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On a second-order evolution inclusion

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Abstract

We study a class of second-order evolution inclusions and we obtain a sufficient condition for f -local controllability along a reference trajectory.

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1. Introduction

In this note we are concerned with the following problem

$$x'' \in A(t)x + F(t, x), \quad x(0) \in X_0, \quad x'(0) \in X_1, \quad (1.1)$$

where $F : [0, T] \times X \rightarrow \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $X_0, X_1 \subset X$ and $\{A(t)\}_{t \geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t, s)\}_{t, s \in [0, T]}$.

The general framework of evolution operators $\{A(t)\}_{t \geq 0}$ that define problem (1.1) has been developed by Kozak ([14]) and improved by Henriquez ([12]). In several recent papers ([2-5], [8-11]) existence results and qualitative properties of solutions for problem (1.1) have been obtained by using several techniques.

The aim of the present paper is to obtain a sufficient condition for f -local controllability of inclusion (1.1). We denote by S_F be the set of all mild solutions of (1.1) and by $R_F(T)$ the reachable set of (1.1). If $y(\cdot) \in S_F$ is a mild solution and if $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a locally Lipschitz function then we say that the differential inclusion (1.1) is f -locally controllable around $y(\cdot)$ if $h(y(T)) \in \text{int}(f(R_F(T)))$. In particular, if f is the identity map the above definitions reduces to the usual concept of local controllability of systems around a solution.

The proof of our result is based on an approach of Tuan ([16]). More precisely, we prove that inclusion (1.1) is f -locally controllable around the solution $y(\cdot)$ if a certain variational inclusion is h -locally controllable

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around the null solution for every $h \in \partial f(z(T))$, where $\partial f(\cdot)$ denotes Clarke's generalized Jacobian of the locally Lipschitz function f . The main tools in the proof of our result is a continuous version of Filippov's theorem for mild solutions of problem (1.1) obtained in [8] and a certain generalization of the classical open mapping principle in [17].

We note that similar results for other classes of differential inclusions may be found in our previous papers [6,7].

The paper is organized as follows: in Section 2 we present some preliminary results to be used in the sequel and in Section 3 we present our main results.

2. Preliminaries

Let us denote by I the interval $[0, T]$ and let X be a real separable Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot, \cdot)$. Denote by $\mathcal{L}(I)$ the σ -algebra of all Lebesgue measurable subsets of I , by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X . Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(\cdot) : I \rightarrow X$ endowed with the norm $\|x(\cdot)\|_C = \sup_{t \in I} \|x(t)\|$, by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(\cdot) : I \rightarrow X$ endowed with the norm $\|x(\cdot)\|_1 = \int_I \|x(t)\| dt$ and by $B(X)$ the Banach space of linear bounded operators on X .

In what follows $\{A(t)\}_{t \geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t, s)\}_{t, s \in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t .

Definition 2.1. ([12,14]) A family of bounded linear operators $\mathcal{U}(t, s) : X \rightarrow X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

- i) For any $x \in X$, the map $(t, s) \rightarrow \mathcal{U}(t, s)x$ is continuously differentiable and
 - a) $\mathcal{U}(t, t) = 0$, $t \in I$.
 - b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t} \mathcal{U}(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s} \mathcal{U}(t, s)x|_{t=s} = -x$.
- ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s} \mathcal{U}(t, s)x \in D(A(t))$, the map $(t, s) \rightarrow \mathcal{U}(t, s)x$ is of class C^2 and
 - a) $\frac{\partial^2}{\partial t^2} \mathcal{U}(t, s)x \equiv A(t)\mathcal{U}(t, s)x$.
 - b) $\frac{\partial^2}{\partial s^2} \mathcal{U}(t, s)x \equiv \mathcal{U}(t, s)A(t)x$.
 - c) $\frac{\partial^2}{\partial s \partial t} \mathcal{U}(t, s)x|_{t=s} = 0$.
- iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s} \mathcal{U}(t, s)x$, $\frac{\partial^3}{\partial s^2 \partial t} \mathcal{U}(t, s)x$ and
 - a) $\frac{\partial^3}{\partial t^2 \partial s} \mathcal{U}(t, s)x \equiv A(t)\frac{\partial}{\partial s} \mathcal{U}(t, s)x$ and the map $(t, s) \rightarrow A(t)\frac{\partial}{\partial s} \mathcal{U}(t, s)x$ is continuous.
 - b) $\frac{\partial^3}{\partial s^2 \partial t} \mathcal{U}(t, s)x \equiv \frac{\partial}{\partial t} \mathcal{U}(t, s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [12])

