical content. These particular mathematical ideas are historical and cultural tools and can be appropriated through awareness.

Whether particular necessary mathematical content is appropriated depends on awarenesses already possessed and attentive noticing. Awareness and noticing are elements that need to be accounted for in the design of tasks. As Hewitt (1999) notes

If a student does have the required awareness for something, then I suggest the teacher's role is not to inform the student but to introduce tasks which help students to use their awareness in coming to know what is necessary. (p. 4)

Within this pedagogic paradigm, if students do not have requisite awareness, then they are invited to engage tasks that enable them to construct the required awareness. Constructing the awareness involves thinking mathematically. The teacher informs them of those cultural tools that are arbitrary and, by definition, do not entail mathematical reasoning and invite them to use their existing awareness to notice and reason about necessary relations and relations of relations so as to appropriate new mathematical ideas through their discursive interaction.

To increase the probability that the discourse of interlocutors is mathematically productive, it is useful that they employ individual and collaborative discursive means to make sense of mathematical situations. For this purpose, we invite interlocutors to employ particular epistemic tools. That is, to ask questions of themselves and of their interlocutors that query what they perceive, how it connects to what they already know, and what they want to know more about it. Specifically, these tools include three questions that interlocutors explicitly or implicitly engage: (1) What do you notice? (2) What does it mean to you? (3) What do you wonder about? The first and third questions come directly from work of The Math Form @ Drexel University (see, Ray, 2013). The second question is one that we have added. The purpose of these questions is to foster generative discussions within small groups of interlocutors that are grounded in their attention on perceivable, not necessarily visible, contents of experience that can be described as objects, relations among objects, and dynamics linking different relations. Using the epistemic tools, interlocutors' responses become public, relevant, and accountable. The idea is for interlocutors' to practice consciously these epistemic tools and over time become incorporated into their mathematical habits of mind.

The epistemic tools, among other things, are useful for enabling reflection on perceived infrastructural reactions of a dynamic geometry environment to interlocutors' actions in the environment. As they drag (click, hold, and slide) a base point of an object in a constructed figure, the environment redraws and updates information on the screen, preserving constructed geometrical relations among the figure's objects. This reaction to learners' dragging establishes a dialectical co-active relationship as the learner and the environment react to each other (Hegedus & Moreno-Armella, 2010). As learners attend to the environment's reaction, they experience and, since it responds in ways that are valid in Euclidean geometry, may become aware of underlying mathematical relations among objects such as dependencies.

Another role of the epistemic tools is to scaffold interlocutors' activity directed to understand and solve a mathematical task. We view tasks and activity from a sociocultural perspective. Within this perspective, Christiansen and Walther (1986) distinguish between task and activity in that "the *task* (the assignment set by the teacher) becomes the object for the student's activity" (p. 260). A task is the challenge or set of instructions that a teacher sets. An activity is the set of actions learners perform directed toward accomplishing the task. The activity is what students do and what they build and act upon such as material, mental, or semiotic objects and relations among the objects. The task initiates activity and is the object of students' activity.

Given the new digital, collaborative environments in which teaching and learning can occur, we find it theoretically useful to extend Christiansen and Walther's (1986) distinction of task and activity beyond analog environments: The purpose of a mathematical task in collaborative digital environments is to initiate and foster productive mathematical, discursive activity. The discursive activity is what learners communicate and do, what they build and act upon such as material, mental, or semiotic objects and relations. The digital, mathematical task is the object of learners' collective and coordinated activity.

