Pell-Lucas Collocation Method for Solving High-Order Functional Differential Equations with Hybrid Delays

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Abstract

In this study, the Pell-Lucas collocation method has been presented to solve high-order linear functional differential equations with hybrid delays under mixed conditions. The proposed method is based on the matrix forms of Pell-Lucas polynomials and their derivatives, along with the collocation points. The used technique reduces the problem to a matrix equation corresponding to a set of algebraic equations with the unknown Pell-Lucas coefficients. In addition, an error analysis based on residual function is performed and some numerical examples are presented to show the efficiency and accuracy of the method.

Keywords: Pell-Lucas polynomials, collocation method, functional differential-equations, residual error analysis, matrix method.

1. Introduction

In this study, we consider the high-order linear functional-differential equations with variable coefficients and mixed delays in the generalized form [1-12]

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{r=0}^{m_1}\sum_{s=0}^{m_2} F_{r\,s}(x)y^{(r)}(\alpha_{r\,s} + \beta_{r\,s}) = g(x), m \geq m_1 \qquad (1.1)$$

under the conditions

$$\sum_{k=0}^{m-1} \left(a_{k\,j}y^{(k)}(a) + b_{k\,j}y^{(k)}(b)\right) = \lambda_j\,, \quad j = 0,1,2,\dots,m-1. \qquad (1.2)$$

where the functions $P_k(x), F_{r\,s} and\ g(x)$ are known function having m-th derivatives on interval [a,b]; the constants $\alpha_{r\,s}, b_{k\,j},\ a_{k\,j}, \beta_{r\,s}\ and\ \lambda_j$ are appropriate constants.

In the context of the modeling of dynamical systems, the functional differential equations in the form (1.1) play a central role in various fields such as biology, economy, electrodynamics, potential theory, electrostatics, astronomy, chemistry, mechanics, physics, etc.[1,4,9,11,12]. Most of these equations have not analytical solution and so, numerical methods may be required to obtain their approximate solutions. For example, some functional differential equations have been solved by using the numerical methods such as one-Leg θ method [9], the spline function method [2], Legendre-Gauss collocation method [4], Chebyshev operational matrix method [5], Optimal residual method [10], Dickson Collocation Method Based on Residual Error [11], Jacobian elliptic function method [12], Chebyshev Collocation method with residual correction [13], Legendre spectral collocation method [14] and Lagrange-collocation method [15].

In recent years, some matrix and collocation methods to solve linear and nonlinear differential, integral, integro-differential, integro-differential-difference and pantograph equations have been presented in many articles by Sezer and Coworkers [7, 8, 11, 16-18]. The purpose in this paper, by means of the above mentioned methods, is to develop a new collocation method based on Pell-Lucas polynomials and to find the approximate solution of the problem (1.1)-(1.2) as the truncated Pell-Lucas series defined by

$$y(x) \cong y_N(x) = \sum_{n=o}^{N} a_n Q_n(x) \ \ , N \geq m\,, \ , a \leq x \leq b \qquad (1.3)$$

where $Q_n(x), n = 0,1,\dots,N,$ denote the Pell-Lucas polynomials [19,20]; $a_n, n = 0,1,\dots,N,$ are unknown Pell-Lucas coefficients and N is any positive integer chosen such that $N \geq m.$ Besides, the collocation points are defined by

$$x_i = a + \frac{b-a}{N}i, \ \ (standard)$$

and

$$x_i = \frac{b+a}{2}\frac{b-a}{2}\cos\left(\frac{\pi i}{N}\right)(Cheb.Lobatto) \qquad (1.4)$$

2. Materials and Methods

2.1 Some Important Properties of the Pell-Lucas Polynomials [19,20]

Pell-Lucas defined the set of polynomials$\{Q_n(x)\}$. The polynomials $\{Q_n(x)\}$ are recursively defined by the following relationships:

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x)\,, n \geq 2 \qquad (2.1)$$

with $Q_0(x) = 2$ *and* $Q_1(x) = 2x$.

The Pell-Lucas polynomials $Q_n(x)$ can also be given explicitly by

$$Q(x) = \sum_{k=0}^{[n/2]} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}. \qquad (2.2)$$

The first four Pell-Lucas polynomial $Q_n(x)$:

$$Q_0(x) = 2, \qquad Q_1(x) = 2x\,, \qquad Q_2(x) = 4x + 2,$$
$$Q_3(x) = 8x^3 + 6x, \ldots$$