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2}(t, \tau) &= \frac{\partial^2 z}{\partial \tau^2}(t, \tau) + a(t) \frac{\partial z}{\partial t}(t, \tau), \quad t \in [0, T], \tau \in [0, 2\pi], \\ z(t, 0) &= z(t, \pi) = 0, \quad \frac{\partial z}{\partial \tau}(t, 0) = \frac{\partial z}{\partial \tau}(t, 2\pi), \quad t \in [0, T], \end{aligned}$$

where $a(\cdot) : I \rightarrow \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π -periodic 2-integrable functions from \mathbf{R} to \mathbf{C} , $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R}, \mathbf{C})$ the Sobolev space of 2π -periodic functions whose derivatives belong to $L^2(\mathbf{R}, \mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X . Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}$, $n \in \mathbf{N}$. The set z_n , $n \in \mathbf{N}$ is an orthonormal basis of X . In particular, $A_1 z = \sum_{n \in \mathbf{Z}} -n^2 \langle z, z_n \rangle z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n \in \mathbf{Z}} \cos(nt) \langle z, z_n \rangle z_n$ with the associated sine function $S(t)z = t \langle z, z_0 \rangle z_0 + \sum_{n \in \mathbf{Z}^*} \frac{\sin(nt)}{n} \langle z, z_n \rangle z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t) \frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R}, \mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [12] that this family generates an evolution operator as in Definition 1.

Definition 2.2. A continuous mapping $x(\cdot) \in C(I, X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t)) \quad \text{a.e. } (I), \quad (2.2)$$

$$x(t) = -\frac{\partial}{\partial s} \mathcal{U}(t, 0)x_0 + \mathcal{U}(t, 0)y_0 + \int_0^t \mathcal{U}(t, s)f(s)ds, \quad t \in I. \quad (2.3)$$

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (1.1) if $f(\cdot)$ verifies (2.2) and $x(\cdot)$ is defined by (2.3).

Hypothesis H1. i) $F(\cdot, \cdot) : I \times X \rightarrow \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $l(\cdot) \in L^1(I, \mathbf{R}_+)$ such that, for any $t \in I$, $F(t, \cdot)$ is $l(t)$ -Lipschitz in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \leq l(t)|x_1 - x_2| \quad \forall x_1, x_2 \in X.$$

Hypothesis H2. Let S be a separable metric space, $X_0, X_1 \subset X$ are closed sets, $a_0(\cdot) : S \rightarrow X_0$, $a_1(\cdot) : S \rightarrow X_1$ and $c(\cdot) : S \rightarrow (0, \infty)$ are given continuous mappings.

The continuous mappings $g(\cdot) : S \rightarrow L^1(I, X)$, $y(\cdot) : S \rightarrow C(I, X)$ are given such that

$$(y(s))''(t) = A(t)y(s)(t) + g(s)(t), \quad y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1.$$

and there exists a continuous function $q(\cdot) : S \rightarrow L^1(I, \mathbf{R}_+)$ such that

$$d(g(s)(t), F(t, y(s)(t))) \leq q(s)(t) \quad \text{a.e. } (I), \quad \forall s \in S. \quad (2.4)$$

Theorem 2.3. ([10]) Assume that Hypotheses H1 and H2 are satisfied.

Then there exist $M > 0$ and the continuous functions $x(\cdot) : S \rightarrow L^1(I, X)$, $h(\cdot) : S \rightarrow C(I, X)$ such that for any $s \in S$ $(x(s)(\cdot), h(s)(\cdot))$ is a trajectory-selection of (1.1) satisfying for any $(t, s) \in I \times S$

$$x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s),$$

$$|x(s)(t) - y(s)(t)| \leq M[c(s) + |a_0(s) - y(s)(0)| + |a_1(s) - (y(s))'(0)| + \int_0^t q(s)(u)du]. \quad (2.5)$$

In what follows we assume that $X = \mathbf{R}^n$.

A closed convex cone $C \subset \mathbf{R}^n$ is said to be *regular tangent cone* to the set X at $x \in X$ ([16]) if there exists continuous mappings $q_\lambda : C \cap B \rightarrow \mathbf{R}^n$, $\forall \lambda > 0$ satisfying

$$\lim_{\lambda \rightarrow 0+} \max_{v \in C \cap B} \frac{|q_\lambda(v)|}{\lambda} = 0,$$

$$x + \lambda v + q_\lambda(v) \in X \quad \forall \lambda > 0, v \in C \cap B,$$

where B is the closed unit ball in \mathbf{R}^n .

