Results in Nonlinear Analysis **5** (2022) No. 1, 1–11.
https://doi.org/10.53006/rna.1055249
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Remarks on generalized weak KKM multimaps

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Abstract

The concept of generalized KKM maps was initiated by Kassay-Kolumbán in 1990 and Chang-Zhang in 1991. Recently, Balaj and his colleagues extended generalized KKM maps w.r.t. a multimap to weak KKM maps and generalized weak KKM maps w.r.t. a multimap, and applied them to various problems in the KKM theory. However, their results are mainly concerned within the realm of topological vector spaces. Our aim in this article is to extend some of them to abstract convex spaces. Some related facts are also discussed.

Keywords: 47H04, 47H10, 49J27, 49J35, 54C60, 54H25, 91B50.
2010 MSC: Abstract convex space; KKM maps; generalized KKM map; weak KKM map; generalized weak KKM map.

1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained an intersection theorem which is known to be equivalent to the Brouwer fixed point theorem in 1912, the weak Sperner combinatorial lemma in 1928, and many important theorems. The KKM theory is first named by ourselves in 1992 as the study of applications of extensions or equivalents of the KKM theorem. Nowadays the theory is mainly concerned with abstract convex spaces and (partial) KKM spaces due to ourselves, and became a large scale logical system called the Grand KKM Theory; see [26] in 2021.

One of the topics in the KKM theory is related to generalized KKM maps initiated by Kassay-Kolumbán in 1990 [16] and Chang-Zhang in 1991 [11]. Since then many authors studied generalized KKM maps on various types of spaces and applied them to extend or refine well-known previous results. In fact, it has been followed by Chang-Ma in 1993, Yuan in 1995, Cheng in 1997, Tan in 1997, Lin-Chang in 1998, Lee-Cho-Yuan in 1999, Kirk-Sims-Yuan in 2000 for various classes of abstract convex spaces; see [24]. All of those authors

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Received November 30, 2021; Accepted: December 21, 2021; Online: January 10, 2022

applied their results on KKM type theorems and others to extend or refine well-known previous results in the KKM theory; for example, variational or quasi-variational inequalities, fixed point theorems, the Ky Fan type minimax inequalities, the von Neumann type minimax or saddle point theorems, Nash equilibrium problems, and others.

In our previous review [24], we gave a unified account for generalized KKM maps on abstract convex spaces in the previous works of Kim-Park [17], Lee [18], and Park-Lee [27]. We were mainly concerned with results closely related to the KKM type theorems and characterizations of generalized KKM maps on various types of abstract convex spaces. In short, we showed that generalized KKM maps can be reduced to the usual KKM maps in our abstract convex spaces. Some related topics were also added in [24].

Recently, Balaj and his colleagues extended generalized KKM maps w.r.t. a multimap to weak KKM maps and generalized weak KKM maps w.r.t. a multimap, and applied them to various problems in the KKM theory; see [1, 4, 5, 7, 8]. However, their works are mainly concerned within the realm of topological vector spaces.

Recall that we have recently established the Grand KKM Theory mainly on abstract convex spaces; see [26]. Since the recent results of Balaj and his colleagues in [1, 4, 5, 7, 8] are on topological vector spaces or G-convex spaces, it is better to extend them to abstract convex spaces. Our aim in this article is simply to try this task.

This article is organized as follows: Section 2 devotes to preliminary for concepts on abstract convex spaces, partial KKM spaces and their subclasses in our previous works. In Section 3, we extend the concepts of generalized KKM maps, weak KKM maps, and generalized weak KKM maps to the realm of abstract convex spaces. Sections 4 is to improve certain facts on weak KKM maps mainly in Balaj [5] in 2004.

Section 5 deals with generalized equi-KKM families of Balaj [6] in 2010. In Section 6, we deal with variational problems of Agarwal-Balaj-O'Regan [1] in 2016. Section 7 concerns with the works of Agarwal-Balaj-O'Regan [2, 3, 4] in 2017-19 on weak KKM maps.