2.2 Fundamental Matrix Relations and Pell-Lucas Collocation Method

Firstly, we approximate the solution (1.3) of the problem (1.1)-(1.2) by the matrix form

$$y(x) \cong y_N(x) = \boldsymbol{Q}(x)\boldsymbol{A} \qquad (2.3)$$

where

$$\boldsymbol{Q}(x) = [Q_0(x) \;\; Q_1(x) \;\; \ldots \;\; Q_N(x)]$$
$$\boldsymbol{A} = [a_0 \;\; a_1 \;\; \cdots \;\; a_N]^T.$$

Now we clearly write the matrix form $Q_n(x)$, by using the Pell-Lucas polynomials $Q_n(x)$ given by (2.1) or (2.2), as

$$\boldsymbol{Q}(x) = \boldsymbol{X}(x)\boldsymbol{M} \qquad (2.4)$$

where

$$X(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^N \end{bmatrix}.$$

If N is odd,

$$\boldsymbol{M}^T = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 2^1\frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\
2^0\frac{2}{1}\binom{1}{1} & 0 & 2^2\frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\
0 & 2^1\frac{3}{2}\binom{1}{1} & 0 & 2^3\frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\
2^0\frac{4}{2}\binom{2}{2} & 0 & 2^2\frac{4}{3}\binom{3}{1} & 0 & 2^4\frac{4}{4}\binom{4}{0} & 0 & \cdots & 0 \\
0 & 2^1\frac{5}{3}\binom{3}{2} & 0 & 2^1\frac{5}{4}\binom{4}{1} & 0 & 2^3\frac{5}{5}\binom{5}{0} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 2^1\frac{N}{\frac{N+1}{2}}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & 2^3\frac{N}{\frac{N+3}{2}}\binom{\frac{N+3}{2}}{\frac{N-3}{2}} & 0 & 2^5\frac{N}{N+1}\binom{\frac{N+5}{2}}{\frac{N-5}{2}} & \cdots & 2^N\frac{N}{N}\binom{N}{0}
\end{bmatrix}$$

If N is even,

$$\boldsymbol{M}^T = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 2^1\frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\
2^0\frac{2}{1}\binom{1}{1} & 0 & 2^2\frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\
0 & 2^1\frac{3}{2}\binom{1}{1} & 0 & 2^3\frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\
2^0\frac{4}{2}\binom{2}{2} & 0 & 2^2\frac{4}{3}\binom{3}{1} & 0 & 2^4\frac{4}{4}\binom{4}{0} & 0 & \cdots & 0 \\
0 & 2^1\frac{5}{3}\binom{3}{2} & 0 & 2^1\frac{5}{4}\binom{4}{1} & 0 & 2^3\frac{5}{5}\binom{5}{0} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2^0\frac{N}{\frac{N}{2}}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & 2^2\frac{N}{\frac{N+2}{2}}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & 0 & 2^2\frac{N}{\frac{N+2}{2}}\binom{\frac{N+4}{2}}{\frac{N-4}{2}} & 0 & \cdots & 2^N\frac{N}{N}\binom{N}{0}
\end{bmatrix}.$$

From the relations (2.3) and (2.4), we obtain the matrix form

$$y_N(x) = \boldsymbol{X}(x)\boldsymbol{M}\boldsymbol{A}\;\; . \qquad (2.5)$$

Also, the relations between the matrix $\boldsymbol{X}(x)$ and its derivative $\boldsymbol{X}^{(k)}(x)$ are

$$\boldsymbol{X}^{(k)}(x) = \boldsymbol{X}(x)\boldsymbol{B}^k\,, \qquad k = 0,1,\ldots \qquad (2.6)$$

so that

$$B^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using the matrices (2.5) and (2.6), we have the matrix relation

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \boldsymbol{Q}^{(k)}(x)\boldsymbol{A} = \boldsymbol{X}^{(k)}(x)\boldsymbol{MA} = \boldsymbol{X}(x)\boldsymbol{B}^k\boldsymbol{MA}, \quad k = 0,1,2,\dots,m \qquad (2.7)$$

and by substituting $x \to \alpha_{r\,s}x + \beta_{r\,s}$ and $k \to r$ in (2.7), the matrix relation

$$y^{(r)}(\alpha_{r\,s}x + \beta_{r\,s}) = \boldsymbol{X}(\alpha_{r\,s}x + \beta_{r\,s})\boldsymbol{B}^r\boldsymbol{MA} = \boldsymbol{X}(x)\boldsymbol{B}(\alpha_{r\,s},\beta_{r\,s})\boldsymbol{B}^r\boldsymbol{MA} \qquad (2.8)$$

