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TOTALLY REDUCIBLE FOCAL SET WITH STEREOGRAPHIC PROJECTION

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Abstract

In [1], Carter and the author introduced the idea of an immersion $f: M \to R^n$ with totally reducible focal set (TRFS). Such an immersion has the property that, for all $p \in M$, the focal set with base p is a union of hyperplanes in the normal plane to $f(M)$ at $f(p)$. In this study, we show that the property of an immersion having TRFS is preserved under inverse image of stereographic projection.

1. Introduction

Let $f: M \to R^n$ be a smooth immersion of connected smooth m-dimensional manifold without boundary into Euclidean n-space. For each $p \in M$, the focal set of f with base p is an algebraic variety. In this study we consider immersions for which this variety is a union of hyperplanes. Trivially, this holds if $n = m+1$ so we only consider $n > m+1$.

For $p \in M$, let U be a neighborhood of p in M such that $f|_U : U \to R^n$ is an embedding. Let $\upsilon_f(p)$ denote the (n-m)-plane which is normal to $f(U)$ at $f(p)$. Then the total space of normal bundle is $N_f = \{ (p,x) \in M \times R^n \mid x \in \upsilon_f(p) \}$. The projection map $\eta_f : N_f \to R^n$ is defined by $\eta_f(p,x) = x$ and the set of focal points with base p is $\Gamma_f(p) = \{ p \in R^n \mid (p,x) \text{ is a singularity of } \eta_f \}$. The focal set of f, $\Gamma_f = \bigcup_{p \in M} \Gamma_f(p)$ is the image by η_f. For each $p \in M$, $\Gamma_f(p)$ is a real algebraic variety in $\upsilon_f(p)$ which can be defined as the zeros of polynomial on $\upsilon_f(p)$ of degree $\le m$. The focal point of f has weight (multiplicity) k if $\text{rank}(\text{jac}\, \eta_f) = n - k$ [3].

Definition 1. The immersion $f: M \to R^n$ has totally reducible focal set (TRFS) property if for all $p \in M$, $\Gamma_f(p)$ can be defined as the zeros of real polynomial which is a product of real linear factors [1].

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So, each irreducible component of $\Gamma_f(p)$ is an affine in $\upsilon_f(p)$, and $\Gamma_f(p)$ is a union of $(n-m-1)$-planes (possible $\Gamma_f(p)=\Phi$). There are other ways of describing this property, it is shown in ([2], [4], and [5]) that f has TRFS property if and only if f has flat normal bundle, where M is thought of as a Riemannian manifold with metric g induced from R^n. We will give explicit ways of constructing immersions with TRFS property.

In calculating focal sets it is often easiest to work with distance functions. For $x \in R^n$ the distance function $L_x : M \to R$ is defined by $L_x(p)=\|x-f(p)\|^2$. Then $x \in R^n$ is a focal point of f with base p if and only if p is a degenerate critical point of L_x, where at p, $\frac{\partial L_x}{\partial p_i}=0$ and $H=\left[\frac{\partial^2 L_x}{\partial p_i \partial p_j}\right]$ is singular for $i,j=1,2,\cdots,m$ [Mi].

Proposition 1. [1] If $f : M \to R^n$ has TRFS property and $g : M \to R^{n+k}$ is defined by $g(p)=(f(p),t) \in R^n x R^k$. Then g has TRFS property.

Proposition 2. [1] Let $f : M^m \to S^{m+1} \subset R^{m+2}$ be an immersion. Then f has TRFS property.

In this study it has been shown that TRFS property of an immersion, with arbitrary codimension is preserved under composition with stereographic projection, or the inverse image of stereographic projection.

2. Stereographic Projection

As we know that any immersion $f : M^m \to R^{m+1}$ has TRFS property. Let $\pi : S^{m+1} \backslash (0,\cdots,0,1) \to R^{m+1}$ be stereographic projection. Then the composition of an immersion $f : M^m \to R^{m+1}$ with the inverse of stereographic projection gives an immersion $\pi^{-1} \circ f : M^n \to S^{m+1} \subset R^{m+2}$ with TRFS property.

