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McCoy Rings and Matrix Rings with McCoy 0-Multiplication

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Abstract

In this study, we consider a construction of subrings with McCoy 0-multiplication of matrix rings of McCoy rings which is a unifed generalization of the ring $R[x]/(x^n)$, where $n \geq 0$. One objective is to extend the various known results to this new extension from the rings such as $R[x]/(x^n)$, Hurwitz extension $H(R)$.

Keywords: Armendariz Ring, McCoy Ring, Simple 0-multiplication, McCoy 0-multiplication.

McCoy Halkaları ve McCoy-0 Çarpımlı Matris Halkaları

Özet

Bu çalışmada, $n \geq 0$ için $R[x]/(x^n)$, halkasının bir genellemesi olan McCoy halkalarının matris halkalarının McCoy 0-Çarpımlı alt halkalarını ele aldık. Bu doğrultuda, $R[x]/(x^n)$, $H(R)$ Hurwitz genişlemeleri gibi halkalardaki bilinen bazı sonuçları bu yeni genişlemeye aktarmayı amaçladık.

Anahtar Kelimeler: Armendariz Halka, McCoy Halka, Basit 0-çarpım, McCoy 0-çarpım.

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1. Introduction

Throughout this paper, we will assume that R is an associative ring with nonzero identity and the polynomial ring over R is denoted by $R[x]$ with x its indeterminate. For notation, $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$ denote the $n \times n$ full matrix ring over R and full upper triangular matrix ring over R, respectively.

In 1942, McCoy observed that if R is a commutative ring, then whenever $g(x)$ is a zero divisor in $R[x]$, there exists a nonzero element $c \in R$ such that $cg(x) = 0$ (see [10, Theorem 2]). But it is only in 2006 when Nielsen [11] started a systematic study of McCoy rings. According to Nielsen, a ring R is said to be right McCoy, when the equation $f(x)g(x) = 0$ over $R[x]$, where $f(x), g(x) \neq 0$, implies that there exists a nonzero element $c \in R$ with $f(x)c = 0$. The definition of left McCoy ring is similar. If R is both a left and right McCoy, then R is called a McCoy ring. In the literature, there are several different studies on this topic. For instance, among other interesting manuscripts and results, it is shown in [6, Theorem 2.8] that R is a right McCoy ring if and only if $R[x]$ is a right McCoy ring and if R is a right McCoy ring then $R[x]/(x^n)$ is a right McCoy ring where $n \geq 2$ is a positive integer. This implies that R is a right McCoy ring if and only if the trivial extension $T(R,R)$ is a right McCoy ring.

Let R be a domain (commutative or not) and $R[x]$ its polynomial ring. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be two elements of $R[x]$. It is easy to see that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j, since $f(x) = 0$ or $g(x) = 0$. Armendariz [2] noted that the above result can be extended the class of reduced rings. Note that a ring R is called reduced if it has no nonzero nilpotent elements. A ring R is symmetric if $a_1 a_2 \cdots a_n = 0$, then $a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} = 0$, for all $n \in \mathbb{N}$, $a_i \in R$ and $\sigma \in S_n$. Note that reduced rings are symmetric. A ring R is said to be Armendariz if $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j (see [1]). Anderson and Camillo [1], showed that R is an Armendariz ring if and only if $R[x]$ is an Armendariz ring. In [8, Corollary 1.5], Lee and Zhou showed that R is a reduced ring if and only if $R[x]/(x^n)$ is an Armendariz ring. It is well known that $R[x]/(x^n)$ is isomorphic to the subring $S_n(R)$ of the ring $\mathbb{T}_n(R)$ over R consisting of matrices of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

Since $\mathbb{T}_n(R)$ is not an Armendariz ring by [5, Example 3], Lee and Zhou studied many specific Armendariz subrings of $\mathbb{T}_n(R)$ in [8]. This was a starting point of the notion of simple 0-multiplication. According to Wang, Puczylowski and Li [13], a subring S of the ring $\mathbb{M}_n(R)$ of $n \times n$ matrices over R is with simple 0-multiplication if for arbitrary $(a_{ij}), (b_{ij}) \in S$ satisfying $(a_{ij})(b_{ij}) = 0$ implies that $a_{il}b_{lj} = 0$ for arbitrary $1 \leq i, j, l \leq n$.