We recall, also, some well known intrinsic tangent cones in the literature (e.g. [1]); namely, the *contingent*, the *quasitangent* and *Clarke's tangent cones*, defines, respectively, by

$$\begin{aligned} K_x X &= \{v \in \mathbf{R}^n; \exists s_m \rightarrow 0+, x_m \in X : \frac{x_m - x}{s_m} \rightarrow v\} \\ Q_x X &= \{v \in \mathbf{R}^n; \forall s_m \rightarrow 0+, \exists x_m \in X : \frac{x_m - x}{s_m} \rightarrow v\} \\ C_x X &= \{v \in \mathbf{R}^n; \forall (x_m, s_m) \rightarrow (x, 0+), x_m \in X, \exists y_m \in X : \frac{y_m - x_m}{s_m} \rightarrow v\}. \end{aligned}$$

It is known that, unlike $K_x X$, $Q_x X$, the cone $C_x X$ is convex and one has $C_x X \subset Q_x X \subset K_x X$.

The results in the next section will be expressed, in the case when the mapping $f(\cdot) : X \subset \mathbf{R}^n \rightarrow \mathbf{R}^m$ is locally Lipschitz at x , in terms of the Clarke generalized Jacobian, defined by ([11])

$$\partial f(x) = \text{co}\{\lim_{i \rightarrow \infty} f'(x_i); \quad x_i \rightarrow x, \quad x_i \in X \setminus J_f\},$$

where J_f is the set of points at which f is not differentiable.

Corresponding to each type of tangent cone, say $\tau_x X$ one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset \mathbf{R}^n \rightarrow \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \text{graph}(G)$ as follows

$$\tau_y G(x; v) = \{w \in \mathbf{R}^n; (v, w) \in \tau_{(x,y)} \text{graph}(G)\}, \quad v \in \tau_x X.$$

We recall that a set-valued map, $A(\cdot) : \mathbf{R}^n \rightarrow \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, *closed convex*) *process* if $\text{graph}(A(\cdot)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

Hypothesis H3. i) *Hypothesis H1 is satisfied and $X_0, X_1 \subset \mathbf{R}^n$ are closed sets.*

ii) *$(y(\cdot), g(\cdot)) \in C(I, \mathbf{R}^n) \times L^1(I, \mathbf{R}^n)$ is a trajectory-selection pair of (1.1) and a family $L(t, \cdot) : \mathbf{R}^n \rightarrow \mathcal{P}(\mathbf{R}^n)$, $t \in I$ of convex processes satisfying the condition*

$$L(t, u) \subset Q_{g(t)} F(t, \cdot)(y(t); u) \quad \forall u \in \text{dom}(P(t, \cdot)), \text{ a.e. } t \in I \quad (2.6)$$

is assumed to be given.

The family of convex processes in Hypothesis H3 defines the variational inclusion

$$v'' \in A(t)v + L(t, v). \quad (2.7)$$

Remark 2.4. We point out that Hypothesis H3 is not restrictive, since for any set-valued map $F(\cdot, \cdot)$, one may find an infinite number of families of convex processes $L(t, \cdot)$, $t \in I$, satisfying condition (2.6). Any family of closed convex subcones of the quasitangent cones, $\bar{L}(t) \subset Q_{(y(t), g(t))} \text{graph}(F(t, \cdot))$, defines the family of closed convex processes

$$L(t, u) = \{v \in \mathbf{R}^n; (u, v) \in \bar{L}(t)\}, \quad u, v \in \mathbf{R}^n, \quad t \in I$$

that satisfy condition (2.6). For example one may take an "intrinsic" family of such closed convex process given by Clarke's convex-valued directional derivatives $C_{g(t)} F(t, \cdot)(y(t); \cdot)$.

Since $F(t, \cdot)$ is assumed to be Lipschitz a.e. on I , the quasitangent directional derivative is given by ([1])

$$Q_{g(t)} F(t, \cdot)((y(t); u)) = \{w \in \mathbf{R}^n; \lim_{\theta \rightarrow 0+} \frac{1}{\theta} d(g(t) + \theta w, F(t, y(t) + \theta u)) = 0\}. \quad (2.8)$$

In what follows $B_{\mathbf{R}^n}$ denotes the closed unit ball in \mathbf{R}^n and 0_n denotes the null element in \mathbf{R}^n . Consider $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ an arbitrary given function.