In Section 8, we extend some key results on generalized weak KKM maps due to Balaj [7] in 2021 to the corresponding ones in abstract convex spaces. Finally in Section 9, we recall some basic known results and indicate the difficulty of extending Balaj's results to abstract convex spaces.

Note that terminology in the original sources different from our current usage is preserved in the present article. This may not give any confusion to the readers.

2. Preliminaries on Abstract Convex Spaces

For the concepts on abstract convex spaces, partial KKM spaces and their subclasses, we follow [22, 23, 25, 26] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *(partial) KKM space* if it satisfies the (partial) KKM principle, resp.

The (partial) KKM principle has a large number of equivalent formulations and, for example, the following (0) and (1) are equivalent:

(0) The KKM principle: *For any closed-valued [resp. open-valued] KKM map $G : D \multimap E$, the family $\{G(z) : z \in D\}$ has the finite intersection property.*

(I) The Fan matching property: *Let $S : D \multimap E$ be a map satisfying*
(i) $S(z)$ is open [resp. closed] for each $z \in D$; and
(ii) $E = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$.
Then there exists an $N \in \langle M\rangle$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

There are plenty of examples of KKM spaces; see [25] and the references therein.

Now we have the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}$$
$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space}$$
$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

Definition 2.4. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map with respect to F*. A *KKM map $G : D \multimap E$* is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp. a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly, $\mathfrak{KO}(E, Z)$ is defined.

3. Various types of generalized weak KKM maps

In the KKM theory, many authors adopted the concept of generalized KKM maps and applied it to extend or refine well-known previous results. In this section, we give a unified account for such maps in abstract convex spaces. Our results include the KKM type theorems and characterizations of generalized KKM maps.

From now on all numbers attached to Definitions, Theorems and other statements are the same one in the original sources, and the one attached * marks are our corresponding generalizations.

The following definition is given in Balaj [7]:

Definition 3.1. ([7]) Let X be a convex set in a vector space, let Z be a nonempty set and let $S, T : X \multimap Z$ be two set-valued mappings. We say that:

(i) S is a generalized KKM mapping w.r.t. T if for each nonempty finite subset $\{x_1, \ldots, x_n\}$ of X,

$$T(\mathrm{co}\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} S(x_i).$$

(ii) S is a weak KKM map w.r.t. T (see [5]) if for each nonempty finite subset $\{x_1, \ldots, x_n\}$ of X, and any $x \in \mathrm{co}\{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}\}$,

$$T(x) \cap (\bigcup_{i=1}^{n} S(x_i)) \neq \emptyset.$$

We can extend these definitions as follows:

Definition 3.1.* Let $(E, D; \Gamma)$ be an abstract convex space and Z a set [or a topological space]. For two multimaps $F : E \multimap Z$ and $G : D \multimap Z$ with nonempty values,

(i) G is a KKM map w.r.t. F if for each nonempty finite subset $\{x_1, \ldots, x_n\}$ of D,

$$F(\Gamma\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} G(x_i).$$

(ii) If a multimap $G : D \multimap Z$ satisfies

$$F(x) \cap G(A) \neq \emptyset \qquad \text{for all } A \in \langle D \rangle \text{ and } x \in \Gamma_A,$$

then G is called a *weak KKM map w.r.t. F*. A *KKM map* $G : D \multimap E$ is a weak KKM map w.r.t. the identity map 1_E.

Balaj [7] introduced a new concept, more general than those mentioned above, by the following definition:

Definition 3.2. ([7]) Let X and Z be two nonempty sets, let Y be a convex set in a vector space and let $S : X \multimap Z$, $T : Y \multimap Z$ be two set-valued mappings. We say that S is a generalized weak KKM mapping w.r.t. T if for each nonempty finite subset $\{x_1, \ldots, x_n\}$ of X there exists a subset $\{y_1, \ldots, y_n\}$ of Y such that for each nonempty index set $I \subset \{1, \ldots, n\}$ and any $y \in \mathrm{co}\{y_i : i \in I\}$,

$$T(y) \cap (\bigcup_{i\in I} S(x_i)) \neq \emptyset.$$

We can extend this definition as follows:

Definition 3.2.* Let X and Z be two nonempty sets, and $(E, D; \Gamma)$ be an abstract convex space. For a multimap $F : E \multimap Z$ with nonempty values and a multimap $G : D \multimap Z$, we say that F is a *generalized weak KKM map* w.r.t. G if for each nonempty finite subset $\{x_1, \ldots, x_n\}$ of X there exists a subset $\{y_1, \ldots, y_n\}$ of D such that for each nonempty index set $I \subset \{1, \ldots, n\}$ and any $y \in \Gamma\{y_i : i \in I\}$,

$$F(y) \cap (\bigcup_{i\in I} G(x_i)) \neq \emptyset.$$

The following characterization of generalized KKM maps given in [24] extends many other previously given versions by other authors:

Theorem 3.3. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set, and $T : Y \multimap X$ a map with closed [resp. open] values.*

(i) If T is a generalized KKM map, then the family of its values has the finite intersection property.

(ii) The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.

4. Weak KKM maps [5]

In 1998, Lin, Ko and Park [19] introduced the concepts of generalized G-KKM mapping (w.r.t. T) and weakly G-KKM mapping (w.r.t. T). Relating to this, they obtained some intersection results and minimax inequalities of Ky Fan type. They also gave a new class of mappings with G-KKM property and a new Sion type minimax inequality.

Later in 2008, Park [21] obtained variants of the KKM principle for KKM spaces related to weak KKM maps and indicated some applications of them. These results properly generalize the corresponding ones in G-convex spaces and ϕ_A-spaces. Consequently, results by Balaj in 2004, Liu in 1991, and Tang et al. in 2007 can be properly generalized and unified.

All results in Park [21] are stated for KKM spaces and they can be extended to partial KKM spaces.

We recall some earlier works of Balaj [5].

The following extension to G-convex spaces of Fan's matching theorem is well known. For instance, it is equivalent to the assertion (i) of Theorem 1 in [21].

Lemma 1. ([5]) *Let $(X, D; \Gamma)$ be a G-convex space, $A \in \langle D\rangle$ and $\{M_z : z \in A\}$ an open or closed cover of X. Then there exists a nonempty subset B of A such that $\Gamma(B) \cap \bigcap\{M_z : z \in B\} \neq \emptyset$.*

Lemma 1.* *Lemma 1 holds for a partial KKM space $(X, D; \Gamma)$ instead of a G-convex space.*

Note that this is simply the Fan matching property.

Theorem 2. ([5]) *Let $(X, D; \Gamma)$ be a compact G-convex space, Y a nonempty set and $T : X \multimap Y$, $S : D \multimap Y$ two maps satisfying the following conditions:*

(i) S is weakly G-KKM map w.r.t. T;

(ii) for each $z \in D$ the set $\{x \in X : T(x) \cap S(z) \neq \emptyset\}$ is closed.

Then there exists an $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in D$.

This is extended to KKM spaces in Park [21, Theorem 4.3]. But we have more general one as follows:

Theorem 2.* *Theorem 2 holds for a partial KKM space $(X, D; \Gamma)$ instead of a G-convex space.*

Proof. Suppose the conclusion does not hold and for every $z \in D$ put

$$M_z = \{x \in X : T(x) \cap S(z) = \emptyset\}.$$

Then the family $\{M_z : z \in D\}$ is an open cover of X and since X is compact there is a set $A \in \langle D\rangle$ such that $\bigcup\{M_z : z \in A\} = X$. By Lemma 1* there exists a nonempty subset B of A and a point

$$x_0 \in \Gamma(B) \cap \bigcap\{M_z : z \in B\}.$$

Since S is weak KKM map w.r.t. T , by $x_0 \in \Gamma(B)$ we get $T(x_0) \cap S(B) \neq \emptyset$. On the other hand, since $x_0 \in \bigcap\{M_z : z \in B\}$, we have $T(x_0) \cap S(z) = \emptyset$ for each $z \in B$, hence $T(x_0) \cap S(B) = \emptyset$. The obtained contradiction completes the proof. □

Remark 1. Condition (ii) in Theorem 2* is fulfilled if Y is a topological space, T is upper semicontinuous and S has closed values.