In the present paper, we define the ring with McCoy 0-multiplication as follows: a subring S of the ring $\mathbb{M}_n(R)$ is with McCoy 0-multiplication if for arbitrary $(a_{ij}) \in S$ and $(b_{ij}) \in S\backslash\{0\}$ such that $(a_{ij})(b_{ij}) = 0$ implies that for arbitrary $1 \leq i, j \leq n$ there exists $0 \neq c \in R$ with $a_{ij}c = 0$. We give many descriptions of subrings with McCoy 0-multiplication and McCoy subrings of matrix rings.

In Section 2, we gave several properties of this new notion. For many subrings, if R is a reduced ring, then $S_4(R)$ is a ring with McCoy 0-multiplication (see Theorem 2.10). We also prove that if a subring S_1 of $\mathbb{M}_n(R_1)$ and a subring S_2 of $\mathbb{M}_n(R_2)$ are rings with McCoy 0-multiplication, then the subring $S_1 \times S_2$ of $\mathbb{M}_n(R_1 \times R_2)$ is a ring with McCoy 0-multiplication (see Theorem 2.6). Sequentially, we will argue the property McCoy 0-multiplication of some kinds of ring extensions.

2. Results

We start this section with an example showing that the definition below, which is main focus of the paper, is not meaningless.

Example 2.1 Let R be a McCoy ring, $f(x)=\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}x$ and $g(x)=\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&-1\end{pmatrix}x$ be two elements of $\mathbb{M}_2(R)[x]$. Clearly $f(x)g(x)=0$. But, there is only one element $C=(0)\in\mathbb{M}_2(R)$ such that $f(x)C=0$.

Example 2.2 Let R be a McCoy ring and $S=\left\{\begin{pmatrix}a&b&c\\0&d&e\\0&0&f\end{pmatrix}:a,b,c,d,e,f\in R\right\}$. Now we consider the elements $f(x)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}x\in S[X]$ and $g(x)=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&-1&0\\0&0&0\end{pmatrix}x\in S[X]$. A simple computation gives that $f(x)g(x)=0$. Taking $C=\begin{pmatrix}0&0&0\\0&-1&-1\\0&0&0\end{pmatrix}\in S$ gives $f(x)C=0$.

By this vein, we can mention the following defition.

Definition 2.3 The subring S of the ring $\mathbb{M}_n(R)$ of $n\times n$ matrices over R is with McCoy 0-multiplication if for arbitrary $(a_{ij})\in S$ and $(b_{ij})\in S\backslash\{0\}$, $(a_{ij})(b_{ij})=0$ implies that for arbitrary $1\le i,j\le n$, there exists a nonzero element $c\in R$ such that $a_{ij}c=0$.

Example 2.4 Let R be a ring. Then

$$S=\left\{\begin{pmatrix}a&0&0&\cdots&0\\0&a&0&\cdots&0\\0&0&a&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&a\end{pmatrix}:a\in R\right\}\subseteq\mathbb{M}_n(R)$$

is a subring of $\mathbb{M}_n(R)$ with McCoy 0-multiplication.

We denote the set of all nilpotent elements of R by $nil(R)$. Note that a ring R is semicommutative if $ab=0$ implies $aRb=0$.

Theorem 2.5 If R is semicommutative, then the subring

$$S=\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}: a\in R, b\in nil(R)\right\}$$

of $\mathbb{M}_2(R)$ is a subring of $\mathbb{T}_2(R)$ with McCoy 0-multiplicationin case R is a domain.

Proof. Since R is semicommutative, $nil(R)$ is an ideal of R by [9, Lemma 3.1]. So, S is a subring of $\mathbb{T}_2(R)$. Let $A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B=\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\in S$ with $AB=0$. We may assume that both A and B are nonzero. Then we have

$$ac=0, ad+bc=0. \qquad (2.0)$$

If $a=0$, then $b\neq 0$ and $b^m=0$ for some minimal integer m. Let $r=b^{m-1}(\neq 0)$. Then $a_{ij}r=0$ for $1\leq i,j\leq 2$. If $b=0$, then both c and d annihilate a on the right by (2.0).

