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Existence of Solutions for Nonlinear Fractional Order Differential Equations with Quadratic Perturbations

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Abstract

In this work, we prove the existence of a solution for the initial value problem of nonlinear fractional differential equation with quadratic perturbations involving the Caputo fractional derivative

$\left({}^{c}D^{\alpha}_{0^+} - \rho t\, {}^{c}D^{\beta}_{0^+} \right)\left(\frac{x(t)}{f(t,x(t))} \right) = g(t, x(t)), \quad t \in J = [0, 1],$

$1 < \alpha < 2, 0 < \beta < \alpha$ with conditions $x_0 = \dfrac{x(0)}{f(0, x(0))}$ and

$x_1 = \dfrac{x(1)}{f(1, x(1))}$. Dhage's fixed-point the theorem was used to establish this existence. As an application, we have given example to demonstrate the effectiveness of our main result.

Keywords: Fractional differential equation Quadratic perturbations Dhage fixed point.
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1. Introduction

The fractional derivation theory has recently attracted the attention of several famous authors as is well known this theory comes to generalize the theory of integer order derivation to non-integer orders and the differential equations involving these fractional derivatives are more realistic and accurately describe several phenomena in nature (modeling process that keeps memory). In physics, problems that can be solved exactly often correspond to strongly idealized phenomena [1, 7, 9, 12, 16, 18, 23]. When we look for a more realistic description, we usually end up with a set of

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analytically insoluble equations. From a mathematical point of view, the formalization of such a problem then results in a differential equation with one or more disturbing terms. In this work, we are interested in the so-called quadratic perturbations, which recently attracted of certain researchers' interest [11, 13, 14, 15, 17, 24]. We call them fractional hybrid differential equations.

In [11] Dhage and Lakshmikantham discussed the following first-order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)) \quad a.e. \quad t \in [0,T] \\ x(t_0) = x_0 \end{cases}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0,T] \times \mathbb{R}, \mathbb{R})$. They established the existence, and uniqueness results under mixed Lipschitz and Carathéodory conditions. They also gave some fundamental differential inequalities for hybrid differential equations initiating the study of the theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions, and comparison results.

Zhao, Sun, Han and al in [24] introduced a hybrid disturbance about this same problem which is the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$\begin{cases} D^{\alpha}_{R}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)) \quad a.e. \quad t \in [0,T] = J \\ x(0) = 0 \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$ ($C(J \times \mathbb{R}, \mathbb{R})$ is called the Caratheodory class of functions). The authors of in [24] established the existence theorem for fractional hybrid differential equations and some fundamental differential inequalities, they also established the existence of extremal solutions for this fractional problem.

Hilal and Kajouni in [17] have studied the Boundary value problems for hybrid differential equations with fractional order

$$\begin{cases} D^{\alpha}_{c}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)) \quad a.e. \quad t \in [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c. \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in Car(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.
An existence theorem for this equation is proved under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities which are utilized to prove the existence of extremal solutions are also established. Necessary tools are considered and the comparison principle is proved, which will be useful for further study of the qualitative behavior of solutions.

In [5] Babakhani and Baleanu discussed the existence and uniqueness of solution for nonlinear fractional order differential equations

$$\begin{gathered} (D^{\alpha} - \rho t D^{\beta})x(t) = f(t, x(t), D^{\gamma}x(t)), \quad t \in (0,1), \\ x(0) = x_0, \qquad x(1) = x_1 \qquad or \qquad \{x(0) = x_0, \qquad x'(0) = 1\}. \end{gathered} \tag{1}$$

In this parer we introduce a quadratic perturbation to problem (1), this perturbation sometimes she is called hybrid perturbation he attracted the attention of several famous authors, this attention is caused by the intensive development of the theory of fractional calculus itself and by the recent applications of hybrid fractional differential equations. To be more precise, we develop the theory of hybrid fractional boundary differential equations involving the Caputo differential operator of order $\alpha \in (1,2)$ and $\beta \in (0,\alpha)$, consider the following problem

$$\begin{cases} \Big({}^{c}D^{\alpha}_{0^+} - \rho t\, {}^{c}D^{\beta}_{0^+}\Big)\Bigg(\frac{x(t)}{f(t,x(t))}\Bigg) = g(t,x(t)) \quad a.e. \quad t \in J = [t_{ini}, t_{fin}] = [0,1], \\ x_0 = \frac{x(0)}{f(0,x(0))}, \quad x_1 = \frac{x(1)}{f(1,x(1))}. \end{cases} \tag{2}$$