Definition 2.5. Differential inclusion (1.1) is said to be *f-locally controllable* around $y(\cdot)$ if $f(y(T)) \in \text{int}(f(R_F(T)))$.

In particular, differential inclusion (1.1) is said to be *locally controllable* around the solution $y(\cdot)$ if $y(T) \in \text{int}(R_F(T))$.

Finally a key tool in the proof of our results is the following generalization of the classical open mapping principle due to Warga ([17]).

For $k \in \mathbf{N}$ we define

$$\Sigma_k := \{\beta = (\beta_1, \dots, \beta_k); \sum_{i=1}^k \beta_i \leq 1, \quad \beta_i \geq 0, \quad i = 1, 2, \dots, k\}.$$

Lemma 2.6. ([17]) Let $\delta \leq 1$, let $g(\cdot) : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a mapping that is C^1 in a neighborhood of 0_n containing $\delta B_{\mathbf{R}^n}$. Assume that there exists $\beta > 0$ such that for every $\theta \in \delta \Sigma_n$, $\beta B_{\mathbf{R}^m} \subset g'(\theta) \Sigma_n$. Then, for any continuous mapping $\varphi : \delta \Sigma_n \rightarrow \mathbf{R}^m$ that satisfies $\sup_{\theta \in \delta \Sigma_n} |g(\theta) - \varphi(\theta)| \leq \frac{\delta \beta}{32}$ we have $\varphi(0_n) + \frac{\delta \beta}{16} B_{\mathbf{R}^m} \subset \varphi(\delta \Sigma_n)$.

3. The main result

In order to prove our result we assume that Hypothesis H3 is satisfied, C_0 is a regular tangent cone to X_0 at $y(0)$ and C_1 is a regular tangent cone to X_1 at $y'(0)$. We denote by S_L the set of all solutions of the differential inclusion

$$w'' \in A(t)w + L(t, w), \quad w(0) \in C_0, \quad w'(0) \in C_1$$

and by $R_L(T) = \{x(T); x(\cdot) \in S_L\}$ its reachable set at time T .

Theorem 3.1. Assume that Hypothesis H3 is satisfied and let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a Lipschitz function with m its Lipschitz constant.

Then, differential inclusion (1.1) is f -locally controllable around the solution $y(\cdot)$ if

$$0_m \in \text{int}(hR_L(T)) \quad \forall h \in \partial f(y(T)). \quad (3.1)$$

Proof. $hR_L(T)$ is a convex cone, thus, by (3.1), it follows that $hR_L(T) = \mathbf{R}^m \quad \forall h \in \partial f(y(T))$. Taking into account that the set $\partial f(y(T))$ is compact (e.g., [11]), we have that for every $\gamma > 0$ there exist $k \in \mathbf{N}$ and $w_j \in R_L(T)$ $j = 1, 2, \dots, k$ such that

$$\gamma B_{\mathbf{R}^m} \subset h(w(\Sigma_k)) \quad \forall h \in \partial f(y(T)), \quad (3.2)$$

with

$$w(\Sigma_k) = \{w(\beta) := \sum_{j=1}^k \beta_j w_j, \quad \beta = (\beta_1, \dots, \beta_k) \in \Sigma_k\}.$$

Using an usual separation theorem we deduce the existence of $\gamma_1, r_1 > 0$ such that for all $h \in L(\mathbf{R}^n, \mathbf{R}^m)$ with $d(h, \partial f(y(T))) \leq r_1$ we have

$$\gamma_1 B_{\mathbf{R}^m} \subset h(w(\Sigma_k)). \quad (3.3)$$

Since $w_j \in R_L(T)$, $j = 1, \dots, k$, there exist $(w_j(\cdot), q_j(\cdot))$, $j = 1, \dots, k$ trajectory-selection pairs of (2.7) such that $w_j = w_j(T)$, $j = 1, \dots, k$. We note that $\gamma > 0$ can be taken small enough such that $|w_j(0)| \leq 1$, $j = 1, \dots, k$.

Define

$$w(t, s) = \sum_{j=1}^k s_j w_j(t), \quad \bar{q}(t, s) = \sum_{j=1}^k s_j q_j(t), \quad \forall s = (s_1, \dots, s_k) \in \mathbf{R}^k.$$

Obviously, $w(\cdot, s) \in S_L$, $\forall s \in \Sigma_k$.