Next we suppose that $a\neq 0, b\neq 0$.

If $d=0$, then $c\neq 0$ and $ac=bc=0$. If $c=0$, then $d\neq 0$ and $ad=0$ by (2.0). In this case, if $bd=0$, then we are done, otherwise (i.e., $bd\neq 0$), since $b\in nil(R)$, there exists an integer n such that $b^n d=0$ but $b^{n-1}d\neq 0$. Take $r=b^{n-1}d$. Then $br=0$ and $ar=ab^{n-1}d=0$ since R is semicommutative and $ad=0$. Thus, we now assume that all of a, b, c, d are nonzero.

If $ad=0$, then $bc=0$ by (2.0). Thus $ac=bc=0$. So, we only need to check the case that $ad\neq 0$. Assume that $ad\neq 0$. Then $bc\neq 0$. Since $b\in nil(R)$, there exists an integer k such that $b^k c=0$ but $b^{k-1}c\neq 0$ Take $r=b^{k-1}c\neq 0$. Then $ar=a(b^{k-1})c=0$ by the semicommutativity of R and $br=b^k c=0$. The proof is now complete.

Theorem 2.6 If a subring S_1 of $\mathbb{M}_n(R_1)$ and a subring S_2 of $\mathbb{M}_n(R_2)$ are rings with McCoy 0-multiplication, then the subring $S_1 \times S_2$ of $\mathbb{M}_n(R_1 \times R_2)$ is a ring with McCoy 0-multiplication.

Proof. Let $(A_{ij}) = ((a_{ij}, b_{ij}))$ and $(B_{ij}) = ((c_{ij}, d_{ij})) \in \mathbb{M}_n(R_1 \times R_2)$ such that $(A_{ij})(B_{ij}) = 0$ for all $1 \leq i, j \leq n$. Then

$$
\begin{aligned}
(A_{ij}) &= \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & (a_{13}, b_{13}) & \cdots & (a_{1n}, b_{1n}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & (a_{23}, b_{23}) & \cdots & (a_{2n}, b_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_{n1}, b_{n1}) & (a_{n2}, b_{n2}) & (a_{n3}, b_{n3}) & \cdots & (a_{nn}, b_{nn}) \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}, 0) & \cdots & (a_{1n}, 0) \\ (a_{21}, 0) & \cdots & (a_{2n}, 0) \\ \vdots & \vdots & \vdots \\ (a_{n1}, 0) & \cdots & (a_{nn}, 0) \end{pmatrix} + \begin{pmatrix} (0, b_{11}) & \cdots & (0, b_{1n}) \\ (0, b_{21}) & \cdots & (0, b_{2n}) \\ \vdots & \vdots & \vdots \\ (0, b_{n1}) & \cdots & (0, b_{nn}) \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
(B_{ij}) &= \begin{pmatrix} (c_{11}, d_{11}) & (c_{12}, d_{12}) & (c_{13}, d_{13}) & \cdots & (c_{1n}, d_{1n}) \\ (c_{21}, d_{21}) & (c_{22}, d_{22}) & (c_{23}, d_{23}) & \cdots & (c_{2n}, d_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (c_{n1}, d_{n1}) & (c_{n2}, d_{n2}) & (c_{n3}, d_{n3}) & \cdots & (c_{nn}, d_{nn}) \end{pmatrix} \\
&= \begin{pmatrix} (c_{11}, 0) & \cdots & (c_{1n}, 0) \\ (c_{21}, 0) & \cdots & (c_{2n}, 0) \\ \vdots & \vdots & \vdots \\ (c_{n1}, 0) & \cdots & (c_{nn}, 0) \end{pmatrix} + \begin{pmatrix} (0, d_{11}) & \cdots & (0, d_{1n}) \\ (0, d_{21}) & \cdots & (0, d_{2n}) \\ \vdots & \vdots & \vdots \\ (0, d_{n1}) & \cdots & (0, d_{nn}) \end{pmatrix}
\end{aligned}
$$

Set $(a_{ij})' = ((a_{ij}, 0))$, $(b_{ij})' = ((0, b_{ij}))$, $(c_{ij})' = ((c_{ij}, 0))$ and $(d_{ij})' = ((0, d_{ij}))$, for every $1 \leq i, j \leq n$. Then $(a_{ij})'(d_{ij})' = 0 = (b_{ij})'(c_{ij})'$ for $1 \leq i, j \leq n$.

