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On new modular sequence space defined over 2-normed spaces

Gülcan ATICİ TURAN*

Department of Mathematics, University of Muş Alparslan, Muş, Turkey

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Abstract

In this paper, a new sequence space $F(\|.,.\|,\mathcal{M},p,u)$ is defined by using a sequence of Orlicz functions in 2-normed spaces. Some various properties and some inclusions are also examined on this space.

Keywords: *Orlicz function, sequence spaces, 2-norm, paranormed spaces.*

2-normlu uzaylarda tanımlı yeni modular dizi uzayı

Öz

Bu çalışmada, 2-normlu uzaylarda Orlicz fonksiyonlarının bir dizisi kullanılarak $F(\|.,.\|,\mathcal{M},p,u)$ yeni dizi uzayı tanımlanmıştır. Ayrıca bu uzayın bazı özellikleri ve bazı kapsama bağıntıları incelenmiştir.

Anahtar Kelimeler: *Orlicz fonksiyon, dizi uzayları, 2-norm, paranormlu uzaylar.*

1. Introduction

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language published in Mathematische Nachrichten, see for example references [1,2]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. of USA in 1969 entitled 2-Banach spaces [3]. In the same year Gähler published another paper on this theme in the same journal [1]. A.H. Siddiqi

* Gülcan ATICİ TURAN, gatici23@hotmail.com, https://orcid.org/0000-0002-1009-6072

delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with Gähler et al. [4] of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [5].

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a function, which is continuous, nondecreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Note that for M is an Orlicz function, we have $M(\lambda x) \leq \lambda M(x)$ where $0 \leq \lambda \leq 1$
ℓ_M sequence space defined as following:

$$\ell_M = \left\{x \in w: \sum_{k=1}^{\infty}\left(M\left(\frac{|x_k|}{\rho}\right)\right) < \infty, for\ some\ \rho > 0\right\} [6].$$

Let X be a real linear space and $\|.,.\|$ is defined a real valued mapping on $X \times X$. For $x, y, z \in X$ and $\lambda \in \mathbb{R}$, the function $\|.,.\|$, which satisfies the following conditions is called 2-norm and the pair $(X, \|.,.\|)$ is called a linear 2-normed space or shortly 2-normed space. $\|.,.\|$ is a non–negative function.

(N_1) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(N_2) $\|x, y\| = \|y, x\|$;
(N_3) $\|\lambda x, y\| = |\lambda|\|x, y\|$, $\lambda \in \mathbb{R}$;
(N_4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$(X, \|.,.\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X [7].

Let X be a linear metric space. A function $g: X \to \mathbb{R}$ is called paranorm, if

(i) $g(x) \geq 0$, for all $x \in X$
(ii) $g(-x) = g(x)$, for all $x \in X$
(iii) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$
(iv) if (μ_n) is a sequence of scalars with $\mu_n \to \mu$ as $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$, then $g(\mu_n x_n - \mu x) \to 0$ as $n \to \infty$ [8].

A scalar valued paranormed sequence space (F, g_F), where g_F is a paranorm on F is called monotone paranormed space if $x = (x_k)$, $y = (y_k) \in F$ and $|x_k| \leq |y_k|$ for all k implies $g_F(x) \leq g_F(y)$ [8].

Definition 1.1. Let X be a sequence space.
(i) If $y = (y_k) \in X$ whenever $|y_i| \leq |x_i|$, $i \geq 1$ for some $x = (x_k) \in X$, then X is called solid (or normal).
(ii) If $(x_k) \in X$ implies $(X_{\pi(k)}) \in X$ such that $\pi(k)$ is a permutation of $\mathbb{N}$, then X is called symmetric [9].

U is showed as the set of all real sequences $u = (u_k)$, where $u_k > 0$ for all $k \in \mathbb{N}$.

Throughout this study the following inequality will be used. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le sup\, p_k = H, D = max(1, 2^{H-1})$, then for all $a_k, b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \tag{1}$$

2. Main results

Let (F, g_F) be a normal paranormed sequence space with paranorm g_F which satisfies the following properties:

(i) g_F is a monotone paranorm;
(ii) coordinatewise convergence implies convergence in paranorm g_F, which implies that for each $(X^n) = (X_k^n) \in F, n, k \in \mathbb{N}, X_k^n \to 0$ as $n \to \infty$ (for each k) $\Rightarrow g_F(X^n) \to 0$ as $n \to \infty$ [10].

Let $(N, \|.,.\|)$ be a 2-normed space and $\mathcal{M} = (\mathrm{M}_k)$ be a sequence of Orlicz functions. Further, let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. We define the set:

$$F(\|.,.\|, \mathcal{M}, p, u) = \left\{X = (X_k) : X_k \in N, \left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right) \epsilon F, for\ some\ \rho > 0\right\}$$

for every $Z \in N$.