where

$$\boldsymbol{B}(\boldsymbol{\alpha}_{r\,s},\boldsymbol{\beta}_{r\,s}) = \begin{bmatrix} \binom{0}{0}\alpha_{rs}^{0}\beta_{rs}^{0} & \binom{1}{0}\alpha_{rs}^{0}\beta_{rs}^{1} & \binom{2}{0}\alpha_{rs}^{0}\beta_{rs}^{2} & \cdots & \binom{N}{0}\alpha_{rs}^{0}\beta_{rs}^{N} \\ 0 & \binom{1}{1}\alpha_{rs}^{1}\beta_{rs}^{0} & \binom{2}{1}\alpha_{rs}^{1}\beta_{rs}^{1} & \cdots & \binom{N}{1}\alpha_{rs}^{1}\beta_{rs}^{N-1} \\ 0 & 0 & \binom{2}{2}\alpha_{rs}^{2}\beta_{rs}^{0} & \cdots & \binom{N}{2}\alpha_{rs}^{2}\beta_{rs}^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}\alpha_{rs}^{N}\beta_{rs}^{0} \end{bmatrix}$$

Note that the matrix $\boldsymbol{X}(\alpha_{r\,s}x + \beta_{r\,s})$ can be written as

$$\boldsymbol{X}(\alpha_{r\,s}x + \beta_{r\,s}) = \boldsymbol{X}(x)\boldsymbol{B}(\alpha,\beta).$$

By substituting the relations (2.7) and (2.8) into Eq.(1.1),we obtain the matrix equation

$$\left\{\sum_{k=0}^{m} P_k(x)X(x)B^k + \sum_{r=0}^{m_1}\sum_{s=0}^{m_2} F_{r\,s}(x)X(x)B(\alpha_{r\,s},\beta_{r\,s})B^r\right\}MA = g(x)$$

and then, by placing the collocation points (1.4),the system of the matrix equations

$$\left\{\sum_{k=0}^{m} P_k(x_i)\boldsymbol{X}(x_i)\boldsymbol{B}^{\boldsymbol{k}} + \sum_{r=0}^{m_1}\sum_{s=0}^{m_2} F_{r\,s}(x_i)\boldsymbol{X}(\boldsymbol{x_i})\boldsymbol{B}(\alpha_{r\,s},\beta_{r\,s})\boldsymbol{B}^{\boldsymbol{r}}\right\}\boldsymbol{MA} = g(x_i),$$
$$i = 0,1,\dots,N.$$

The compact form of this system can be written as

$$\left\{\sum_{k=0}^{m} \boldsymbol{P}_k\boldsymbol{X}\boldsymbol{B}^k + \sum_{r=0}^{m_1}\sum_{s=0}^{m_2} \boldsymbol{F}_{r\,s}\boldsymbol{X}\boldsymbol{B}(\alpha_{r\,s},\beta_{r\,s})\boldsymbol{B}^r\right\}\boldsymbol{MA} = \boldsymbol{G} \qquad (2.9)$$

where

$$\boldsymbol{X} = \begin{bmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_n) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}$$

$$\boldsymbol{P}_k = diag[P_k(x_0) \quad P_k(x_1) \quad \cdots \quad P_k(x_N)]$$

$$\boldsymbol{F}_{r\,s} = diag[F_j(x_0) \quad F_j(x_1) \quad \cdots \quad F_j(x_N)].$$

In Eq. (2.9), the general forms of the matrices $P_k, X, B, F_{r\,s}, B(\alpha,\beta), M, A$ and G, respectively, are $(N+1)x(N+1), (N+1)x(N+1), (N+1)x(N+1), (N+1)x(N+1), (N+1)x(N+1), (N+1)x(N+1), (N+1)x1$ and $(N+1)x1$.

The fundamental matrix equation (2.9) can be expressed in the form

$$\boldsymbol{WA} = \boldsymbol{G} \text{ or } [\boldsymbol{W};\boldsymbol{G}] \qquad (2.10)$$

where

$$\boldsymbol{W} = [\boldsymbol{w}_{pq}] = \left\{\sum_{k=0}^{m} \boldsymbol{P}_k\boldsymbol{XB}^k + \sum_{r=0}^{m_1}\sum_{s=0}^{m_2} \boldsymbol{F}_{r\,s}\boldsymbol{XB}(\alpha_{r\,s},\beta_{r\,s})\boldsymbol{B}^r\right\}\boldsymbol{M},$$
$$p,q = 0,1,\dots,N.$$

On the other hand, by mean of the relation (2.7), we can write the matrix forms of the conditions (1.2) as

$$\boldsymbol{U}_j\boldsymbol{A} = \boldsymbol{\lambda}_j \text{ or } [\boldsymbol{U}_j,\boldsymbol{\lambda}_j], j = 0,1,2,\dots,m-1, \qquad (2.11)$$

such that

$$\boldsymbol{U}_{\boldsymbol{j}} = [u_{j\,0} \quad u_{j\,1} \quad \cdots \quad u_{j\,N}] = \sum_{k=0}^{m-1}\left(a_{k\,j}\boldsymbol{X}(a) + b_{k\,j}\boldsymbol{X}(b)\right)(\boldsymbol{B})^{\boldsymbol{k}}\boldsymbol{M},$$
$$j = 0,1,2,\dots,m-1.$$