If $(a_{ij})' = 0$, then $(0, b_{ij})(1,0) = (0,0)$ for $1 \leq i, j \leq n$.

If $(b_{ij})' = 0$, then $(a_{ij}, 0)(0,1) = (0,0)$ for $1 \leq i, j \leq n$.

Now assume that $(a_{ij})' \neq 0$ and $(b_{ij})' \neq 0$. Since $B_{ij} \neq (0)$, we have $(c_{ij})' \neq 0$ or $(d_{ij})' \neq 0$ for $1 \leq i, j \leq n$.

If $(c_{ij})' \neq 0$, then $c_{ij} \neq 0$ for some i, j. So, there exists a nonzero $u_1 \in R_1$ for $1 \leq i, j \leq n$ such that $a_{ij}u_1 = 0$. Hence $(a_{ij}, 0)(u_1, 0) = 0$.

If $(d_{ij})' \neq 0$, then $d_{ij} \neq 0$ for some i, j. So, there exists a nonzero $u_2 \in R_2$ for $1 \leq i, j \leq n$ such that $b_{ij}u_2 = 0$. Hence $(0, b_{ij})(0, u_2) = 0$.

Corollary 2.7 If $\mathbb{M}_n(R_1)$ and $\mathbb{M}_n(R_2)$ are rings with McCoy 0-multiplication, then $\mathbb{M}_n(R_1 \times R_2)$ is also a ring with McCoy 0-multiplication.

By the same notation of authors in [13], ϕ denotes the canonical isomorphism of $\mathbb{M}_n(R)[x]$ onto $\mathbb{M}_n(R[x])$. It is given by

$$\phi(A_0 + A_1x + \cdots + A_mx^m) = (f_{ij}(x)),$$

where

$$f_{ij}(x) = (a_{ij}^{(0)} + a_{ij}^{(1)}x + \cdots + a_{ij}^{(m)}x^m)$$

and $a_{ij}^{(k)}$ denotes the (i, j)-entry of A_k. In what follows E_{ij} will denote the usual matrix unit.

According to Nielsen and Camillo [12], a ring R is said to be right linearly McCoy if given nonzero linear polynomials $f(x), g(x) \in R[X]$ with $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ with $f(x)r = 0$.

Theorem 2.8 Let R be an integral domain.

(1) If a subring S of $\mathbb{M}_n(R)$ is a ring with McCoy 0-multiplication, then S is a linearly McCoy ring.

(2) If for a subring S of $\mathbb{M}_n(R)$, $\phi(S[X])$ is a subring of $\mathbb{M}_n(R[X])$ with McCoy 0-multiplication, then S is a McCoy ring.

Proof. (1) Assume that $(A_0+A_1x)(B_0+B_1x)=0$, where A_i, B_i's are nonzero matrices in S. Then $A_0B_0=0$ and $A_1B_1=0$. Since S is a ring with McCoy 0-multiplication, we have $0\neq c_0, c_1\in R$ such that $a_{ij}^{(0)}c_0=0$ and $a_{ij}^{(1)}c_1=0$. Set $c_2=c_0c_1$. Since R is symmetric, we have $(a_{ij}^{(0)}+a_{ij}^{(1)}x)c_2=0$. Consequently, if we choose $0\neq C=(c_{ij})$ where $(c_{ij})=c_2$ for $1\leq i,j\leq n$, then we get $(A_0+A_1x)C=0$.