For $p_k = 1$ for all $k \in \mathbb{N}$, we write this space as $F(\|.,.\|, \mathcal{M}, u)$.

Theorem 2.1. If $\mathcal{M} = (\mathrm{M}_k)$ is a sequence of Orlicz functions then $F(\|.,.\|, \mathcal{M}, p, u)$ is a linear space.

Proof. Let $X = (X_k), Y = (Y_k) \in F(\|.,.\|, \mathcal{M}, p, u)$ and $a, b \in \mathbb{R}$, thus there are some positive numbers ρ_1 and ρ_2 such that

$$\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho_1}\right)\right]^{p_k}\right) \epsilon F \quad \text{and} \quad \left(u_k\left[\mathrm{M}_k\left(\frac{\|Y_k, Z\|}{\rho_2}\right)\right]^{p_k}\right) \epsilon F$$

for every $Z \in N$. Define $\rho = max\{2|a|\rho_1, 2|b|\rho_2\}$. Because of the definition of the Orlicz function, we can write

$$\begin{aligned} u_k\left[\mathrm{M}_k\left(\frac{\|aX_k + bY_k, Z\|}{\rho}\right)\right]^{p_k} &\le u_k\left[\mathrm{M}_k\left(\frac{\|aX_k, Z\| + \|bY_k, Z\|}{\rho}\right)\right]^{p_k} \\ &< u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho_1}\right) + \mathrm{M}_k\left(\frac{\|Y_k, Z\|}{\rho_2}\right)\right]^{p_k} \\ &\le Du_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho_1}\right)\right]^{p_k} + Du_k\left[\mathrm{M}_k\left(\frac{\|Y_k, Z\|}{\rho_2}\right)\right]^{p_k} \in F, \end{aligned}$$

such that $D = max\{1, 2^{H-1}\}$. Therefore $aX + bY \in F(\|.,.\|, \mathcal{M}, p, u)$. Hence $F(\|.,.\|, \mathcal{M}, p, u)$ is a linear space.

Theorem 2.2. For any sequence $\mathcal{M} = (\mathrm{M}_k)$ of Orlicz function, $F(\|.,.\|, \mathcal{M}, p, u)$ is a paranormed space with

$$g_T(X) = inf\left\{\rho^{\frac{p_k}{T}} > 0 : \left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \le 1, k = 1,2, \ldots\right\} \qquad (2)$$

such that $T = max(1, H)$, $H = sup_k p_k < \infty$ and $inf\ p_k > 0$ and for $Z \in N$.

Proof. It is easy to prove that $g_T(\theta) = 0$ and $g_T(-X) = g_T(X)$. Since g_F is monotone and when $a = b = 1$ is taken in the proof of Theorem 2.1, we write $g_T(X + Y) \le g_T(X) + g_T(Y)$ for $X = (X_k), Y = (Y_k) \in F(\|.,.\|, \mathcal{M}, p, u)$.
Let $\lambda \ne 0$ be any complex number. Because of the continuity of the scalar multiplication, we can write

$$g_T(\lambda X) = inf\left\{\rho^{\frac{p_k}{T}} > 0 : \left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|\lambda X_k, Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \le 1, k = 1,2, \ldots\right\}$$
$$= inf\left\{(|\lambda| r)^{\frac{p_k}{T}} > 0 : \left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{r}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \le 1, k = 1,2, \ldots\right\}$$

where $r = \rho/|\lambda|$.
Since $|\lambda|^{p_k} \le \max(1, |\lambda|^H)$. We have $|\lambda|^{\frac{p_k}{T}} \le \left(max(1, |\lambda|^H)\right)^{\frac{1}{T}}$. Thus $g_T(\lambda X)$ converges to zero when $g_T(X)$ converges to zero in $F(\|.,.\|, \mathcal{M}, p, u)$.
Let $X = (X_k) \in F(\|.,.\|, \mathcal{M}, p, u)$ and assume that $\lambda_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and K be a positive integer. Then we can write

$$g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right) < \left(\frac{\varepsilon}{2}\right)^T$$

every some $\rho > 0$ and for $k > K$ such that $k \in N$,

$$\left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \le \frac{\varepsilon}{2}.$$