Recall that [21, Theorem 4.3] is the case for KKM spaces of Theorem 2*. Note that many results of [5] on G-convex spaces can be extended to partial KKM spaces as above.

5. Generalized equi-KKM family [6]

Using the Brouwer fixed point theorem, Balaj [6] in 2010 established a common fixed point theorem for a family of multimaps. As applications of this result he obtained existence theorems for the solutions of two types of vector equilibrium problems, a Ky Fan type minimax inequality and a generalization of a known result due to Iohvidov.

Inspired by generalized KKM map, Balaj [6] introduced the following:

Definition 1. ([6]) Let X be a nonempty set, Z be a convex subset of a vector space and $\mathcal{T}$ be a family of set-valued mappings with nonempty values from X into Z. We say that $\mathcal{T}$ is *generalized equi-KKM* if for any nonempty finite subset $\{x_1, \dots, x_n\}$ of X there is $\{z_1, \dots, z_n\} \subset Z$ such that $\mathrm{co}\{z_i : i \in I\} \subset \bigcup_{i \in I} T(x_i)$, for each nonempty subset I of $\{1, \dots, n\}$ and for all $T \in \mathcal{T}$.

Definition 1.* Let X be a nonempty set, (E, D, Γ) be a abstract convex space and $\mathcal{T}$ be a family of multimaps with nonempty values from X into E. We say that $\mathcal{T}$ is *generalized equi-KKM* if for any nonempty finite subset $\{x_1, \dots, x_n\}$ of X there is $\{z_1, \dots, z_n\} \subset D$ such that $\Gamma\{z_i : i \in I\} \subset \bigcup_{i \in I} T(x_i)$, for each nonempty subset I of $\{1, \dots, n\}$ and for all $T \in \mathcal{T}$.

Remark 1. ([6]) If Z is a convex subset of a topological vector space and $\mathcal{T}$ is generalized equi-KKM then, according to Lemma 3.3 in [15], for each $T \in \mathcal{T}$, $\{\overline{T(x)} : x \in X\}$ has the finite intersection property.

Remark 1.* Suppose that $(E, D; \Gamma)$ is a partial KKM space. Every element $T \in \mathcal{T}$ is a generalized KKM map and hence $\{\overline{T(x)} : x \in X\}$ has the finite intersection property by Theorem 3.3.

6. Variational problems [1]

In 2008, Luc [20] proposed a general model for a large class of problems in optimization and nonlinear analysis and called his model a variational relation problem. A variational relation R is represented as a subset of a product space $X \times Y \times Z$, so that $R(x, y, z)$ holds if and only if the point (x, y, z) belongs to that set.

In 2016, Agarwal-Balaj-O'Reagan [1] considered variational relation problems involving a binary relation. The framework presented is more general than that in Luc[20] and in other recent papers which deal with this subject.

The next lemma is a particular case of Ky Fan's 1981 KKM theorem:

Lemma 4.1. ([1]) *Let X be a nonempty convex subset of a topological vector space and let $T : X \multimap X$ be a KKM mapping such that for each $x \in X$, $T(x)$ is a relatively closed subset of X. If there exist a compact convex subset C of X and a compact subset K of X such that $\bigcap_{x \in C} T(x) \subset K$, then $\bigcap_{x \in X} T(x) \neq \emptyset$.*

Fan's 1981 KKM theorem was generalized in the later work of Park [23], where we can find several generalizations of Lemma 4.1.

Definition 4.2. ([1]) Let X be a convex set in a vector space, D a nonempty subset of X and $\rho(x, y)$ a relation linking elements $x, y \in X$. We say that the relation ρ is KKM w.r.t. D if, for every finite subset $\{y_1, y_2, \dots, y_n\}$ of X and for every $x \in \mathrm{co}\{y_1, y_2, \dots, y_n\} \cap D$, one can find some index i such that $\rho(x, y_i)$ holds.

Definition 4.2.* Let $(X \supset D; \Gamma)$ be an abstract convex space. Definition 4.2 can be extended to this space.