Consequently, in order to obtain the Pell-Lucas polynomial solution of Eq. (1.1) under the condition (1.2), we replace the m row matrices (2.11) by any m rows of the augmented matrix (2.10). Thereby we obtain the new augmented matrix

$$[\widetilde{\boldsymbol{W}},\widetilde{\boldsymbol{G}}] \ or \ \widetilde{\boldsymbol{W}}\boldsymbol{A}=\widetilde{\boldsymbol{G}}\,. \qquad (2.12)$$

If $\widetilde{W}=rank[\widetilde{W},\widetilde{G}]=N+1$, then we can write $A=(\widetilde{W})^{-1}\widetilde{G}$. Thus the matrix A (thereby the Pell-Lucas coefficients $a_0,a_1,\dots,a_N$) is uniquely determined. Also, Eq. (1.1) under the conditions (1.2) has an unique solution. This solution is given by the truncated Pell-Lucas series (1.3).

3. Results and Discussion

3.1 Accuracy of Solutions and Residual Error Estimation

In this section, we investigate the accuracy of the obtained Pell-Lucas solutions. When $y_N(x)$ and its derivatives are substituted in Eq. (1.1), the solved equation is required to satisfy approximately. For $x=x_r\in[a,b], j=0,1,2,\dots,$

$$R_N(x_j)=\sum_{k=0}^{m}P_k(x_j)\,y_N^{(k)}(x_j)$$

$$+\sum_{r=0}^{m_1}\sum_{s=0}^{m_2}F_{r\,s}(x_j)\,y_N^{(r)}(\alpha_{r\,s}x_j+\beta_{r\,s})-g(x_j)\cong 0$$

or

$$R_N(x_j)\le 10^{-x_j}(,x_j \ is\ any\ positive\ integer).$$

If $\max 10^{-k_j}=10^{-k}$ (k_j is an positive integer) is determined, then the truncation $limit\ N$ is increased until the difference $R_N(x_j)$ at each of the points becomes smaller than the prescribed $10^{-k}[16-18]$.

On the other hand, the accuracy of the solution can be determined and the error can be estimated by means of the residual function $R_N(x)$ and the mean value of $|R_N(x)|$ on $[a,b]$, If $R_N(x)\to 0$ and N is sufficiently enough, then the error decreases. Also, by using the Mean-Value Theorem for the residual function [18], we can estimate the upper bound of the mean error, $\overline{R_N}$:

$$\left|\int_a^b R_N(x)\,dx\right|\le\int_a^b|R_N(x)|dx$$

and

$$\int_a^b R_N(x)\,dx=(b-a)R_N(c)\,,\quad a\le c\le b$$

$$\Rightarrow\left|\int_a^b R_N(x)\,dx\right|=(b-a)|R_N(c)|$$

$$\Rightarrow(b-a)|R_N(c)|\le\int_a^b|R_N(x)|dx$$

$$\Rightarrow|R_N(c)|\le\frac{\int_a^b|R_N(x)|dx}{b-a}=\overline{R}_N$$

3.2 Numerical Examples

Example 1. Consider the second order pantograph equation

$$y''(x)-\frac{3}{4}y(x)-y\left(\frac{x}{2}\right)=-x^2+2\,,0\le x\le 1$$

with initial conditions

$$y(0)=0\,,\qquad y'(0)=0.$$

The exact solution of this problem is $y(x)=x^2$ and the coefficients in Eq. (1.1) are defined as

$$m=2\,,P_0=\frac{-3}{4}\,,P_1=0\,,P_2=1\,,F_{00}=-1\,,$$

$$\alpha_{00}=\frac{1}{2}\,,\beta_{00}=0\,,g(x)=-x^2+2.$$

We find the solution $y(x)$ with truncated Pell-Lucas series for N=2

$$\boldsymbol{y}_2(x)=\sum_{n=0}^{2}\boldsymbol{a}_n\boldsymbol{Q}_n(x)$$

and the collocation points for $N=2;\left\{x_0=0\,,\ x_1=\frac{1}{2},\ x_2=1\right\}$ are obtained. The fundamental matrix equation of the problem can be written, using Eq. (2.9), as

$$\Big\{\boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0\boldsymbol{M}+\boldsymbol{P}_1\boldsymbol{X}\boldsymbol{B}^1\boldsymbol{M}+\boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2\boldsymbol{M}$$
$$+\boldsymbol{F}_{00}\boldsymbol{X}\boldsymbol{B}(\frac{1}{2},0)\boldsymbol{B}^0\boldsymbol{M}\Big\}\boldsymbol{A}=\boldsymbol{G}$$