Let $0 < |\lambda| < 1$. Because of the definition of the Orlicz function and by the condition (iii) of 2-norm, we have

$$g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|\lambda X_k, Z\|}{\rho}\right)\right]^{p_k}\right) = g_F\left(u_k\left[\mathrm{M}_k\left(|\lambda|\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right)$$
$$< g_F\left(u_k\left[|\lambda|\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k}\right)$$

$$< g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k,Z\|}{\rho}\right)\right]^{p_k}\right)$$
$$< \left(\frac{\varepsilon}{2}\right)^T$$

for $k > K$. Since M is continuous everywhere in $[0,\infty)$ and by the definition of g_F, it follows that for $k \leq K$

$$\varphi(t) = g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|tX_k,Z\|}{\rho}\right)\right]^{p_k}\right)$$

is continuous at 0. Therefore $|\varphi(t)| < \frac{\varepsilon}{2}$ for $0 < t < \delta$ such that $0 < \delta < 1$. Let L be any integer such that $|\lambda_n| < \delta$ for $n > L$, then

$$\left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|\lambda_n X_k,Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} < \frac{\varepsilon}{2}$$

for $n > L$ and $k \leq K$. Therefore

$$\left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|\lambda_n X_k,Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} < \varepsilon$$

for $n > L$ and for all k. So $\lambda_n X \to \theta$ as $n \to \infty$. This completes the prof of the theorem.

Theorem 2.3. Let $(N,\|.,.\|)$ be a 2-Banach space, then the space $F(\|.,.\|,\mathcal{M},p,u)$ is a complete paranormed space with $g_T(X)$, where F is a K-space.

Proof. The proof is routine verification by using standard arguments and therefore omitted.

Theorem 2.4. If F is a K-space, then $F(\|.,.\|,\mathcal{M},p,u)$ is a K-space.

Proof. Let us define a mapping $\tau_n: F(\|.,.\|,\mathcal{M},p,u) \to N$ by $\tau_n(X) = X_n$, $\forall n \in \mathbb{N}$. Our aim is to show τ_n is continuous.

Let (X^m) be a sequence in $F(\|.,.\|,\mathcal{M},p,u)$ such that $X^m \overset{g}{\to} 0$ as $m \to \infty$. Then for some suitable choice of $\rho > 0$,

$$\left[g_F\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k^m,Z\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{T}} \to 0$$

as $m \to \infty$. Since F is a K $-$space, this implies that for each k and as m tending to ∞,

$$u_k\left[\mathrm{M}_k\left(\frac{\|X_k^m,Z\|}{\rho}\right)\right]^{p_k} \to 0$$

for some $\rho > 0$. Since M_k be a sequence of Orlicz functions, it follows that $\|X_k^m, Z\| \to 0$ as $m \to \infty$. Consequently, $X^m \to 0$ in N.

Theorem 2.5. Let $\mathcal{M}$ and $\mathcal{T}$ be two sequence of Orlicz functions. Then

$$F(\|.,.\|, \mathcal{M}, p, u) \cap F(\|.,.\|, \mathcal{T}, p, u) \subseteq F(\|.,.\|, \mathcal{M} + \mathcal{T}, p, u)$$

where F is a normal sequence space.

Proof. Let $X = (X_k) \in F(\|.,.\|, \mathcal{M}, p, u) \cap F(\|.,.\|, \mathcal{T}, p, u)$. Then we can choose $\rho_1, \rho_2 > 0$ such that

$$\left(u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho_1}\right)\right]^{p_k}\right) \epsilon F \text{ and } \left(u_k\left[\mathrm{T}_k\left(\frac{\|X_k, Z\|}{\rho_2}\right)\right]^{p_k}\right) \epsilon F.$$

Define $\rho = max\{\rho_1, \rho_2\}$. We can write

$$u_k\left[(\mathrm{M}_k + \mathrm{T}_k)\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k} \leq u_k D\left\{\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho_1}\right)\right]^{p_k} + \left[\mathrm{T}_k\left(\frac{\|X_k, Z\|}{\rho_2}\right)\right]^{p_k}\right\} \in F,$$

where $D = max\{1, 2^{H-1}\}$. Since F is normal, $X \in F(\|.,.\|, \mathcal{M} + \mathcal{T}, p, u)$.

Theorem 2.6. Let $\mathcal{M} = (\mathrm{M}_k)$ be a sequence of Orlicz functions. Then $c_0(\|.,.\|, \mathcal{M}, p, u) \subset c(\|.,.\|, \mathcal{M}, p, u) \subset \ell_\infty(\|.,.\|, \mathcal{M}, p, u)$.

Proof. It is obvious that $c_0(\|.,.\|, \mathcal{M}, p, u) \subset c(\|.,.\|, \mathcal{M}, p, u)$. The second inclusion follows from the following inequality. Let $X = (X_k) \in c(\|.,.\|, \mathcal{M}, p, u)$ and for some $\rho = 2\mu > 0$, we obtain

$$u_k\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k} \leq u_k D\left[\mathrm{M}_k\left(\frac{\|X_k - L, Z\|}{\mu}\right)\right]^{p_k} + u_k D max\left\{1, \left[\mathrm{M}_k\left(\frac{\|L, Z\|}{\mu}\right)\right]^H\right\}.$$

Thus $X = (X_k) \in \ell_\infty(\|.,.\|, \mathcal{M}, p, u)$.