Where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in Car(J \times \mathbb{R}, \mathbb{R})$, ${}^{c}D^{\alpha}_{0^+}$ denotes the Caputo fractional derivative, ρ is constant, $1 < \alpha \leq 2$ and $0 < \beta \leq \alpha - 1$.

Our manuscript is organized as follows: In Section 2, we give some notations, definitions and preliminary facts that will be used further in this work used in the rest of our paper. In Section 3, we establish the existence of solutions of the Caputo fractional hybrid differential equation 2 by using some Lipschitz and Caratheodory conditions. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we present some notations, definitions and preliminary facts that will be used further in this work.

Definition 2.1. *[18] The fractional integral of the function $h \in L^1([a,b],\mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds$$

where Γ is the gamma function.

Definition 2.2. *[18] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h : (0,\infty) \to \mathbb{R}$ is given by*

$$\begin{aligned} ({}^R D_{a^+}^{\alpha} h)(t) &= \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds \\ &= \frac{d^n}{dt^n} I_a^{n-\alpha} h(t) \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Theorem 2.3. *[23]*
The Riemann-Liouville fractional derivative of power function satisfies

$${}^R D^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, \quad n-1 < \alpha < n, \quad p > n-1, \quad p \in \mathbb{R}.$$

Definition 2.4. *[18]*
For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h is defined by

$$\begin{aligned} ({}^c D_{a^+}^{\alpha} h)(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds \\ &= I_a^{n-\alpha} \frac{d^n}{dt^n} h(t) \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations. Therefore, in this work we will use the Caputo fractional derivative proposed by Caputo in his work on the theory of viscoelasticity [8]. From the above definition one can get :

Theorem 2.5. *[23] The Caputo fractional derivative of power function satisfies*

$$\begin{cases} {}^c D^{\alpha} x^p = \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, \quad n-1 < \alpha < n, \quad p > n-1, \quad p \in \mathbb{R} \\ {}^c D^{\alpha} x^p = 0, \quad n-1 < \alpha < n, \quad p \leq n-1, \quad p \in \mathbb{N}. \end{cases}$$

For example we have

Example 2.6.

$$For \quad \alpha = \frac{1}{2}, \qquad {}^cD^{\frac{1}{2}}x^2 = \frac{2}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}$$

$$^cD^\alpha Cste = 0, \qquad {}^RD^\alpha Cste \neq 0.$$

Furthermore we will use the following important results [18].
The fractional integral of $x(t) = (t-a)^\beta$, $a \geq 0, \beta \geq -1$ is given as

$$I_a^\alpha x(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \alpha,\beta \geq 0.$$

The fractional order integral satisfies the semigroup property

$$I^\alpha(I^\beta x(t)) = I^\beta(I^\alpha x(t)). \tag{3}$$

The integer order derivative operator D^m commutes with fractional order D^α, that is

$$D^m(D^\alpha x(t)) = D^{m+\alpha}x(t) = D^\alpha(D^m x(t)). \tag{4}$$

The fractional operator and fractional derivative operator do not commute in general.

Lemma 2.7. *[18, 6]*
For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha x(t) = 0$ is given by

$$x(t) = \sum_{i=0}^{r-1} c_i t^i \; ; \; c_i \in \mathbb{R} \;, \; i = 0,1,2,3,...,r-1 \;\; and \;\; r = [\alpha]+1. \tag{5}$$

where $[\alpha]$ denotes the integer part of the real number α.

In view of Lemma 2.1 it follows that

$$I^\alpha(D^\alpha x(t)) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{r-1}t^{r-1},$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,...,r-1$, $r = [\alpha]+1$.