where

$$\boldsymbol{W} = \boldsymbol{P}_0\boldsymbol{X}\boldsymbol{B}^0\boldsymbol{M} + \boldsymbol{P}_1\boldsymbol{X}\boldsymbol{B}^1\boldsymbol{M} + \boldsymbol{P}_2\boldsymbol{X}\boldsymbol{B}^2\boldsymbol{M} + \boldsymbol{F}_{00}\boldsymbol{X}\boldsymbol{B}\left(\frac{1}{2},0\right)\boldsymbol{B}^0\boldsymbol{M}.$$

Substituting the numerical values yields

$$W = \begin{bmatrix} -3.5 & 0 & 4.5 \\ -3.5 & -1.25 & 3.5 \\ -3.5 & -2.5 & 0.5 \end{bmatrix} \; and \; G = \begin{bmatrix} 2 \\ 1.75 \\ 1 \end{bmatrix}.$$

The matrix forms for the initial conditions in

(2.11) are

$$[u_0;\lambda_0] = [2 \quad 0 \quad 2 \quad ; \quad 0]$$
$$[u_1;\lambda_1] = [0 \quad 2 \quad 0 \quad ; \quad 0].$$

Thus,

$$\tilde{W}A = \tilde{G} \;\; ; \left[\tilde{W};\tilde{G}\right] = \begin{bmatrix} -3.5 & 0 & 4.5 & ; & 2 \\ 2 & 0 & 2 & ; & 0 \\ 0 & 2 & 0 & ; & 0 \end{bmatrix}.$$

By solving this system, the Pell-Lucas coefficients matrix is determined as

$$A = [-0.25 \quad 0 \quad 0.25]^T.$$

This system yields the exact solution of the problem $y(x) = x^2$.

Example 2. Let us now consider the first order linear differential-difference equation with variable coefficients

$$y''(x) - y(x) + 2y(x-1) = 2e^{1-x}, -1 \le x \le 0$$

under the initial conditions $y(0) = 1$, $y'(0) = -1$. A complete solution of the problem

$y(x) = e^{-x}$. Here the coefficients of the equation are

$$m = 2, P_0 = -1, P_2 = 1, F_{00} = 2,$$

$$\alpha_{00} = 1, \beta_{00} = -1, g = 2e^{1-x}.$$

We find the solution $y(x)$ with truncated Pell-Lucas series for N=2

$$y_2(x) = \sum_{n=o}^{2} a_n Q_n(x)$$

and the collocation points for $N = 2; \left\{ x_0 = -1,\ x_1 = \frac{-1}{2},\ x_2 = 0 \right\}$ are obtained. The fundamental matrix equation of the problem can be written using, Eq. (2.9), as

$$\{\boldsymbol{P_0XB^0} + \boldsymbol{P_2XB^2} + \boldsymbol{F_{00}XB(1,-1)B^0}\}\boldsymbol{MA} = \boldsymbol{G}$$

where

$$\boldsymbol{W} = \boldsymbol{P_0XB^0M} + \boldsymbol{P_2XB^2MF_{00}XB(1,-1)B^0M}\ .$$

Here,

$$\boldsymbol{P_0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\boldsymbol{B(1,-1)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{F_{00}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} 0.27067 \\ 0.44626 \\ 0.73576 \end{bmatrix}, \boldsymbol{P_2} = \boldsymbol{B^0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting the numerical values yields

$$[W] = \begin{bmatrix} 2 & -6 & 38 \\ 2 & -5 & 27 \\ 2 & -4 & 18 \end{bmatrix}.$$

The matrix form for the initial condition in (2.11) is

$$[U_0;\lambda_0] = [2 \quad 0 \quad 2 \quad ; \quad 1]$$
$$[U_1;\lambda_1] = [0 \quad 2 \quad 0 \quad ; \quad -1].$$

Thus, the new matrix equation based on the condition is obtained as

$$\tilde{W}A = \tilde{G}$$

or

$$\left[\tilde{W};\tilde{G}\right] = \begin{bmatrix} 2 & -6 & 38 & ; & 14.778 \\ 2 & 0 & 2 & ; & 1 \\ 0 & 2 & 0 & ; & -1 \end{bmatrix}.$$