Theorem 2.7. Let $\mathcal{M} = (\mathrm{M}_k)$ be a sequence of Orlicz functions. Then
(i) If $0 < inf\, p_k \leq p_k \leq 1$, then $c_0(\|.,.\|, \mathcal{M}, u) \subset c_0(\|.,.\|, \mathcal{M}, p, u)$;
(ii) If $1 \leq p_k \leq supp_k < \infty$, then $c_0(\|.,.\|, \mathcal{M}, p, u) \subset c_0(\|.,.\|, \mathcal{M}, u)$.

Proof. (i) Let $X = (X_k) \in c_0(\|.,.\|, \mathcal{M}, u)$. Since $0 < inf\, p_k \leq p_k \leq 1$, then we have

$$\left[\mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right)\right]^{p_k} \leq \mathrm{M}_k\left(\frac{\|X_k, Z\|}{\rho}\right).$$

Therefore $X = (X_k) \in c_0(\|.,.\|, \mathcal{M}, p, u)$.
(ii) Let $1 \leq p_k \leq supp_k < \infty$ and $X = (X_k) \in c_0(\|.,.\|, \mathcal{M}, p, u)$. Then for each $0 < \varepsilon < 1$ there is a positive integer L such that

$$u_k\left[\mathrm{M}_k\left(\frac{\|X_k,Z\|}{\rho}\right)\right]^{p_k} \leq \varepsilon < 1, \qquad \forall k \geq L.$$

Since $1 \leq p_k \leq supp_k < \infty$, then we have

$$u_k\left[\mathrm{M}_k\left(\frac{\|X_k,Z\|}{\rho}\right)\right] \leq u_k\left[\mathrm{M}_k\left(\frac{\|X_k,Z\|}{\rho}\right)\right]^{p_k}.$$

Therefore $X = (X_k) \in c_0(\|.,.\|, \mathcal{M}, u)$. This completes the proof of the theorem.

Theorem 2.8. The space $F(\|.,.\|, \mathcal{M}, p, u)$ is both solid(normal) and symmetric.

Proof. The proof is similar to [10].

References

[1] Gähler, S., Uber 2-Banach räume, **Mathematische Nachrichten**, 42, 335-347, (1969).

[2] Diminnie, C., Gähler, S. and White Jr. A.G., 2-inner product spaces, **Demonstrario Mathematica**, 6, 525-536, (1973).

[3] White Jr. A.G., 2-Banach spaces, **Mathematische Nachrichten**, 42, 43-60, (1969).

[4] Gähler, S., Siddiqi, A.H. and Gupta, S.C., Contributions to non-archimedean functional analysis, **Mathematische Nachrichten**, 69, 162-171, (1975).

[5] Siddiqi, A.H., 2-normed spaces, **The Aligarh Bulletin of Mathematics,** 53-70, (1980).

[6] Lindenstrauss, J., Tzafriri, L. On Orlicz sequence spaces, **Israel Journal of Mathematics,** 101, 379–390, (1971).

[7] Gähler, S., Lineare 2-normierte Räume, **Mathematische Nachrichten,** 28, 1–43, (1965).

[8] Maddox, I. J., **Elements of functional analysis**, Cambridge University Press, Cambridge, (1970).

[9] Kamthan, P.K., Gupta, M., **Sequence spaces and series**, Marcel Dekker, New York, (1981).

[10] Dutta, H., Kılıçman, A., Altun, Ö., Topological properties of some sequences defined over 2-normed spaces, **SpringerPlus**, 5, 2-16, (2016).

[11] Dutta, H., Some statistically convergent difference sequence spaces defined over real 2-normed linear spaces, **Applied Sciences**, 12, 37–47, (2010).

[12] Bektaş, Ç.A., Altin, Y., The sequence space $l_M(p,q,s)$ on seminormed spaces, **Indian Journal of Pure Applied Mathematics**, 34(4), 529–534, (2003).

[13] Rao, M.M., Ren, Z.D., **Theory on Orlicz spaces**, Marcel Dekker, New York, (1991).

[14] Gürdal, M. and Şahiner, A., Ideal Convergence in n-normed spaces and some new sequence spaces via n-norm, **Journal of Fundamental Sciences**, 4(1), 233-244, (2008).

[15] Şahiner, A., Gürdal, M., Saltan, S. and Gunawan, H., Ideal convergence in 2-normed spaces, **Taiwanese Journal of Mathematics**, 11(5), 1477-1484, (2007).

[16] Şahiner, A. and Gürdal, M., New sequence spaces in n-normed spaces with respect to an Orlicz function, **The Aligarh Bulletin of Mathematics**, 27(1), 53-58, (2008).