But in the opposite way we have

$$D^\alpha(I^\beta x(t)) = D^{\alpha-\beta}x(t). \tag{6}$$

Proposition 2.8. *[19]*
Assume that $x : [0,+\infty) \longrightarrow \mathbb{R}$ is continuous and $0 < \beta \leq \alpha$. Then
i) *$I^\alpha(tx(t)) = tI^\alpha x(t) - \alpha I^{\alpha+1}x(t)$.*
ii) *$I^\alpha(tD^\beta x(t)) = tI^{\alpha-\beta}x(t) - \alpha I^{\alpha-\beta+1}x(t)$.*

Theorem 2.9 (7,**Dhage fixed point**).
Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : S \longrightarrow S$ and $B : S \longrightarrow S$ be two operators such that

(a) A is Lipschitzian with a Lipschitz constant α;
(b) B is completely continuous;
(c) $x = AxBy \Longrightarrow x \in S$ for all $y \in S$, and
(d) $\alpha M < 1$, where $M = \|B(S)\| = sup\{\|B(x)\| : x \in S\}$,

then the operator equation $AxBx = x$ has a solution in S.

3. Main Results

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $X = C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0, 1]$ into $\mathbb{R}$ with the norm $||y|| = sup\{|y(t)|, \ t \in J\}$ and let $Car(J \times R, R)$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that
(i) the map $t \to g(t, x)$ is measurable for each $x \in \mathbb{R}$,and
(ii) the map $x \to g(t, x)$ is continuous for each $t \in J$.
The class $Car(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\| \ . \ \|_1$ defined by

$$\| \ x \ \|_1 = \int_0^T | \ x(t) \ | \ dt.$$

Before presenting our main results, we introduce the following assumptions:
(H_0) $\quad x \longrightarrow \frac{x}{f(x,t)}$ is continuous.
(H_1) $\quad f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ is continuous and there exists a constant $L > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq L_1 |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}.$$

(H_2) $\quad g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t, x)| \leq h(t), \quad \forall x \in \mathbb{R}.$$

We prove the existence of solution for the problem (2) by a fixed point theorem in Banach algebra due to Dhage [10].

Lemma 3.1. *The function $x \in C([0, 1], \mathbb{R})$ is a solution of the problem*

$$\begin{cases} \left({}^cD^\alpha_{0^+} - \rho t \, {}^cD^\beta_{0^+} \right) = g(t, x(t)) \quad a.e. \quad t \in J = [0, 1] \quad 1 < \alpha < 2, \quad 0 < \beta < \alpha \\ x_0 = \frac{x(0)}{f(0,x(0))}, \quad x_1 = \frac{x(1)}{f(1,x(1))} \end{cases} \tag{7}$$

if and only if x satisfies the hybrid integral equation

$$x(t) \;\; = \;\; f(t, x(t))\Big(x_0 + (x_1 - x_0)t + \int_0^1 G(t, s)ds \Big), \tag{8}$$

where $G(t, s)$ given by

$$G(t, s) = \begin{cases} G_1(t, s) & 0 \leq s < t \\ G_2(t, s) & t \leq s < 1 \end{cases}$$

with

$$\begin{aligned} G_1(t, s) \;\; &= \;\; \rho\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \Big) \frac{x(s)}{f(s, x(s))} \\ &+ \;\; \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big) g(s, x(s)) \\ G_2(t, s) \;\; &= \;\; \rho t \Big(\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \Big) \frac{x(s)}{f(s, x(s))} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s)) \end{aligned}$$

Proof. Applying the operator I^α on the equation

$$\left({}^cD^\alpha_{0^+} - \rho t \, {}^cD^\beta_{0^+} \right) \frac{x(t)}{f(t, x(t))} = g(t, x(t)),$$

we get $I^{\alpha}\Big({}^{c}D_{0^+}^{\alpha} - \rho t\, {}^{c}D_{0^+}^{\beta}\Big)\frac{x(t)}{f(t,x(t))} = I^{\alpha}(g(t,x(t)))$.
Then,

$$\frac{x(t)}{f(t,x(t))} = \rho t I^{\alpha-\beta}\Big(\frac{x(t)}{f(t,x(t))}\Big) - \alpha\rho I^{\alpha-\beta+1}\Big(\frac{x(t)}{f(t,x(t))}\Big) + I^{\alpha} g(t,x(t)) - c_0 - c_1 t. \tag{9}$$

for some constants c_0 and c_1. Hence using the boundary conditions. We obtain $c_0 = -x_0$ and

$$\begin{aligned} c_1 &= x_0 - x_1 + \int_0^1 \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} g(s,x(s))ds + \rho \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \frac{x(s)}{f(s,x(s))}ds \\ &- \alpha\rho \int_0^1 \frac{(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \frac{x(s)}{f(s,x(s))}ds. \end{aligned}$$