By solving this system, the Pell-Lucas coefficients matrix is determined as

$$A = [0.20061 \quad -0.5 \quad 0.29939]^T.$$

Thus, the approximate of the problem is obtained as

$$y_2(x) \cong 1 - x + 1.1976x^2.$$

Similarly, for N=3 and N=5, we find the following solutions:

$$y_3(x) = 1 - 0.99998x + 0.17341x^2 - 0.43898x^3$$

$$y_5(x) = 1 - x + 0.48844x^2 - 0.19319x^3 + 0.01452x^4 - 2.0491x10^{-2}x^5.$$

The residual errors for N=2, 3 and 5 is obtained as follows;

$$\overline{R_2} = \int_{-1}^{0} \frac{|R_2(x)|\,dx}{|0-(-1)|} = 2.7279x10^{-1}$$

$$\overline{R_3} = \int_{-1}^{0} \frac{|R_3(x)|\,dx}{|0-(-1)|} = 1.7061x10^{-3}$$

$$\overline{R_5} = \int_{-1}^{0} \frac{|R_5(x)|\,dx}{|0-(-1)|} = 8.42103x10^{-5}$$

Thereby, the results related with exact solution, approximate solution and residual error obtained by our method for Example 2 are demonstrated in Table 1, Figure 1 and Figure 2.

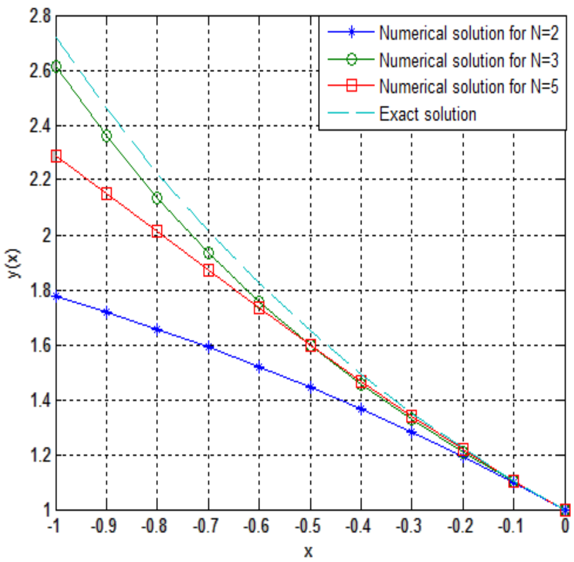

Figure 1. Exact and numerical solutions of Example 2 for N=2, 3 and 5

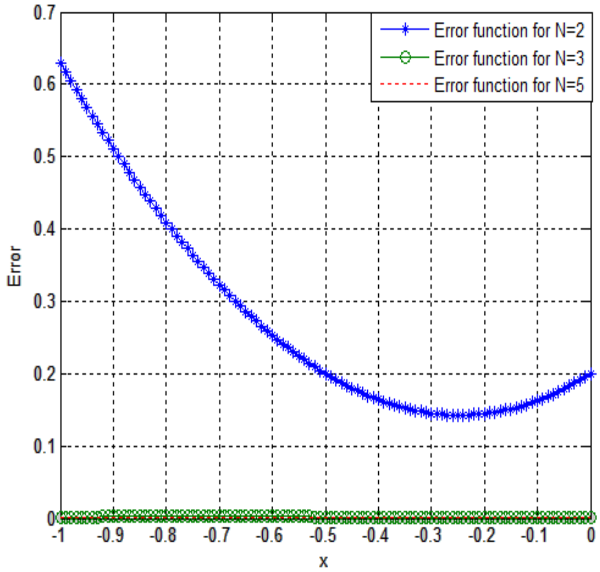

Figure 2. Residual error functions of Example 2 for N=2, 3 and 5.

Example 3. Let us consider the third order linear delay differential equation with variable coefficients

$$y'''(x) - xy''(x) + y'(x) + y\left(\frac{x}{2}\right) = x\cos(2x) + \cos\left(\frac{x}{2}\right) \quad ,0 \le x \le 1$$

under the initial conditions

$$y(0) = 1\,, \qquad y'(0) = 0\,, \qquad y''(0) = -1.$$

The exact solution of the problem is $y(x) = \cos x$. Here the coefficients of the equation are

$$m = 3\,, P_0 = 0\,, P_1 = 1\,, P_2 = 0, P_3 = 1\,, F_{00} = 2\,, \alpha_{00} = 1\,, \beta_{00} = -1$$

$$F_{20} = -x, \alpha_{20} = 2\,, \beta_{20} = 0\,, \quad g(x) = x\cos(2x) + \cos(\frac{x}{2}).$$