(2) Suppose that $A_i, B_j\in S$ for $0\leq i,j\leq m$. Let $f(x)=A_0+A_1x+\cdots+A_mx^m$ and $g(x)=B_0+B_1x+\cdots+B_mx^m$ be two elements in $\mathbb{M}_n(R[X])$ and $f(x)g(x)=0$. Then $\phi(f(x))\phi(g(x))=0$ and since $\phi(S[X])$ is with McCoy 0-multiplication, we get $0\neq C\in R[X]$ such that $f_{ij}(x)C=0$. We know that R is a McCoy ring so we have $0\neq k_{ij}\in R$ such that $f_{ij}(x)k_{ij}=0$. If we choose $a_{ij}=\prod_{i,j=1}^{m}k_{ij}$ and $C'=(a_{ij})$, then we get $f(x)C'=0$.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R,M)=R\oplus M$ with the usual addition and multiplication

$$(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2).$$

This is the subring $\left\{\begin{pmatrix} a & m\\ 0 & a\end{pmatrix}: a\in R, m\in M\right\}$ of the formal triangular matrix ring $\begin{pmatrix} R & M\\ 0 & R\end{pmatrix}$.

Let R be a domain. Recall that, an R-module M is called torsion, if $t(M)=M$, where $t(M)=\{x\in M: l_R(x)\neq 0\}$ (see [14]). Let $S[x]$ and $R[x]$ be the polynomial rings over rings S and R, respectively. Given a module M, let $M[x]$ be the set of all formal polynomials with indeterminate x and with coefficients from M. Then $M[x]$ becomes an $(S[x],R[x])$-bimodule under usual addition and multiplication of polynomials. Assume that M is an R-module such that $ma=0$ implies $mRa=0$, for any $m\in M$ and $a\in R$. In [3, Proposition 2.5], it is proved that, if $m'(x)$ is a torsion element in $M[x]$, then there exists a nonzero element $c\in R$ such that $m'(x)c=0$. An R-module M is called a

McCoy module if $m(x)=\sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x)=\sum_{j=0}^{s} a_j x^j \in R[x]$, $m(x)f(x)=0$ implies $m(x)c=0$ for some nonzero $c \in R$.

Theorem 2.9 Let M be an (R,R)-bimodule and R a domain such that M is torsion as a right R-module. If for any $h(x) \in R[x]$ and $n(x) \in M[x]\backslash\{0\}$, $h(x)=0$ whenever $h(x)n(x)=0$, then the trivial extension $T(R[X],M[X])$ is a ring with McCoy 0-multiplication.

Proof. Let

$$f(x)=a_0+a_1x+\cdots+a_nx^n,\ g(x)=b_0+b_1x+\cdots+b_sx^s,$$

$$m(x)=m_0+m_1x+\cdots+m_nx^n, l(x)=l_0+l_1x+\cdots+l_sx^s.$$

Then $f(x), g(x) \in R[X]$ and $m(x), l(x) \in M[X]$. Suppose that

$$\begin{pmatrix} f(x) & m(x) \\ 0 & f(x) \end{pmatrix}\begin{pmatrix} g(x) & l(x) \\ 0 & g(x) \end{pmatrix}=0,$$

where at least one of $g(x)$ and $l(x)$ is nonzero. Then $f(x)g(x)=0$ and $f(x)l(x)+m(x)g(x)=0$. Since $R[x]$ is a domain, one of $f(x)$ and $g(x)$ is equal to 0. So, we have $f(x)l(x)=m(x)g(x)=0$. Next we separate the proof into two cases:

Case 1: Let $m(x)=0$. We shall show that $f(x)=0$. If $f(x)\neq 0$, then $g(x)=0$ since $f(x)g(x)=0$ and $R[x]$ is a domain. By assumption, we have $l(x)\neq 0$. Now we can obtain that $f(x)l(x)=0$. This is a contradiction.

Case 2: Let $m(x)\neq 0$, we conclude that $f(x)=0$. Otherwise, if $f(x)\neq 0$, then $g(x)=0$. So $l(x)\neq 0$ and $f(x)l(x)=0$ which contradicts the hypotesis. Thus we have $f(x)=0$. Since M is a torsion right R-module, there exists a nonzero $c_i \in R$ such that $m_i c_i=0$, where $i=0,1,2,\ldots,n$. Let $c=c_0,\ldots,c_n$. Then $c\neq 0$ and $m(x)c=0$.