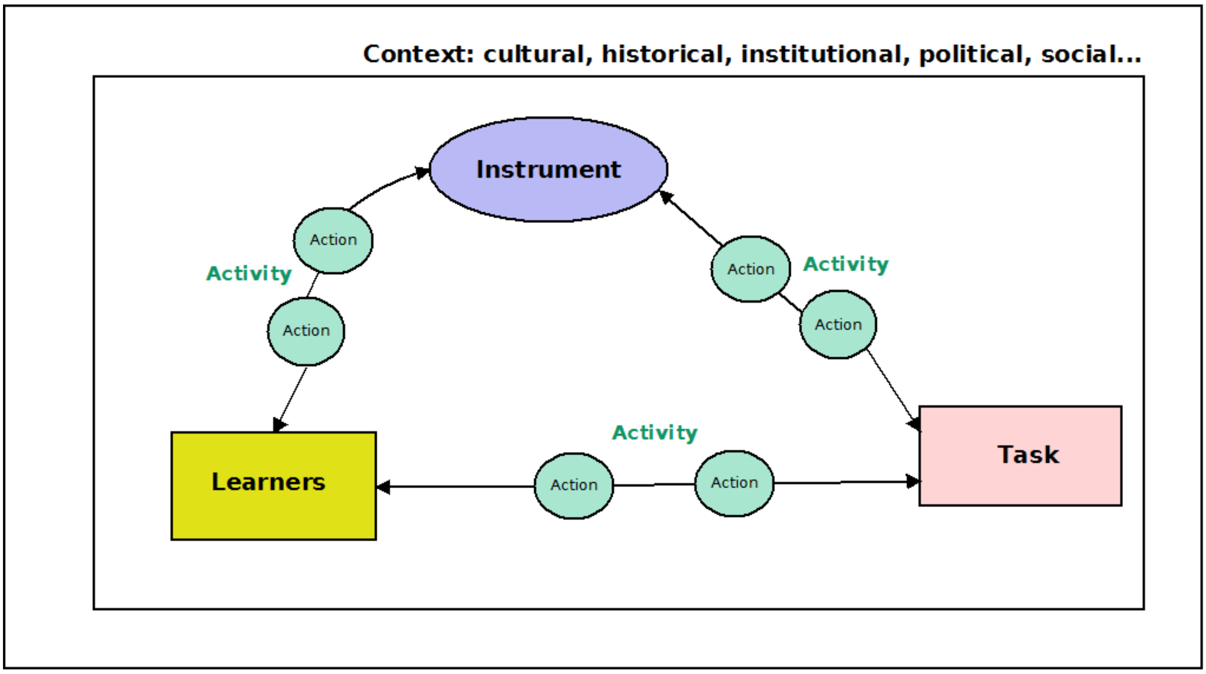

Figure 1. Relational Model Of Learners Engaged In Instrument-Mediated Activity Initiated By A Task.

Learners' activity directed toward a task is mediated by instruments. Before an instrument achieves its instrumental status, it is an artifact or tool. According to Rabardel and Beguin (2005) "the instrument is a composite entity made up of a tool component and a scheme component" (p. 442). The scheme component concerns how learners use the tool. Therefore, an instrument is a two-fold entity, part artifactual and part psychological. The transformation of an artifact into an instrument occurs through a dialectical process. One part accounts for potential changes in the instrument and the other accounts for changes in learners, respectively, instrumentalization and instrumentation. In instrumentalization, learners' interactions with a tool change how it is used, and consequently, learners enrich the artifact's properties. In instrumentation, the structure and functionality of a tool influence how learners use it, shaping, therefore, learners' cognition (Rabardel & Beguin, 2005). The processes of instrumentalization, instrumentation, and activity as well as the interaction of learners with themselves and the task reside within a particular, evolving context that is cultural, historical, institutional, political, social, and so on (see Figure 1).

In what follows, we present our design methodologies for mathematical tasks and a category of specialized tasks and provide examples of each.

TASK-DESIGN METHODOLOGY

Our methodology of task design embodies particular intentionalities for a virtual synchronous, collaborative environment, such as VMTwG, that has representation infrastructures (GeoGebra, a dynamic mathematics environment) and communication infrastructures (social network and chat features). The intentions are for mathematical tasks to be vehicles "to stimulate creativity, to encourage collaboration and to study learners' untutored, emergent ideas" (Powell et al., 2009, p. 167) and to be sequenced so as to influence the co-emergence of learners instrumentation and building of mathematical ideas. To these ends, rooted in our theoretical perspective and sensitive to the infrastructural features of VMTwG, we developed and tested the following seven design principals for digital tasks that are intended to promote productive mathematical discourse by encouraging collaboration in virtual environments around constructing necessary mathematical content (Hewitt, 1999):