Now, we look for the Pell-Lucas solution in the form

$$y_3(x) = \sum_{n=0}^{3} \boldsymbol{a}_n \boldsymbol{Q}_n(x)$$

and obtain the collocation points for

$$N = 3\,\{x_0 = 0\,,\ x_1 = \tfrac{1}{3}\,,\ x_2 = \tfrac{2}{3}\,, x_3 = 1\}.$$

The fundamental matrix equation of the problem, Eq. (2.9), can be written as

$$\left\{\begin{array}{l} P_0XB^0 + P_1XB^1 + P_2XB^2 + P_3XB^3 \\ +F_{00}XB_{00}\left(\frac{1}{2},0\right)B^0 + F_{20}XB(2,0)B^2 \end{array}\right\} MA = G$$

where

$$W = P_0XB^0M + P_1XB^1M + P_2XB^2M$$
$$+P_3XB^3M + F_{00}XB_{00}\left(\tfrac{1}{2},0\right)B^0M$$
$$+F_{20}XB(2,0)B^2M.$$

Substituting the numerical values yields

$$[W]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 7/3 & 19/9 & 1270/27 \\ 2 & 8/3 & 22/9 & 656/27 \\ 2 & 3 & 3 & -14 \end{bmatrix}.$$

The matrix forms for the initial conditions in are

$$[U_0;\lambda_0]=[2 \quad 0 \quad 2 \quad 0 \quad ; \quad 1]$$
$$[U_1;\lambda_1]=[0 \quad 2 \quad 0 \quad 6 \quad ; \quad 0]$$
$$[U_2;\lambda_2]=[0 \quad 0 \quad 4 \quad 0 \quad ; \quad -1].$$

Thus, the new matrix equation based on the conditions are obtained as

$$\left[\tilde{W} \quad ; \quad \tilde{G}\right]=\begin{bmatrix} 2 & 0 & 2 & 0 & ; & 1 \\ 2 & 7/3 & 19/3 & 1270/27 & ; & 1.24811 \\ 0 & 2 & 0 & 6 & ; & 0 \\ 0 & 0 & 4 & 0 & ; & -1 \end{bmatrix}.$$

By solving this system, the Pell-Lucas coefficients matrix is determined as

$$A=[0.75 \quad -0.026722 \quad -0.25 \quad 0.006891]^T$$

and the approximate the solution becomes:

$$y_3(x) = 1 - x^2 + 5.51256x10^{-2}x^3.$$

Similarly we find other solutions, for N=4 and N=5;

$$y_4(x) = 1 - x^2 + 7.33548x10^{-2}x^3 + 3.895632x10^{-3}x^4$$

$$y_5(x) = 1 - x^2 + 5.6289x10^{-2}x^4 - 8.7786x10^{-3}x^5.$$

The residual errors for N=2, 3 and 5:

$$\overline{R_3} = \int_0^1 \frac{|R_2(x)|\,dx}{|1-0|} = 9.9656x10^{-2}$$

$$\overline{R_4} = \int_0^1 \frac{|R_3(x)|\,dx}{|1-0|} = 1.18579x10^{-2}$$

$$\overline{R_5} = \int_0^1 \frac{|R_5(x)|\,dx}{|1-0|} = 2.32283x10^{-2}$$

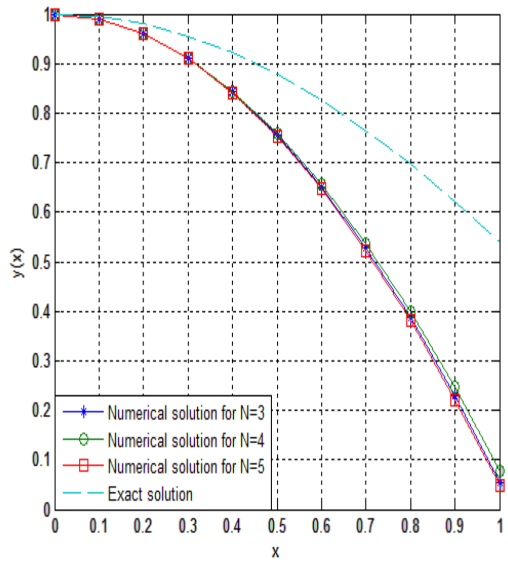

Figure 3. Comparisons of the exact and the numerical solutions of Example 3 for different N values

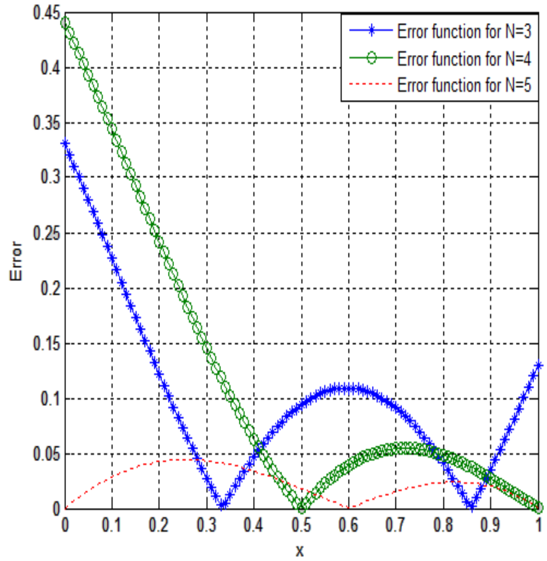

Figure 4. Residual error functions of Example 3 for N=2, 3 and 5

Consequently, the results related with exact solution, approximate solution and residual error obtained by our method for Example 3 are demonstrated in Table 2, Figure 3 and Figure 4.