From the definition of C_0 and C_1 we find that for every $\varepsilon > 0$ there exists a continuous mapping $o_\varepsilon : \Sigma_k \rightarrow \mathbf{R}^n$ such that

$$y(0) + \varepsilon w(0, s) + o_\varepsilon(s) \in X_0, \quad y'(0) + \varepsilon \frac{\partial w}{\partial t}(0, s) + o_\varepsilon(s) \in X_1 \quad (3.4)$$

$$\lim_{\varepsilon \rightarrow 0+} \max_{s \in \Sigma_k} \frac{|o_\varepsilon(s)|}{\varepsilon} = 0. \quad (3.5)$$

Define

$$\begin{aligned} \rho_\varepsilon(s)(t) &:= \frac{1}{\varepsilon} d(\bar{q}(t, s), F(t, y(t) + \varepsilon w(t, s)) - g(t)), \\ d(t) &:= \sum_{j=1}^k [||q_j(t)|| + l(t)||w_j(t)||], \quad t \in I. \end{aligned}$$

Then, for every $s \in \Sigma_k$ one has

$$\begin{aligned} \rho_\varepsilon(s)(t) &\leq |\bar{q}(t, s)| + \frac{1}{\varepsilon} d_H(0_n, F(t, y(t) + \varepsilon w(t, s)) - g(t)) \leq |\bar{q}(t, s)| + \\ &\frac{1}{\varepsilon} d_H(F(t, y(t)), F(t, y(t) + \varepsilon w(t, s))) \leq |\bar{q}(t, s)| + l(t)||w(t, s)|| \leq d(t). \end{aligned} \quad (3.6)$$

Next, if $s_1, s_2 \in \Sigma_k$ one has

$$\begin{aligned} |\rho_\varepsilon(s_1)(t) - \rho_\varepsilon(s_2)(t)| &\leq |\bar{q}(t, s_1) - \bar{q}(t, s_2)| + \frac{1}{\varepsilon} d_H(F(t, y(t) + \varepsilon w(t, s_1)), \\ &F(t, y(t) + \varepsilon w(t, s_2))) \leq |s_1 - s_2| \cdot \max_{j=1, \dots, k} [||q_j(t)|| + l(t)||w_j(t)||], \end{aligned}$$

thus $\rho_\varepsilon(\cdot)(t)$ is Lipschitz with a Lipschitz constant not depending on ε .

At the same time, from (2.8) it follows that

$$\lim_{\varepsilon \rightarrow 0} \rho_\varepsilon(s)(t) = 0 \quad a.e. (I), \quad \forall s \in \Sigma_k$$

and hence

$$\lim_{\varepsilon \rightarrow 0+} \max_{s \in \Sigma_k} \rho_\varepsilon(s)(t) = 0 \quad a.e. (I). \quad (3.7)$$

Lebesgue's dominated convergence theorem, (3.6) and (3.7) imply that

$$\lim_{\varepsilon \rightarrow 0+} \int_0^T \max_{s \in \Sigma_k} \rho_\varepsilon(s)(t) dt = 0. \quad (3.8)$$

From (3.4), (3.5), (3.8) and the upper semicontinuity of the Clarke generalized Jacobian we can find $\varepsilon_0, e_0 > 0$ such that

$$\max_{s \in \Sigma_k} \frac{||o_{\varepsilon_0}(s)||}{\varepsilon_0} + \int_0^T \max_{s \in \Sigma_k} \rho_{\varepsilon_0}(s)(t) dt \leq \frac{\gamma_1}{2^8 m^2}, \quad (3.9)$$

$$\varepsilon_0 w(T, s) \leq \frac{e_0}{2} \quad \forall s \in \Sigma_k. \quad (3.10)$$

We define

$$y(s)(t) := y(t) + \varepsilon_0 w(t, s), \quad g(s)(t) := g(t) + \varepsilon_0 \bar{q}(t, s) \quad s \in \mathbf{R}^k,$$

$$a_0(s) := y(0) + \varepsilon_0 w(0, s) + o_{\varepsilon_0}(s), \quad a_1(s) := y'(0) + \varepsilon_0 \frac{\partial w}{\partial t}(0, s) + o_{\varepsilon_0}(s), \quad s \in \mathbf{R}^k,$$

and we apply Theorem 2.3 in order to obtain that there exists a continuous function $x(\cdot) : \Sigma_k \rightarrow C(I, \mathbf{R}^n)$ such that for any $s \in \Sigma_k$ the function $x(s)(\cdot)$ is a mild solution of the differential inclusion $x'' \in A(t)x + F(t, x)$, $x(s)(0) = a_0(s)$, $(x(s))'(0) = a_1(s)$ $\forall s \in \Sigma_k$ and one has

$$||x(s)(T) - y(s)(T)|| \leq \frac{\varepsilon_0 \gamma_1}{2^6 m} \quad \forall s \in \Sigma_k. \quad (3.11)$$