1. Provide a pre-constructed figure or instructions for constructing a figure.
2. Invite participants to interact with a figure by looking at and dragging objects (their base points) to notice how the objects behave, relations among objects, and relations among relations.
3. Invite participants to reflect on the mathematical meaning or consequence of what they notice.
4. Invite participants to wonder or raise questions about what they notice or the mathematical meaning or consequence of it.
5. Pose suggestions as hints or new challenges that prompt participants to notice particular objects, attributes, or relationships without explicitly stating what observation they are to make. Each hint has one or more of these three characteristics:

a. Suggest issues to discuss.
b. Suggest objects or behaviors to observe.
c. Suggest GeoGebra tools to use to explore relations, particularly dependencies.

6. Provide formal mathematical language that corresponds to awarenesses that they are likely to have explored and discussed or otherwise realized.
7. Respond with feedback based on participants' work in the spirit of the following:
   a. Pose new situations as challenges that extend what participants have likely noticed, wondered, or constructed or that follow from an earlier task and that involve the same awarenesses or logical extensions of awarenesses they have already acquired.
   b. Invite participants to revisit a challenge or a task on which they already worked to gain awareness of other relationships.
   c. Invite participants to generalize noted relationships and to construct justifications and proofs of conjectures.
   d. Invite participants to consider the attributes of a situation (theorem, figure, actions such as drag) in order to generate a "what if?" question and explore the new question.

The purpose of the hints is to maintain learners' engagement with a task and to encourage them to extend what they know. The hints support participants' discourse by eliciting from them statements that reveal what they observe and what they understand about the mathematical meanings or consequences of their observations. The challenges are available to provide opportunities for participants to further their exploration by investigating new, related situations. Hidden initially, the hints and challenges can be revealed by learners clicking a check box.

These design principals guided how we developed tasks in our research project, a collaboration among investigators at Rutgers University and Drexel University. We employed VMTwG, which contains chat rooms for small teams to collaborate with tools for mathematical explorations, including a multi-user, dynamic version of GeoGebra. Team members construct geometrical objects and can explore them for relationships by dragging base points (see Figure 2). VMTwG records users' chat postings and GeoGebra actions. The project participants are middle and high school teachers in New Jersey who have little to no experience with dynamic geometry environments and no experience collaborating in a virtual environment to discuss and resolve mathematics problems. The teachers took part in a semester-long professional development course. They met for 28 two-hour synchronous sessions in VMTwG and worked collaboratively on 55 tasks, Tasks 1 to 55.

Using our design principles, we developed dynamic-geometry tasks that encourage participants to discuss and collaboratively manipulate and construct dynamic-geometry objects, notice dependencies and other relations among the objects, make conjectures, and build justifications.

TASK EXAMPLE

We present the work of a team of two teachers on a task. The task, Task 10, is one that the research team posed. While the teachers worked on it, they posed a wondering that led us to provide feedback of type 7a, inviting them to explore that wondering. Our analysis reveals how using the epistemic tools the teachers noticed and discussed geometric relations and completed a construction task, wondered about the necessity of a foundational object of the construction, and in the following session resolved their wondering, all through the use of the epistemic tools.

In the fourth week of the professional development course, the team worked on Task 10. Employing procedures of Euclid's second proposition (Euclid, 300 BCE/2002), the task engaged the team in constructing the copy of a line segment, without using the built-in compass tool, only using line segments, rays, and circles. The task also requested that they discuss dependencies and other relations among the objects (see Figure 2).