Substituting $c_0 = -x_0$ and c_1 into 9 we get

$$\begin{aligned} &\frac{x(t)}{f(t,x(t))} \\ &= x_0 + (x_1 - x_0)t \\ &+ \rho \int_0^t \Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))}ds \\ &+ \rho \int_t^1 \Big(\frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))}ds + \int_0^t \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big) g(s,x(s))ds \\ &- \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s))ds. \end{aligned}$$

$$x(t) = f(t,x(t))\Big(x_0 + (x_1 - x_0)t + \int_0^1 G(t,s)ds\Big),$$

where $G(t,s)$ given by

$$G(t,s) = \begin{cases} G_1(t,s) & 0 \le s < t \\ G_2(t,s) & t \le s < 1 \end{cases}$$

with

$$\begin{aligned} G_1(t,s) &= \rho\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))} \\ &+ \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big) g(s,x(s)). \\ G_2(t,s) &= \rho t\Big(\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s)). \end{aligned}$$

Conversely, it is clear that if $x(t)$ satisfies the equation 8, then we divide by $f(t,x(t))$ and we apply the Caputo fractional derivative ${}^cD^{\alpha}_{0^+}$ to both sides of equation 8 and we use Proposition 2.8, we obtain

$$\begin{aligned}
{}^cD^{\alpha}_{0^+}\Big(\frac{x(t)}{f(t,x(t))}\Big) &= {}^cD^{\alpha}_{0^+}\Big(x_0+(x_1-x_0)t+\int_0^1 G(t,s)ds\Big)\\
&= {}^cD^{\alpha}_{0^+}\Big(x_0+(x_1-x_0)t\\
&+ \rho\int_0^t\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}-\frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\\
&\quad \frac{x(s)}{f(s,x(s))}ds+\rho\int_t^1\Big(\frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))}ds\\
&+ \int_0^t\Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)g(s,x(s))ds-\int_t^1\frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds.\Big)\\
&= {}^cD^{\alpha}_{0^+}\Big(\rho t I^{\alpha-\beta}\Big(\frac{x(t)}{f(t,x(t))}\Big)-\alpha\rho I^{\alpha-\beta+1}\Big(\frac{x(t)}{f(t,x(t))}\Big)+I^{\alpha}g(t,x(t))-c_0-c_1t\Big)
\end{aligned}$$

Then,

$$ {}^cD^{\alpha}_{0^+}\Big(I^{\alpha}\Big({}^cD^{\alpha}_{0^+}-\rho t\,{}^cD^{\beta}_{0^+}\Big)\frac{x(t)}{f(t,x(t))})\Big) = {}^cD^{\alpha}_{0^+}\Big(I^{\alpha}g(t,x(t))\Big).$$

We obatin,

$$\Big({}^cD^{\alpha}_{0^+}-\rho t\,{}^cD^{\beta}_{0^+}\Big)\Big(\frac{x(t)}{f(t,x(t))}\Big)=g(t,x(t)).$$

Finally, we need to verify that the conditions $x_0=\frac{x(0)}{f(0,x(0))}$ and $x_1=\frac{x(1)}{f(1,x(1))}$ in th equation 7 also holds. For this purpose, we substitute $t=0$ in 8, we obtain

$$\begin{aligned}
x(0) &= f(0,x(0))\Big(x_0+(x_1-x_0)\times 0\\
&+ \rho\int_0^0\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}-\frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))}ds\\
&+ \rho\int_0^1\Big(\frac{\alpha\times 0(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}-\frac{0\times(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s,x(s))}ds+\int_0^0\Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\
&- \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)g(s,x(s))ds-\int_0^1\frac{0\times(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds.
\end{aligned}$$