Let R be a ring. We consider the following subrings of $\mathbb{T}_n(R)$ for any $n\geq 2$:

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\},$$

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} : a_i \in R \right\}$$

Let $m \leq n$ be positive integers and $S_{n,m}(R)$ the set of all $n \times n$ matrices (a_{ij}) with entries in a ring R such that

(a) For $i > j$, $a_{ij} = 0$,

(b) For $i \leq j$, $a_{ij} = a_{kl}$ when $i - k = j - l$ and either $1 \leq i, j, k, l \leq m$ or $m \leq j, k, l \neq n$. Clearly, $S_{n,1}(R) = S_{n,n}(R) = T(R,n)$.

By [6, Example 2.3], $S_3(R)$ is a right McCoy ring if R is a reduced ring and R is a right McCoy ring if and only if $S_4(R)$ is a right McCoy ring by [6, Theorem 2.5]. One may suspect that, if R is a reduced ring, then the subring $S_{n,m}(R)$ of $\mathbb{T}_n(R)$ is a ring with McCoy 0-multiplication. In particular, $S_{n,m}(R)$ is a right McCoy ring. But this is not true. Let R be any ring and $A = B = e_{1n}$, where e_{1n} is a matrix unit (with 1 in $(1, n)$-th entry and 0 elsewhere). Then $AB = 0$, but non of nonzero element in R annihilates $a_{1n} = 1$.

Theorem 2.10 If R is a commutative reduced ring, then the subring $S_4(R)$ of $\mathbb{T}_n(R)$ is a ring with McCoy 0-multiplication.

Proof. Let $A = \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & b_{12} & b_{13} & b_{14} \\ 0 & b & b_{23} & b_{24} \\ 0 & 0 & b & b_{34} \\ 0 & 0 & 0 & b \end{pmatrix}$ with $AB = 0$. Then we get

$$ab = 0, \tag{2.1}$$

$$ab_{12} + a_{12}b = 0, \tag{2.2}$$

$$ab_{13} + a_{12}b_{23} + a_{13}b = 0, \tag{2.3}$$

$$ab_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b = 0, \tag{2.4}$$

$$ab_{23} + a_{23}b = 0, \tag{2.5}$$

$$ab_{24} + a_{23}b_{34} + a_{24}b = 0, \tag{2.6}$$

$$ab_{34} + a_{34}b = 0. \tag{2.7}$$

As R is reduced, multiplying equation (2.2), (2.5) and (2.7) on the left by b gives $1a_{12}b^2 = 0$, $a_{23}b^2 = 0$ and $a_{34}b^2 = 0$. Similarly, multiplying equation (2.3) and (2.6) on the left by b^2 gives $a_{13}b^3 = 0$ and $a_{24}b^3 = 0$. Finally, multiplying equation (2.4) on the left by b^3 gives $a_{14}b^4 = 0$. If we choose $c = b^4$, then we see $S_n(R)$ is a ring with McCoy 0-multiplication.

Let R be a ring. We denote $H(R)$ the ring of Hurwitz series over R which is defined as follows. The elements of $H(R)$ are sequences of the form $a = (a_n) = (a_0, a_1, \dots)$, where $a_n \in R$ for each $n \in \mathbb{N}$. An element in $H(R)$ can be thought as a function from $\mathbb{N}$ to R.

Two elements (a_n) and (b_n) in $H(R)$ are equal if they are equal as functions from $\mathbb{N}$ to R, i.e., if $a_n = b_n$ for all $n \in \mathbb{N}$. The element $a_m \in R$ will be called the mth term of (a_n). Addition in $H(R)$ is defined termwise, such that $(a_n) + (b_n) = (c_n)$, where $c_n = a_n + b_n$ for all $n \in \mathbb{N}$.