In the case of a ternary relation, the previous definition induces two new concepts of KKM relation relative to a set, as follows.

Definition 4.3. ([1]) Let X be a convex set in a vector space, D a nonempty subset of X, Z a nonempty set and $P : X \times X \multimap Z$. A relation $R \subset X \times X \times Z$ is said to be

(i) s-P-KKM with respect to D if, for every finite subset $\{y_1, y_2, \dots, y_n\}$ of X and for every $x \in \mathrm{co}\{y_1, y_2, \dots, y_n\} \cap D$, one can find some index i such that $R(x, y_i, z)$ holds for all $z \in P(x, y_i)$;

(ii) w-P-KKM with respect to D if, for every finite subset $\{y_1, y_2, \dots, y_n\}$ of X and for every $x \in \mathrm{co}\{y_1, y_2, \dots, y_n\} \cap D$, one can find some index i such that $R(x, y_i, z)$ holds for all $z \in P(x, y_i)$.

Remark 4.4. ([1]) (i) When $D = X$, the concept of s-P-KKM relation reduces to the concept of P-KKM relation introduced in [20, Definition 3.2].

(ii) Clearly a relation $R \subset X \times X \times Z$ is s-P-KKM (resp. w-P-KKM) if and only if the relation ρ defined by $\rho(x, y)$ holds iff $R(x, y, z)$ holds for all (resp. for some) $z \in P(x, y)$ is KKM with respect to D.

Regarding the KKM concepts introduced above, the abstract form of Proposition 4.5 in [1] can be easily established and useful in concrete problems. See also Theorem 4.6, Corollaries 4.7, 4.9, Theorems 4.10, 4.11 in [1].

7. Weak KKM maps by Agarwal-Balaj-O'Regan [2-4]

In each year in 2017-19, Agarwal, Balaj, and O'Regan published papers on weak KKM maps.

In 2017 [2], they present two methods for obtaining common fixed point theorems in topological vector spaces. Both methods combine an intersection theorem and a fixed point theorem, but the order in which they are applied differs.

In 2018 [3], they established two intersection theorems which are useful in considering some optimization problems (complementarity problems, variational inequalities, minimax inequalities, saddle point problems).

Finally, in 2919 [4], they obtained open versions of the above mentioned intersection theorems related to a compact convex set in $\mathbf{R^n}$. As applications, they established several minimax inequalities and existence criteria for the solutions of three types of set-valued equilibrium problems.

These three papers based on the following lemma (Lemma 3.1 [2] and Lemma 3.2 [3]):

Lemma 3.1. ([2, 3]) *Let X be a nonempty and convex set and Y be a nonempty, compact and convex set, each in a topological vector space. If $P : X \multimap Y$ is a closed mapping with nonempty convex values and convex cofibers, then $\bigcap_{u\in X} P(u) \neq \emptyset$.*

This is hard to extend to abstract convex spaces.

8. Generalized weak KKM maps [7]

Abstract of [7]: In this paper, we introduce the concept of generalized weak KKM mapping that is more general than many others encountered in the KKM theory. Then, two previous intersection theorems of the author are extended from weak KKM mappings to generalized weak KKM mappings. Applications of these results to set-valued equilibrium problems and minimax inequalities are given in the last two sections.

The lemma below is a particular case of Theorem 2 of Park-Lee [27]:

Lemma 3.1. ([7]) *Let X be a nonempty set and let Y be a nonempty compact convex subset of a topological vector space. If $G : X \multimap Y$ is a generalized KKM set-valued mapping with nonempty closed values, then $\bigcap_{x\in X} G(x) \neq \emptyset$.*

Lemma 3.1.* *Let X be a nonempty set and let $(E, D; \Gamma)$ be a abstract convex space. If $G : X \multimap E$ is a generalized KKM map with nonempty closed values, then $\bigcap_{x\in X} G(x) \neq \emptyset$.*

In [7], using Lemma 3.1, Theorem 1.4 can be generalized as follows:

Theorem 3.2. ([7]) *Let Y be a compact convex subset of a topological vector space, let X and Z be nonempty sets and let $S : X \multimap Z$, $T : Y \multimap Z$ be two nonempty-valued set-valued mappings satisfying the following conditions:*

(i) for each $x \in X$, the set $\{y \in Y : T(y) \cap S(x) \neq \emptyset\}$ is closed;

(ii) S is a generalized weak KKM mapping w.r.t. T.