Then,

$$\frac{x(0)}{f(0,x(0))}=x_0,$$

we substitute $t=1$ in 8, it easy to obtain

$$\frac{x(1)}{f(1,x(1))}=x_1.$$

This completes the proof. □

Now we pose

$$\begin{aligned}
m_1 &= max\Big\{\frac{2|\rho|}{\Gamma(\alpha-\beta+1)},\frac{2|\rho|}{\Gamma(\alpha-\beta)},\frac{2|\rho|}{\Gamma(\alpha)}\Big\},\\
m_2 &= max\Big\{3|\frac{x(s)}{f(s,x(s))}|,|h(s)|\Big\}.
\end{aligned}$$

Theorem 3.2. *Assume that hypotheses* (H_0), (H_1) *and* (H_2) *hold and if*

$$L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big) < \frac{1}{2},$$

then the hybrid fractional-order differential equation (2) *has a solution defined on* J.

Proof. Let S subset of X such that

$$S = \{x \in X / \parallel x \parallel \leq N\},$$

where

$$N = \frac{F_0\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big)}{1 - L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big)}$$

and $F_0 = sup_{s \in [0,1]} |f(t, 0)|$.
It is clear that S satisfies the hypothesis of Theorem 2.9 By an application of Lemma 3.1, equation (2) is equivalent to the nonlinear hybrid integral equation.

Claim 1, Let $x, y \in X$. Then by hypothesis (H_1),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq L_1 |x(t) - y(t)| \leq L_1 \|x - y\|,$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \leq L_1 \|x - y\|,$$

for all $x, y \in X$.
Claim 2, We show that B is continuous in S .
Let (x_n) be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \rho\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x_n(s)}{f(s, x_n(s))} ds \\
&= \int_0^t \rho\Big(\frac{t(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\alpha t(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{t(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s, x(s))} ds
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty} \int_t^1 \rho t\Big(\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x_n(s)}{f(s, x_n(s))} ds \\
&= \int_t^1 \rho t\Big(\frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big)\frac{x(s)}{f(s, x(s))} ds \\
&\lim_{n\to\infty} \int_0^t \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big) g(s, x_n(s)) = \int_0^t \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big) g(s, x(s)) ds \\
&\lim_{n\to\infty} \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x_n(s)) = \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, x(s)) ds
\end{aligned}$$

Then,

$$\lim_{n\to\infty} Bx_n(t) \quad = \quad Bx(t),$$

for all $t \in J$. This shows that B is a continuous operator on S.
Claim 3, B is compact operator on S .

First, we show that $B(S)$ is a uniformly bounded set in X.
Let $x \in S$. Then for $0 \leq s \leq t$ we have

$$\begin{aligned}
|g(t,s)| &\leq |\rho|\Big\{\frac{2\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{2(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\Big\}|\frac{x(s)}{f(s,x(s))}| + \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))| \\
&\leq m_1\Big\{\alpha(1-s)^{\alpha-\beta-1} + (1-s)^{\alpha-\beta-1}\Big\}|\frac{x(s)}{f(s,x(s))}| + (1-s)^{\alpha-\beta-1}|h(s)| \\
&\leq m_1(1-s)^{\alpha-\beta-1}\Big\{3|\frac{x(s)}{f(s,x(s))}| + |h(s)|\Big\} \\
&\leq m_1 m_2 (1-s)^{\alpha-\beta-1}
\end{aligned}$$

where

$$\begin{aligned}
m_1 &= max\Big\{\frac{2|\rho|}{\Gamma(\alpha-\beta+1)}, \frac{2|\rho|}{\Gamma(\alpha-\beta)}, \frac{2|\rho|}{\Gamma(\alpha)}\Big\}, \\
m_2 &= max\Big\{3|\frac{x(s)}{f(s,x(s))}|, |h(s)|\Big\}.
\end{aligned}$$

On the other hand, for $s > t$, we arrive at same conclusion. Therefore,

$$\begin{aligned}
\int_0^1 |G(t,s)|ds &\leq m_1 \times m_2 \int_0^1 (1-s)^{\alpha-\beta-1} ds \\
&\leq \frac{m_1 \times m_2}{\alpha-\beta}
\end{aligned}$$