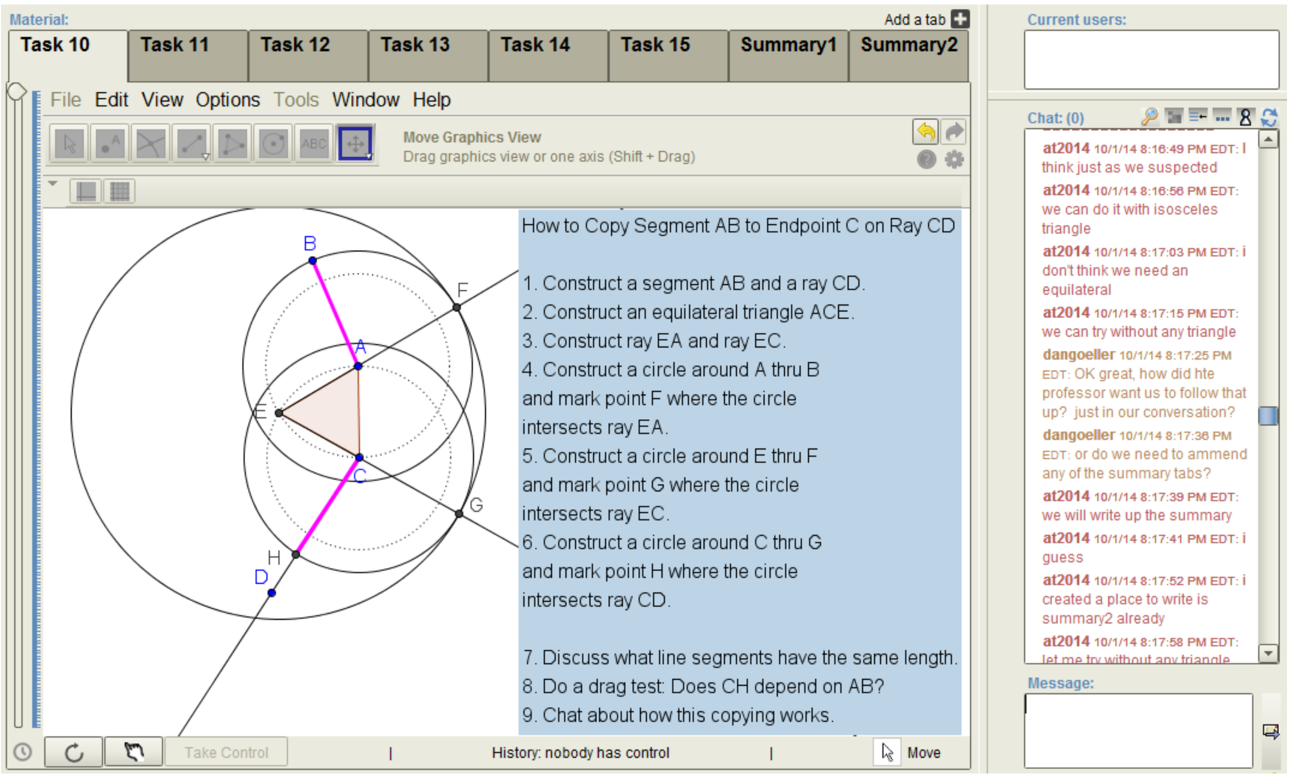

Figure 2: Task 10: Copying A Line Segment.

In the first synchronous session, the teachers successfully followed the construction instructions to copy segment AB onto ray CD. They used the epistemic tools to respond to this task and were attentive to co-active responses of VMTwG to their actions. In their noticings, they chatted about constructed dependencies and other relations among the geometric objects that they constructed. Below, an excerpt of the teachers' discussion illustrates their use of the epistemic tools and how they trigged productive mathematical discourse about a foundational aspect of the construction:

at2014: o what we wonder about
at2014: let's talk about it before we move on
at2014: i am still trying to understand so i am not quite sure whether the equilateral triangle is necessary
at2014: o maybe it does
dangoeller: i agree lets get the others done before sketching this one again
at2014: to get that big circle
at2014: ok
dangoeller: thats a good question
at2014: i am not sure why the equilateral triangle is necessary if it is at all
dangoeller: it appears that it is, but the "why" behind it is unclear to me
at2014: that would be the question for us to put in what we wondered about

In this excerpt, they employed the epistemic tools by wondering about whether an equilateral triangle is necessary in the construction procedure to copy a line segment (see lines 157, 163, and 164). In their session summary, they explicitly stated "We wonder whether the equilateral triangle is necessary or not and if it is necessary, why is it so." In our written feedback, their wondering encouraged us to invite them to explore it in their next synchronous session. In that session, they explored copying a length with an equilateral triangle, an isosceles triangle, and without using any specific type of triangle, which was essentially using a scalene triangle (see Figure 3).