Then, there exists $y_0 \in Y$ such that $T(y_0) \cap S(x) \neq \emptyset$ for all $x \in X$.

This can be generalized as follows:

Theorem 3.2.* *Let $(E, D; \Gamma)$ be a compact abstract convex space, let Z be a nonempty set and let $F : E \multimap Z$, $G : D \multimap Z$ be two nonempty-valued multimap satisfying the following conditions:*

(i) for each $x \in D$, the set $\{y \in E : F(y) \cap G(x) \neq \emptyset\}$ is closed;

(ii) G is a generalized weak KKM map w.r.t. F.

Then, there exists $y_0 \in E$ such that $F(y_0) \cap G(x) \neq \emptyset$ for all $x \in D$.

Proof. For each $x \in D$, set

$$H(X) := \{y \in E : F(y) \cap G(x) \neq \emptyset\}.$$

Since G is generalized weak KKM map w.r.t. F, for each nonempty finite subset $\{x_1, \dots, x_n\}$ of E, there exists $\{y_1, \dots, y_n\} \subset D$ such that for each nonempty index set $I \subset \{1, \dots, n\}$ and any $y \in \Gamma\{y_i : i \in I\}$, $\bigcup_{i=1}^n (F(y) \cap G(x_i)) \neq \emptyset$, hence $y \in \bigcup_{i=1}^n H(x_i)$. This proves that H is a generalized KKM map. Moreover, by (i), H has closed values. By Lemma 3.1*, there exists a point $y_0 \in \bigcap_{x \in X} H(x)$. Clearly, this means that $F(y_0) \cap G(x) \neq \emptyset$ for all $x \in D$. □

Remark 3.3. ([7]) It is obvious that condition (i) in Theorem 3.2 holds whenever Z is a topological space, F is upper semicontinuous and G is closed-valued. This fact will be used in last two sections.

In the sequel we need the following lemma:

Lemma 3.4. (see [2, Lemma 3.1]) *Let X be a nonempty and convex set and let Y be a nonempty, compact and convex set, each in a topological vector space. If $H : X \multimap Y$ is a closed mapping with nonempty convex values and convex cofibers, then $\bigcap_{x \in X} H(x) \neq \emptyset$.*

At present, we are unable to have the abstract convex space version of this Lemma. Consequently, we can not generalize Theorems 3.5, 4.2, and others in [7].

In the sequel, we tried to obtain possible abstract convex space versions of results of Balaj [8].

Given three sets X, Y and Z, a relation R between their elements is represented as a nonempty subset of the product set $X \times Y \times Z$. Adopting Luc's terminology, we say that $R(x, y, z)$ holds, if $(x, y, z) \in R$.

Let X, Y and Z be three nonempty sets, let $P : Y \multimap Z$ be a set-valued mapping with nonempty values and let $R(x, y, z)$ be a relation linking elements $x \in X$, $y \in Y$ and $z \in Z$. The variational relation problems considered in [8] are the following:

(VRP) Find $y_0 \in Y$, such that for each $x \in X$, there exists $z \in P(y_0)$ for which $R(x, y_0, z)$ holds.
(SVRP) Find $y_0 \in Y$ and $z_0 \in P(y_0)$, such that $R(x, y_0, z_0)$ holds for all $x \in X$.

Theorem 4.1. ([8]) *Assume that Y is a compact convex set in a topological vector space and that X and Z are topological spaces. Problem (VRP) has at least a solution if the set-valued mapping P is upper semicontinuous and the relation R satisfies the following conditions:*

(i) for each $x \in X$, the set $\{(y, z) \in Y \times Z : R(x, y, z) \text{ holds}\}$ is closed in $Y \times Z$;

(ii) for each nonempty finite set $\{x_1, \dots, x_n\} \subset X$, there exists $\{y_1, \dots, y_n\} \subset Y$ such that for each nonempty index set $I \subset \{1, \dots, n\}$ and any $y \in conv\{y_i : i \in I\}$, there exist $i \in I$ and $z \in P(y)$ for which $R(x_i, y, z)$ holds.