Then,

$$\begin{aligned}
|\ Bx(t)| &\leq |x_0| + |(x_1 - x_0)| + \int_0^1 |G(t,s)|ds \\
&\leq 2|x_0| + |x_1| + \frac{m_1 \times m_2}{\alpha-\beta}
\end{aligned}$$

Thus,

$$\|\ Bx\ \| \leq 2|x_0| + |x_1| + \frac{m_1 \times m_2}{\alpha-\beta}, \quad \forall x \in S.$$

This shows that B is uniformly bounded on S.
Next, we show that $B(S)$ is an equicontinuous set on X.
Let $k = \max_{s\in[0,1]} g(s,x(s))$ and $M = max_{s\in[0,1]} \frac{1}{f(s,x(s))}$.

Let $t_1,\ t_2 \in [0,1]$, then for any $x \in S$,

$$\begin{aligned}
|\, Bx(t_2) \;&-\; Bx(t_1)\,| \\
&\leq |\, x_1 - x_0 \,||\, t_2 - t_1 \,| + |\, t_2 - t_1 \,| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \,|\, g(s,x(s))\,|\, ds \\
&+ |\,\rho\,||\, t_2 - t_1 \,| \int_0^1 \left| \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right| \left| \frac{x(s)}{f(s,x(s))} \right| ds \\
&+ |\,\rho\,| \int_0^{t_2} \left| \frac{t_2(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_2-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \left(\frac{t_1(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \right| \\
&\quad \left| \frac{x(s)}{f(s,x(s))} \right| ds + \int_0^{t_2} \left| \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right| |\, g(s,x(s))\,|\, ds \\
&\leq |\, x_1 - x_0 \,||\, t_2 - t_1 \,| + k\,|\, t_2 - t_1 \,| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ M\,|\,\rho\,||\, t_2 - t_1 \,|||\, x \,|| \int_0^1 \left| \frac{\alpha(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right| ds \\
&+ M\,|\,\rho\,|||\, x \,|| \int_0^{t_2} \left| \frac{t_2(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{\alpha(t_2-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \left(\frac{t_1(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \right.\right. \\
&- \left.\left. \frac{\alpha(t_1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \right| ds + k \int_0^{t_2} \left| \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right| ds \\
&\leq |\, x_1 - x_0 \,||\, t_2 - t_1 \,| + k \frac{|\, t_2 - t_1 \,|}{\Gamma(\alpha+1)} + M\,|\,\rho\,||\, t_2 - t_1 \,|||\, x \,|| \frac{1-\beta}{\Gamma(\alpha-\beta+2)} \\
&+ M\,|\,\rho\,|||\, x \,|| \frac{t_1\,|\, t_1 - t_2 \,|^{\alpha-\beta} + |\, t_2^{2(\alpha-\beta)} - t_1^{2(\alpha-\beta)} \,|}{\Gamma(\alpha-\beta+1)} \\
&+ \alpha M \,||\, x \,|| \frac{|\, t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1} \,| + |\, t_1 - t_2 \,|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}
\end{aligned}$$

Hence

$$\forall \epsilon > 0,\ \ \exists \eta > 0 :\ \ |\, t_1 - t_2 \,| < \eta \implies |\, Bx(t_1) - Bx(t_2) \,| < \epsilon$$

for all $t_1,\ t_2 \in J$ and for all $x \in X$.
This shows that $B(S)$ is an equicontinuous set in X.
Then, by the Arzelà-Ascoli theorem, B is a continuous and compact operator on S.
Claim 4 Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then

$$\begin{aligned}
|\, x(t) \,| &= |\, Ax(t) \,||\, By(t) \,| \\
&\leq \left| f(t,x(t)) \right| \left| x_0 + (x_1 - x_0)t + \int_0^1 G(t,s) ds \right| \\
&\leq \left| f(t,x(t)) \right| \left(2\,|\, x_0 \,| + |\, x_1 \,| + \int_0^1 |\, G(s,t) \,|\, ds \right) \\
&\leq \left| f(t,x(t)) - f(t,0) + f(t,0) \right| \left(2\,|\, x_0 \,| + |\, x_1 \,| + \frac{m_1 \times m_2}{\alpha-\beta} \right) \\
&\leq \left(L \,||\, x \,|| + F_0 \right) \left(2\,|\, x_0 \,| + |\, x_1 \,| + \frac{m_1 \times m_2}{\alpha-\beta} \right)
\end{aligned}$$