If one identifies a formal power series $\sum_{i=0}^{\infty} a_n x^n \in R[[x]]$ with the sequence of its coefficients (a_n), then multiplication in $H(R)$ is similar to the usual product of formal power series, except that binomial coefficients are introduced at each term in the product as follows by [4]. The (Hurwitz) product of (a_n) and (b_n) is given by $(a_n)(b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} C_k^n a_k b_{k-n}.$$

Hence

$$(a_0, a_1, a_2, a_3, \dots)(b_0, b_1, b_2, b_3, \dots) =$$

$$(a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + 2a_1b_1 + a_2b_0, a_0b_3 + 3a_1b_2 + 3a_2b_1 + a_3b_0, \dots).$$

Set

$$H(R,n) = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} : a_i \in R \text{for} 0 \le i \le n \right\}.$$

We can identify $H(R,n)$ with the set

$$\{(a_0, a_1, \dots, a_n) : a_i \in R \text{for} 0 \le i \le n\}.$$

Then $H(R,n)$ is a ring, with addition defined componentwise and multiplication given by

$$(a_0, a_1, \dots, a_n)(b_0, b_1, \dots, b_n) = (c_0, c_1, \dots, c_n)\ ,$$

$$c_0 = a_0b_0\ ,$$

$$c_m = \sum_{k=o}^{m} C_k^m a_k b_{k-m}\ , \text{ where } 1 \le m \le n.$$

Theorem 2.11 Let R be a commutative reduced ring. Then the subring $H(R,n)$ of $\mathbb{T}_n(R)$is a ring with McCoy 0-multiplication.

Proof. Let $(a_0, a_1, \dots, a_n)$ and $(b_0, b_1, \dots, b_n) \in H(R,n)$ and assume

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix}\begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_0 \end{pmatrix} = 0.$$

We clearly get

$$a_0 b_0 = 0,$$

$$a_0 b_1 + a_1 b_0 = 0,$$

$$C_0^2 a_0 b_2 + C_1^2 a_1 b_1 + C_2^2 a_2 b_0 = 0,$$

$$C_0^3 a_0 b_3 + C_1^3 a_1 b_2 + C_2^3 a_2 b_1 + C_3^3 a_3 b_0 = 0,$$

$$\vdots$$

$$C_0^n a_0 b_n + C_1^n a_1 b_{n-1} + \cdots + C_{n-1}^n a_{n-1} b_1 + C_n^n a_n b_0 = 0.$$

Multiplying the second equation by b_0 and the third one by b_0^2 gives $a_1 b_0^2 = 0$ and $a_2 b_0^3 = 0$, respectively, since R is reduced. After proceeding like this, clearly we can see $a_k b_0^{k+1} = 0$ for all $0 \leq k \leq n$. So, if we choose $0 \neq c = b_0^{n+1} \in R$, then obviously $a_i c = 0$, for all $0 \leq i \leq n$. Hence $H(R, N)$ is a subring of $\mathbb{T}_n(R)$ with McCoy 0-multiplication.

We consider the following ring extension of R with an ideal I:

$$S = (R, I)[x]/(x^{n+1}) = \{\textstyle\sum_{i=0}^{n} a_i x^i, a_0 \in R, a_i \in I, i = 1, \ldots, n\}.$$

We can identify S with

$$S = \left\{\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ 0 & a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} : a_0 \in R, a_i \in I, i \geq 1\right\}.$$

Theorem 2.12 If a commutative ring R is reduced, then the subring S of $\mathbb{T}_n(R)$ is a ring with McCoy 0-multiplication.

Proof. We prove the theorem for 3×3 matrix and other cases can be done similarly. Let

$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{pmatrix}$, $B = \begin{pmatrix} b_0 & b_1 & b_2 \\ 0 & b_0 & b_1 \\ 0 & 0 & b_0 \end{pmatrix} \in S$. Assume that $AB = 0$. Then we get

$$a_0 b_0 = 0,$$

$$a_0 b_1 + a_1 b_0 = 0,$$

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0.$$

Since R is reduced, easily we can write $a_1 b_0^2 = 0$ and $a_2 b_0^3 = 0$. If we choose $0 \neq c = b_0^3 \in R$, then we see that S is a ring with McCoy 0-multiplication.

According to Krempa [7], an endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$ for any $a \in R$. A ring R is a σ-rigid ring if there exists a rigid endomorphism σ of R.

Corollary 2.13 If R is a σ-rigid ring, then $S[x;\sigma]/(x^{n+1})$ is a ring with McCoy 0-multiplication where S is a subring of $M_n(R)$.

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