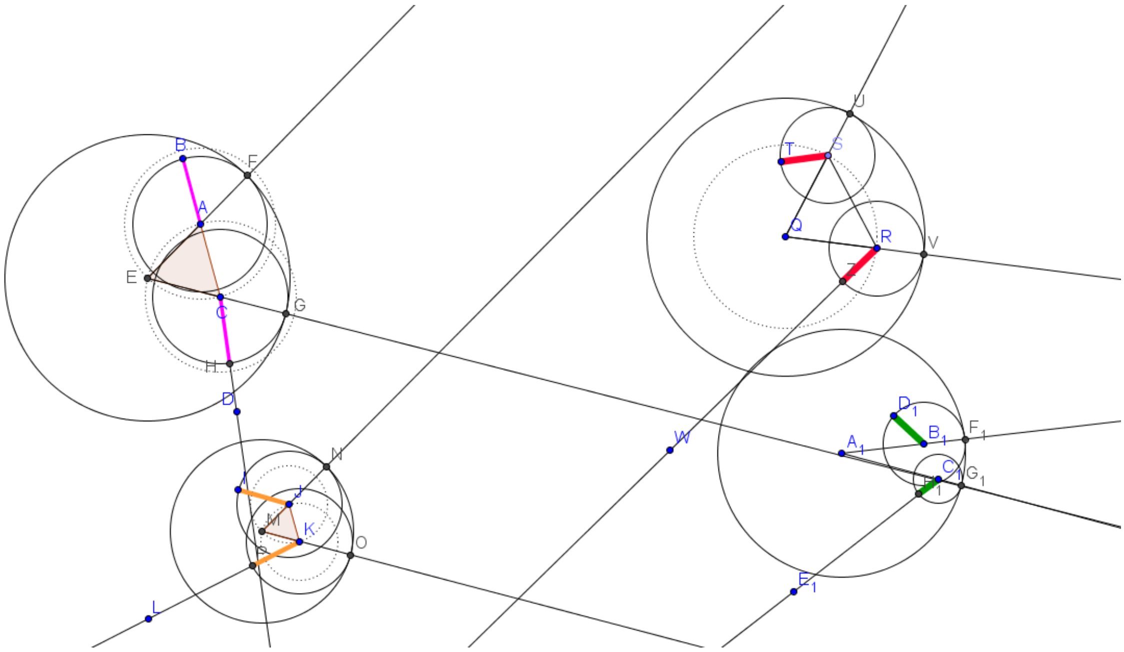

Figure 3: Teachers' Investigation Of Minimal Condition For Copying A Segment Length.

The teachers wrote in their session summary that after conducting drag tests on their constructions, "we found out that if we want the length of one segment to be dependent on another, we need at least the isosceles triangle". Their constructions in Figure 3 include copying a length with an equilateral triangle (lower left corner), using an isosceles triangle (top right corner), and "with no triangle" (lower right corner). They justified their findings by discussing the dependencies each construction has. They make the point that having an equilateral triangle "is only keeping points A and C apart a certain distance, and we can do without it." That is, they demonstrated that to copy the length of the segment AB the distance between A and C is immaterial and that only two congruent sides of a triangle matter.

META-TASK-DESIGN METHODOLOGY

We extend our methodology of task design that promotes productive mathematical discourse to include the design of specialized tasks that encourage reflection on the process and content of mathematical discourse that occurred in prior tasks. We term these specialized tasks, which are reflections on tasks, as meta-tasks. They invite interlocutors of a team to consider and analyze their logged discursive interactions, each time for particular process or content issues such as collaborative norms or mathematical practices. Figure 4 depicts the reflective process in which teams of interlocutors are invited to engage, using the technological structure of VMTwG.