Thus,

$$\left(1 - L \left(2\,|\, x_0 \,| + |\, x_1 \,| + \frac{m_1 \times m_2}{\alpha-\beta} \right) \right) ||\, x \,|| \leq F_0 \left(2\,|\, x_0 \,| + |\, x_1 \,| + \frac{m_1 \times m_2}{\alpha-\beta} \right).$$

Finally,

$$\| x \| \leq \frac{F_0\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big)}{1 - L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big)} = N.$$

Where $F_0 = F(t, 0)$
Then $x \in S$.
Finally, we have

$$M = \|B(S)\| = sup\{\|Bx\| : \quad x \in S\} \leq L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big).$$

Then, $\alpha M \leq 2L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big) < 1$. Thus, all the conditions of theorem 2.9. are satisfied and therefore the operator equation $AxBx = x$ has solution on S. This completes the proof. □

4. An illustrative example.

In this section, we will present example to illustrate the main results.

$$\begin{cases} \Big(({}^cD^{\frac{3}{2}}_{0^+} - \frac{1}{8} t \, {}^cD^{\frac{1}{2}}_{0^+}\Big)\Big(\frac{e^{-t}}{1+e^t} \frac{x(t)}{1+|x(t)|}\Big) = sin(x) \quad a.e. \quad t \in J = [t_{ini}, t_{fin}] = [0, 1], \\ x_0 = \frac{x(0)}{f(0, x(0))} = x_1 = \frac{x(1)}{f(1, x(1))} = 0. \end{cases} \tag{10}$$

Where $f(x, t) = \frac{e^{-t}}{1+e^t} \frac{1}{1+|x(t)|}$, $g(x, t) = sin(x)$, $h(t) = 1$, $\rho = \frac{1}{8}$, $L = 1$, $m_1 = \frac{1}{4}$ and $m_2 = \frac{3}{2}$.
It is clear that the assumptions (H_0) and (H_2) are satisfied
To prove the assumption (H_1), let $t \in J$ and $u, v \in \mathbb{R}$, then we have

$$\begin{aligned} |f(t, u(t)) - f(t, v(t))| &\leq \Big|\frac{e^{-t}}{1 + e^t} \frac{1}{1 + |u(t)|} - \frac{e^{-t}}{1 + e^t} \frac{1}{1 + |v(t)|}\Big| \\ &\leq \Big|\frac{e^{-t}}{1 + e^t}\Big|\Big|\frac{1}{1 + |u(t)|} - \frac{1}{1 + |v(t)|}\Big| \\ &\leq \Big|\frac{e^{-t}}{1 + e^t}\Big|\Big|\frac{v(t) - u(t)}{(1 + |u(t)|)(1 + |v(t)|)}\Big| \\ &\leq \frac{1}{2}\Big|v(t) - u(t)| \end{aligned}$$

Thus, the assumption (H_1) in holds true with $L_1 = \frac{1}{2}$
Moreover, we have

$$L_1\Big(2 \mid x_0 \mid + \mid x_1 \mid + \frac{m_1 \times m_2}{\alpha - \beta}\Big) = \frac{1}{2}\Big(2 \times 0 + 0 + \frac{1}{4} \times \frac{3}{2}\Big) = \frac{3}{16} < \frac{1}{2},$$

Therefore, Finally, all the conditions of Theorem 3.2 are satisfied, thus the fractional hybrid differential equation 10 has a solution .

5. Conclusion

In this article, we have provided a definition of Solution for a Nonlinear Fractional Order Differential Equations with a Quadratic Perturbations using the Caputo fractional derivative of order $\alpha \in (1, 2)$, $\beta \in (1, \alpha)$. Furthermore, using some Lipschitz and Caratheodory conditions we prove the existence of at least one solution to our problem. Finally, a example is presented to illustrate the applicability of our main results.

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