Table 1. Numerical results of the error function of Example 2 for N=2, 3 and 5

x_i	Exact solution	N=2	$\lvert e_2(x_i)\rvert$	N=3	$\lvert e_3(x_i)\rvert$	N=5	$\lvert e_5(x_i)\rvert$
-1.0	2.71828	3.1976	$4.73e-01$	2.61237	$1.05e-01$	2.7166	$164\,e-03$
-0.9	2.4596	2.87006	$4.11e-01$	2.36046	$9.91e-02$	2.4581	$1.51e-03$
-0.8	2.22554	2.56646	$3.41e-01$	2.13572	$8.98e-02$	2.2241	$1.35e-03$
-0.7	2.01375	2.28682	$2.73e-01$	1.93553	$7.82e-02$	2.01253	$1.22e-03$
-0.6	1.82212	2.03114	$2.09e-01$	1.75724	$6.48e-02$	1.82104	$1.07e-03$
-0.5	1.64872	1.7994	$1.51e-01$	1.59822	$5.05e-02$	1.64781	$9.14e-04$
-0.4	1.49182	1.59162	$9.97e-02$	1.45583	$3.59e-02$	1.4911	$7.28e-04$
-0.3	1.34986	1.40778	$5.79e-02$	1.32745	$2.24e-02$	1.43493	$5.15e-04$
-0.2	1.22143	1.2479	$2.65e-02$	1.21044	$1.09e-02$	1.22111	$2.88e-04$
-0.1	1.10517	1.11198	$6.81e-03$	1.10217	$2.99e-03$	1.10508	$9.16e-05$
0.0	1.0	1.0	0	1.0	0	1.0	0

Table 2. Numerical results of the error function of Example 3 for N=2, 3 and 5

x_i	Exact solution	N=3	$\lvert e_3(x_i)\rvert$	N=4	$\lvert e_4(x_i)\rvert$	N=5	$\lvert e_5(x_i)\rvert$
0.0	1.0	1.0	0.0	1.0	0.0	1.0	0.0
0.1	0.99501	$9.91e-01$	$4.94e-03$	$9.91e-01$	$4.93e-03$	$9.91e-01$	$4.99e-03$
0.2	0.98007	$9.60e-01$	$1.96e-02$	$9.61e-01$	$1.94e-02$	$9.60e-01$	$1.99e-02$
0.3	0.95534	$9.11e-01$	$4.38e-02$	$9.12e-01$	$4.33e-02$	$9.10e-01$	$4.49e-02$
0.4	0.92106	$8.43e-01$	$7.75e-02$	$8.44e-01$	$7.62e-02$	$8.41e-01$	$7.97e-02$
0.5	0.8 758	$7.56e-01$	$1.20e-01$	$7.59e-01$	$1.18e-02$	$7.53e-01$	$1.24e-02$
0.6	0.82533	$6.51e-01$	$1.73e-01$	$6.56e-01$	$1.68e-02$	$6.46e-01$	$1.78e-02$
0.7	0.76484	$2.89e-01$	$2.35e-01$	$5.36e-01$	$2.28e-02$	$5.22e-01$	$2.42e-02$
0.8	0.69670	$3.88e-01$	$3.08e-01$	$3.95e-01$	$2.97e-02$	$3.80e-01$	$3.16e-02$
0.9	0.62161	$2.31e-01$	$3.91e-01$	$2.46e-01$	$3.75e-02$	$2.21e-01$	$3.99e-02$
1.0	0.54030	$5.51e-02$	$4.85e-01$	$7.72e-02$	$4.63e-02$	$4.75e-02$	$4.92e-02$

4. Conclusion

A new matrix method based on Pell-Lucas polynomials and collocation points is proposed to solve the high-order linear functional differential equations with hybrid delays under mixed conditions. An error analysis based on residual function is carried out to show the accuracy of the results. It is observed from the tables and figures that the error estimations are very effective. When the exact solution of the problem is not known, the error of the solution can be approximately computed by means of this residual function. In addition, we compared the numerical values of the approximate solutions obtained by the method in tables and figures. Obviously the results of the present method have been compared with the different values of N. It is also clearly seen that the Pell-Lucas matrix collocation method is more convenient to apply to linear and nonlinear integro-differential equations. However, some regularizations are required.

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