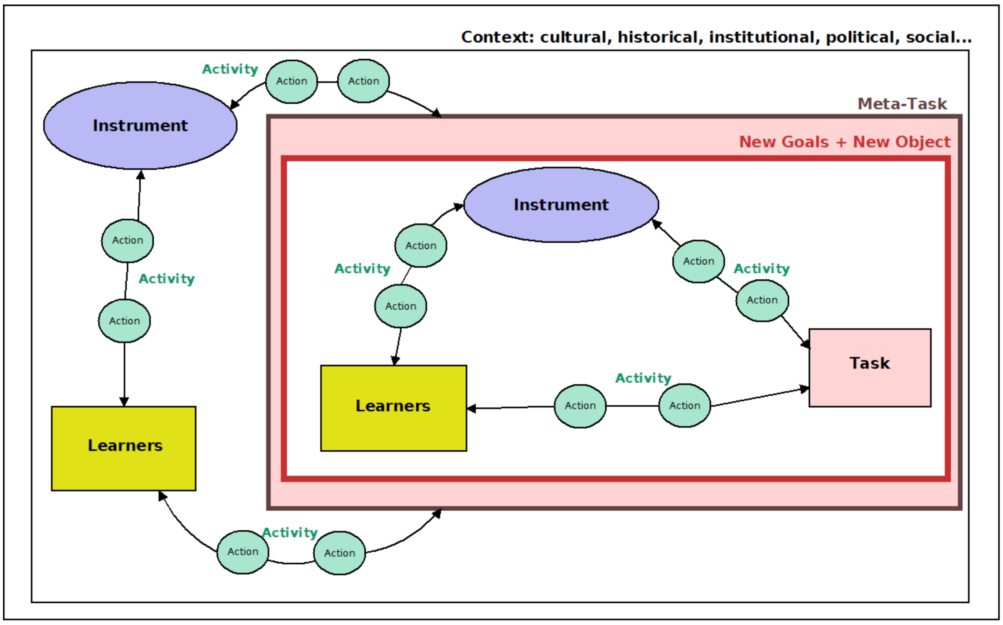

Figure 4. Relational Model Of Learners Engaged In Instrument-Mediated Activity Initiated By A Meta-Task.

In the context of our professional development project, to provide theoretical substance and structure to the meta-tasks, teachers read individually and then in their teams discussed articles about collaboration (Mercer & Sams, 2006; Rowe & Bicknell, 2004), mathematical practices (Common Core State Standards Initiative, 2010), Accountable Talk (Resnick, Michaels, & O'Connor, 2010), technological pedagogical content knowledge or TPACK (Mishra & Koehler, 2006), structures of technology-based mathematics lessons (McGraw & Grant, 2005), and validation of dynamic-geometry constructions (Stylianides & Stylianides, 2005). Teams analyzed previous logs of their VMTwG interactions to examine, reflect, and modify in one meta-task their collaborative norms and in other meta-tasks their mathematical practices and Accountable Talk.

META-TASK EXAMPLE

Part of the goal of professional development project is to promote reflective practices, among teachers' and in turn among their students. During a 14-week semester, given the course readings, the teams of teachers were invited multiple times to reflect on their own VMTwG work. In the second week of the semester, each team was asked to develop their collaborative norms. In the following week, for the first meeting, each team worked for two hours on a mathematical task. In the second meeting that week, each team was asked to select and discuss excerpts from their first meeting that illustrate its collaborative norms. Each team was also invited to modify its norms if team members felt the need to do so. In the fourth week, each team was asked to read about mathematical practices of Common Core State Standards and review its previous work to see where team members employed those practices. Similarly, each team was asked to read about Accountable Talk in the sixth week and review its discourse and discuss whether and how its discursive interactions showed any of the three categories of Accountable Talk.

In this example, we present an excerpt from a team's work and, particularly, their reflection on their collaboration norms. Each team had posted its collaborative norms to an electronic discussion forum in Blackboard, an online course management system that the project uses. Team members then were able to read the norms of other teams. As each team reflected on its discursive interactions, team members became further aware of particular collaborative actions that they felt would be helpful. For instance, after a team discussed and agreed upon their collaborative norms, the team implemented their norms in the following session. For that session, the following is an excerpt of the team members' interactions:

17	sophiak	As we suggested, what should we do first? I am thinking we should make sure we have our norms & then from there we could do this task.
18	sophiak	Just a suggestion & I am open.
19	sabrenam_21	okay great idea i am going to copy and pase the norms suggested that is posted on black board, one second
20	sophiak	Did we post our norms on blackboard? I didn't see them but then again I am not that familar with blackboard & often find myself not sure how to find everything.
21	gouri	I believe the norms were posted by nadine yesterday
22	sophiak	Okay, I didn't see them. I did see her summary of the articles but not the norms. Since we have time Gouri, would you like to take control & comment on what you are doing & understanding about the diagram? This would save a bit of time?
23	sabrenam_21	1. listening while another is talking 2. respect other's opinions 3. respond with respect whether you agree or disagree with group members reasoning 4. Stick to the topic of the talk , some of which are the norms we identified with in team 3.
		.
		.
		.
77	sophiak	Please tell us what you d o Gouri so we can learn from you. This should be a norm too...it is helpful so that we learn from each other. (I think I read this as another groups norm)

The team members started the session by agreeing to state the norms that they established in the previous session (see lines 17-21 and 23). This team's norms include "listening while another is talking", "respect other's opinions", "respond with respect whether you agree or disagree with group members reasoning", and "Stick to the topic of the talk" (line 23). In line 22, sophiak asks Gouri to communicate what she does in the GeoGebra window and what understands about the given figure. Implicit in sophiak's request is the idea that she would find it helpful for Gouri to enact a norm of communicating to other team members what actions (for example, dragging base points) she performs on the figure and what she understands about the figure (such as properties and relations) from the co-active responses of the GeoGebra portion of the VMTwG environment. Later in the session, sophiak requests that the team member in control to of the GeoGebra window communicates to the team the GeoGebra actions she performs (line 77). In the same line, sophiak then suggests that communicating one's GeoGebra actions should be add to the team's norms and gives credit for this norm to another team.

This excerpt illustrates that team members not only reflected on logged interactions but also monitored and negotiated the team's collaboration during the session. It also yields two other results about the meta-task on collaborative norms. First, the course readings and the team's reflection and development of collaborative norms enable team members to make each other accountable to norms of the team. Second, from reading norms of other teams and attending to their work on a new task, a team member suggested new norms that the team considered helpful for the team's geometrical learning. Our meta-task design is aimed to help participants to be more reflective on their own collaboration, mathematical practices, and Accountable Talk. This reflective practice can help individuals be more aware of their own actions and the actions of other interlocutors as well.

DISCUSSION

In this paper, our aim was to describe how we address task-design challenges to promote productive mathematical discourse among interlocutors working in an online synchronous environment. For the purpose of promoting productive mathematical discourse in collaborative digital environments, we detailed our design principles for constructing tasks as well as meta-tasks. In our virtual environment—VMTwG, a classroom teacher or facilitator is present largely as an artifact of the environment's digital tools and most specifically in the structure and content of tasks and meta-tasks. An important feature of our task design is the questions of our epistemic tools since when collaborating interlocutors respond to them they generate propositional statements that can become the focus of their discussions. Their discussions are mathematically productive as their noticings, statements of meaning, and wonderings involve interpretations, procedures, patterns, invariants, conjectures, counterexamples, and justifications about objects, relations among objects, and dynamics linking relations.

Concerning meta-tasks, interlocutors consider and analyze their logged discursive interactions, each time focusing on a particular process or content issue such as collaborative norms or mathematical practices. With tasks, our guiding design principles aim to engage learners in productive mathematical activity through inviting them to explore figures, notice properties, reflect on relations, and wonder about related mathematical ideas. The design provides support through hints and feedback to help learners with certain parts of the tasks. The tasks also include challenges that ask the participants to investigate certain ideas and extend their knowledge. The first example provided above shows that the teachers moved from conjecture to justification through the use of our epistemic tools. They constructed ideas that were new to them. Further investigation is needed to understand how the task-design elements, the affordances of collaborative digital environments, and learners' mathematical discourse interact to shape the development of learners' mathematical activity and understanding.