We define

$$\begin{aligned} f_0(x) &:= \int_{\mathbf{R}^n} f(x - ay) \chi(y) dy, \quad x \in \mathbf{R}^n, \\ \psi(s) &:= f_0(y(T) + \varepsilon_0 w(T, s)), \end{aligned}$$

where $\chi(\cdot) : \mathbf{R}^n \rightarrow [0, 1]$ is a C^∞ function with the support contained in $B_{\mathbf{R}^n}$ that satisfies $\int_{\mathbf{R}^n} \chi(y) dy = 1$ and $a = \min\{\frac{\varepsilon_0}{2}, \frac{\varepsilon_0 \gamma_1}{2^6 m}\}$.

Hence $f_0(\cdot)$ is of class C^∞ and verifies

$$\|f(x) - f_0(x)\| \leq m \cdot a, \quad (3.12)$$

$$f'_0(x) = \int_{\mathbf{R}^n} f'(x - ay) \chi(y) dy. \quad (3.13)$$

In particular,

$$\begin{aligned} f'_0(x) &\in \overline{\text{co}}\{f'(u); \quad \|u - x\| \leq a, \quad f'(u) \text{ exists}\}, \\ \psi'(s)\mu &= f'_0(y(T) + \varepsilon_0 w(T, s)) \varepsilon_0 w(T, \mu) \quad \forall \mu \in \Sigma_k. \end{aligned}$$

If we denote $h(s) := f'_0(y(T) + \varepsilon_0 w(T, s))$, then $\psi'(s)\mu = h(s) \varepsilon_0 w(T, \mu) \quad \forall \mu \in \Sigma_k$.

Taking into account, again, the upper semicontinuity of the Clarke generalized Jacobian we obtain

$$\begin{aligned} d(h(s), \partial f(z(T))) &= d(f'_0(y(T) + \varepsilon_0 w(T, s)), \partial f(y(T))) \leq \sup\{d(f'_0(u), \partial f(y(T)))\}; \\ \|u - y(T)\| &\leq \|u - (y(T) + \varepsilon_0 w(T, s))\| + \|\varepsilon_0 w(t, s)\| \leq e_0, \quad f'(u) \text{ exists} \} < r_1. \end{aligned}$$

The last inequality together with (3.3) gives

$$\gamma_1 B_{\mathbf{R}^m} \subset h(s) w(\Sigma_k).$$

and therefore

$$\varepsilon_0 \gamma_1 B_{\mathbf{R}^m} \subset h(s) \varepsilon_0 w(\Sigma_k) = h(s) \varepsilon_0 w(T, \mu) = \psi'(s)\mu, \quad \forall \mu \in \Sigma_k,$$

i.e.,

$$\varepsilon_0 \gamma_1 B_{\mathbf{R}^m} \subset \psi'(s) \Sigma_k.$$

Finally, for $s \in \Sigma_k$, we put $\varphi(s) = f(x(s)(T))$.

Obviously, $\varphi(\cdot)$ is continuous and from (3.11), (3.12), (3.13) one may write

$$\begin{aligned} |\varphi(s) - \psi(s)| &= |f(x(s)(T)) - f_0(y(s)(T))| \leq |f(x(s)(T)) - f(y(s)(T))| + \\ &|f(y(s)(T)) - f_0(y(s)(T))| \leq m|x(s)(T) - y(s)(T)| + m \cdot a \leq \frac{\varepsilon_0 \gamma_1}{64} + \frac{\varepsilon_0 \gamma_1}{64} = \frac{\varepsilon_0 \gamma_1}{32}. \end{aligned}$$

It remains to apply Lemma 2.6 and to find that

$$f(x(0_k)(T)) + \frac{\varepsilon_0 \gamma_1}{16} B_{\mathbf{R}^m} \subset \varphi(\Sigma_k) \subset f(R_F(T)).$$

Finally, $|f(y(T)) - f(x(0_k)(T))| \leq \frac{\varepsilon_0 \gamma_1}{64}$, so we have $f(z(T)) \in \text{int}(f(R_F(T)))$, which completes the proof.

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