Theorem 4.1.* *Assume that Y is a compact abstract convex space and that X and Z are topological spaces. Then Theorem 4.1 still holds.*

Proof. Let the multimaps $T : Y \multimap Y \times Z, S : X \multimap Y \times Z$ be defined by

$$T(y) = \{y\} \times P(y), \;\; S(x) = \{(y, z) \in Y \times Z : R(x, y, z) \text{ holds}\}.$$

Clearly, T is upper semicontinuous, and from (i), S is closed-valued. Taking into account Remark 3.3*, condition (i) in Theorem 3.2* is satisfied. Note that condition (ii) is nothing else than condition similarly noted in Theorem 3.2.* By Theorem 3.2*, there exists $y_0 \in Y$ such that $T(y_0) \cap S(x) \neq \emptyset$. for all $x \in X$. This means that for each $x \in X$ there exists $z \in P(y_0)$ such that $R(x, y_0, z)$ holds. □

Theorem 5.1. ([8]) *Let X, Y and Z be nonempty convex subsets of three topological vector spaces such that Y and Z are compact. Let $S : X \multimap Z$ and $P : Y \multimap Z$ be two set-valued mappings with nonempty values and f and g be two real functions defined on $Y \times Z$. Assume that:*

(i) g is upper semicontinuous on $Y \times Z$;

(ii) for every $x \in X$ there exists $y \in Y$ such that $\sup_{z \in P(y)} f(y', z) \leq \max_{z \in S(x)} g(y', z)$ for all $y' \in Y$;

(iii) for each $y' \in Y$, the function $y \longmapsto \sup_{z \in P(y)} f(y', z)$ is quasiconvex on Y;

(iv) S is closed-valued.
Then, there exists $y_0 \in Y$ such that

$$\inf_{y\in Y} \sup_{z\in P(y)} f(y,z) \le \inf_{x\in X} \max_{z\in S(x)} g(y_0,z).$$

Theorem 5.1.* *Let X, Y and Z be nonempty abstract convex spaces such that Y and Z are compact. Then Theorem 5.1 holds.*

We have the proof by simply modifying the one of Theorem 5.1 as follows:

Proof. First, let us observe that if g is upper semicontinuous on $Y \times Z$, then for each $y \in Y$, $g(y,\cdot)$ is also an upper semicontinuous function of z on Z and therefore its maximum $\max_{z\in S(x)} g(y,z)$ on the compact set $S(x)$ exists. Assuming that $m := \inf_{y\in Y} \sup_{z\in P(y)} f(y,z) > -\infty$, we define the multimap $T : Y \multimap Z$ by

$$T(y) = \{z \in Z : g(y,z) \le m\}.$$

From (i), the graph of T is closed, and since Z is compact, T is upper semicontinuous. In view of Remark 3.3*, condition (i) of Theorem 3.2* is satisfied.

Let $\{x_1,\dots,x_n\}$ be a finite subset of X. By (ii), for each x_i there exists $y_i \in Y$ such that $\sup_{z\in P(y_i)} f(y',z) \le \max_{z\in S(x_i)} g(y',z)$ for all $y' \in Y$.

We claim that for every nonempty set $I \subset \{1,\dots,n\}$ and any $y \in \Gamma\{y_i : i \in I\}$, $T(y) \cap (\bigcup_{i\in I} S(x_i)) \ne \emptyset$. Assume by way of contradiction that for some index set I and $\bar{y} \in \Gamma\{y_i : i \in I\}$ we have $T(\bar{y}) \cap S(x_i) = \emptyset$, for each $i \in I$. This means that for each $i \in I$ and $z \in S(x_i)$, $g(\bar{y},z) < m$, whence