REFERENCES

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson & M. Otte (Eds.), *Perspectives in mathematics education: Papers submitted by members of the Bacomet group* (pp. 243-307). Dordrecht: Reidel.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics* Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Euclid. (300 BCE/2002). *Euclid's elements* (T. L. Heath, Trans.). Santa Fe, NM: Green Lion.

Gattegno, C. (1987). *The science of education: Part 1: Theoretical considerations*. New York: Educational Solutions.

Grisi-Dicker, L., Powell, A. B., Silverman, J., & Fetter, A. (2012). Addressing transitional challenges to teaching with dynamic geometry in a collaborative online environment. In L. R. Van Zoest, J.-J. Lo & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1024-1027). Kalamazoo, MI: Western Michigan University.

Hegedus, S. J., & Moreno-Armella, L. (2010). Accommodating the instrumental genesis framework within dynamic technological environments. *For the Learning of Mathematics, 30*(1), 26-31.

Hewitt, D. (1999). Arbitrary and necessary: Part 1 A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2-9.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393-425.

Margolinas, C. (2013). *Tasks design in mathematics education. Proceedings of ICMI Study 22*. Oxford, UK.

McGraw, R., & Grant, M. (2005). Investigating mathematics with technology: Lesson structures that encourage a range of methods and solutions. In W. J. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments* (Vol. Sixty-Seventh Yearbook, pp. 303-317). Reston, VA: National Council of Teachers of Mathematics.

Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education, 20*(6), 507-528.

Mishra, P., & Koehler, M. J. (2006). Technological Pedagogical Content Knowledge: A Framework for Teacher Knowledge. *Teachers College Record, 108*(6), 1017-1054.

Pirie, S., & Schwarzenberger, R. (1988). Mathematical discussion and mathematical understanding. *Educational Studies in Mathematics, 19*(4), 459-470.

Powell, A. B., Borge, I. C., Floriti, G. I., Kondratieva, M., Koublanova, E., & Sukthankar, N. (2009). Challenging tasks and mathematics learning. In E. J. Barbeau & P. J. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI study* (pp. 133-170). New York: Springer.

Powell, A. B., Grisi-Dicker, L., & Alqahtani, M. (2013). Letramento matemático: Desenvolvendo as práticas colaborativas, matemáticas, e discursivas com tecnologia [Mathematical literacy: Development of collaborative, mathematical and discusive practices with technology] *XI Encontro Nacional de Educação Matemática, Educação Matemática: Retrospectivas e Perspectivas [XI National Conference of Mathematics Education, Mathematics Education: Retrospectives and Perspectives*. Curitiba, Paraná.

Rabardel, P., & Beguin, P. (2005). Instrument mediated activity: from subject development to anthropocentric design. *Theoretical Issues in Ergonomics Science, 6*(5), 429-461.

Ray, M. (2013). Noticing and wondering *Powerful problem solving: Activities for sense making with the mathematical practices* (pp. 42-55): Heinemann.

Resnick, L. B., Michaels, S., & O'Connor, C. (2010). How (well-structured) talk builds the mind. *Innovations in educational psychology: Perspectives on learning, teaching and human development*, 163-194.

Rowe, K., & Bicknell, B. (2004). *Structured peer interactions to enhance learning in mathematics.* Paper presented at the Proceedings of 27th Annual Conference of the Mathematics Education Research Group of Australasia-Mathematics Education for the Third Millennium, Towards 2010. , Townsville, Australia.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

Stahl, G. (2008). Social practices of group cognition in virtual math teams. In S. Ludvigsen, A. Lund & R. Säljö (Eds.), *Learning in social practices: ICT and new artifacts—transformation of social and cultural practices*: Pergamon.

Stahl, G. (2013). *Translating Euclid: Designing a human-centered mathematics*. San Rafael, CA: Morgan & Claypool.

Stahl, G. (2015). *Constructing dynamic triangles together: The development of mathematical group cognition*. Cambridge, UK: Cambridge.

Stahl, G. (Ed.). (2009). *Studying virtual math teams*. New York: Springer.

Stylianides, G. J., & Stylianides, A. J. (2005). Validation of solutions of construction problems in dynamic geometry environments. *International Journal of Computers for Mathematical Learning, 10*(1), 31–47.