$$\sup_{z\in P(y_i)} f(\bar{y},z) \le \max_{z\in S(x_i)} g(\bar{y},z) < m.$$

Since the function $y \mapsto \sup_{z\in P(y)} f(\bar{y},z)$ is quasiconvex, we infer that

$$\sup_{z\in P(\bar{y})} f(\bar{y},z) < m; \text{ a contradiction.}$$

We have thus proved that S is a generalized weak KKM map w.r.t. T. By Theorem 3.2*, there exists $y_0 \in Y$ such that $T(y_0) \cap S(x) \ne \emptyset$, for all $x \in X$. Then, for every $x \in X$, there is $z_x \in S(x)$ such that $g(y_0,z_x) \ge m$, hence $\max_{z\in S(x)} g(y_0,z) \ge m$. Thus, $\inf_{x\in X} \max_{z\in S(x)} g(y_0,z) \ge m$. □

9. Remarks on related works of Balaj et al.

In 1986 Granas and Liu [12] stated in Section 4 as follows: We are now able to formulate and to prove our most general coincidence result.

Theorem 4.1. ([12]) *Let X be a convex subset of a vector space with finite topology and Y be a topological space. Let $G, S : X \multimap Y$ be two set-valued maps such that $G \in \Phi(X,Y)$ and $S \in V_w^*(X,Y)$. If either,* (i) *Y is compact or* (ii) *the map S is compact, then the maps G and S has a coincidence.*

In 2003 C.W. Ha [14] derived the following intersection theorem on which the main result of his paper based:

Theorem 2.1. ([14]) *Let X, Y be nonempty convex sets, each in a Hausdorff topological vector space, X compact and let $F \subset G \subset H \subset X \times Y$ such that*
(a) $F(x)$, $H(x)$ are convex for each $x \in X$, and $F^{-1}(y)$, $H^{-1}(y)$ are open in X for each $y \in Y$;
(b) $G(x)$ is open in Y for each $x \in X$, and $X \setminus G^{-1}(y)$ is convex for each $y \in Y$.
Then either there exists $x_0 \in X$ such that $F(x_0) = \emptyset$ or $\bigcap_{x\in X} H(x) \ne \emptyset$.

Ha applied this theorem to a three function minimax theorem.

From now on, we briefly introduce the works of Balaj et al. in four papers [1, 2, 7, 8] on applications of the above two theorems.

In [1], the authors consider variational relation problems involving a binary relation. The framework presented is more general than that in [J. Optim. Theory Appl. 138 (2008) 65–76] and in other recent papers which deal with this subject.

In [1], the following lemma was stated as a particular case of the main result in Granas and Liu [12] (see also [9, 10]).

Lemma 3.1. ([12]) *Let X and Y be convex subsets of two topological vector spaces, and let $F, G : Y \multimap X$ be set-valued mappings satisfying*

(i) F is upper semicontinuous and has nonempty closed and convex values;

(ii) G has open values and nonempty convex fibres.

If F is compact, then F and G have a coincidence, that is, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in F(\bar{y}) \cap G(\bar{y})$.

In [2], it is stated that Lemma 3.1 is a reformulation of Theorem 3.1 in [1] (this seems to be incorrect) with a different proof. Lemma 3.1 implies Theorem 3.3, a new common fixed point theorem, with applications to Ky Fan's best approximation theorem, the Stampacchia variational inequality, and existence of better forms of common fixed point theorems. Lemma 3.2 is a dual of Lemma 3.1 and is a particular case of Theorem 2.1 of Ha [14].

In Balaj [7] in 2021, Lemma 3.4 is Ha's one. It implies Theorem 3.5, which in turn implies Theorem 4.2 for variational relation problem (SVRP). Theorem 4.2 is applied to Theorems 4.11 and 4.12 for existence criteria of the strong solutions for two set-valued equilibrium problems. Moreover, Theorem 3.5 implies the minimax inequality in Theorem 5.3.

Finally, Balaj [8] in 2021 applied Lemma 2 [2, Lemma 3.1] to Theorem 1, a quasi-intersection theorem.

Consequently, Balaj showed that the theorem of Ha has numerous applications.

The late Professors A. Granas and C. W. Ha were old friends of the author.

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