Let $f : M^m \to S^n \subset R^{n+1}$ be an immersion with arbitrary codimension and let $x \in R^{n+1}$ and $L_x(p) = \sum_{i=1}^{n+1} (x_i - f_i(p))^2$. We know that $\sum_{i=1}^{n+1} f_i^2 = 1$. Then

$$\sum_{i=1}^{n+1} f_i \frac{\partial f_i}{\partial p_r} = 0 . \tag{1}$$

$x \in \Gamma_f(p)$ if and only if

$$-\frac{1}{2}\frac{\partial L_x}{\partial p_r} = \sum_{i=1}^{n+1} x_i \frac{\partial f_i}{\partial p_r} = 0 , \quad r = 1,2,\cdots,m \tag{2}$$

and

$$\det(A_{rs}) = 0 , \tag{3}$$

where $A_{rs} = -\frac{1}{2}\frac{\partial^2 L_x}{\partial p_r \partial p_s} = \sum_{i=1}^{n+1} x_i \frac{\partial^2 f_i}{\partial p_r \partial p_s}$ (the weight of the focal point x is the number of eigenvalues of (A_{rs})).

Now, we give an immersion $g : M^m \to R^n$ is defined by

$$g(p) = \left(\frac{2f_1(p)}{1-f_{n+1}(p)}, \frac{2f_2(p)}{1-f_{n+1}(p)}, \cdots, \frac{2f_n(p)}{1-f_{n+1}(p)} \right)$$

where $f(p) = (f_1(p), f_2(p), \cdots, f_{n+1}(p))$ and $f_{n+1}(p) \neq 1$ for all $p \in M$. (i.e. g is obtained by composition f with a stereographic projection from S^n to R^n).

Since for $y \in R^n$ and $L_y(p) = \sum_{j=1}^{n} \left(y_i - \frac{2f_j(p)}{1-f_{n+1}}\right)^2$

$y \in \Gamma_g(p)$ if and only if

$$-\frac{1}{4}\frac{\partial L_y}{\partial p_r} = \frac{1}{(1-f_{n+1})}\left(\sum_{j=1}^{n} y_j \frac{\partial f_j}{\partial p_r} + \frac{\left(\sum_{j=1}^{n} y_j f_j\right) - 2}{(1-f_{n+1})} \frac{\partial f_{n+1}}{\partial p_r} \right) = 0 , \tag{4}$$

and

$$\det(B_{rs}) = 0 , \tag{5}$$

where $B_{r,s} = -\frac{1}{4}\frac{\partial^2 L_y}{\partial p_r \partial p_s} = \frac{1}{(1-f_{n+1})}\left(\sum_{j=1}^{n} y_j \frac{\partial^2 f_j}{\partial p_r \partial p_s} + \frac{\left(\sum_{j=1}^{n} y_j f_j\right) - 2}{(1-f_{n+1})}\frac{\partial^2 f_{n+1}}{\partial p_r \partial p_s}\frac{\partial^2}{\partial p_r \partial p_s}\right)$

(the weight of the focal point y is the number of eigenvalues of (B_{rs})).

For fixed $p \in M$, consider the hyperplanes

$P_k = \left\{(u_1, u_2, \ldots, u_{n+1}) \in R^{n+1} \,\middle|\, \sum_{i=1}^{n} u_i f_i(p) - (1 - f_{n+1}(p))u_{n+1} = k\right\}, k \in R.$ Note that, $\forall k \in R$, P_k is orthogonal to line through f(p) and $(0,0,\ldots,0,1)$ the north pole of S^n, since

$$P_O = \left\{(u_1, u_2, \ldots, u_{n+1}) \in R^{n+1} \,\middle|\, \left\langle (u_1, u_2, \ldots, u_{n+1}), (f_1, f_2, \ldots, f_n, f_{n+1} - 1)\right\rangle = 0\right\}$$

and the line through f(p) and $(0,0,\ldots,0,1)$ has direction $(f_1, f_2, \ldots, f_n, f_{n+1} - 1)$.

If $0 \in P_O$ and $f(p) \in P_{1-f_{n+1}}$ $(\neq P_O)$ then $\left(\frac{k}{1-f_{n+1}}\right) f(p) \in (P_k \cap \nu_f(p)), k \in R$, and $0 \notin (P_k \cap \nu_f(p)), k \neq 0$. So $P_k \cap \nu_f(p)$ is an $(n-m)$-plane (hyperplane) in $\upsilon_f(p)$, $\forall k \in R$.

We will consider $P_2 \cap \nu_f(p)$ as in Figure 1. Consider the map $q : \nu_g(p) \to \nu_f(p)$ defined by

$$q(y_1, y_2, \ldots, y_n) = \left(y_1, y_2, \ldots, y_n, \frac{\left(\sum_{j=1}^{n} y_i f_j\right) - 2}{(1-f_{n+1})}\right).$$

Note that $(y_1, y_2, \ldots, y_n)$ satisfies equation (4) then $q(y_1, y_2, \ldots, y_n)$ satisfies equation (2).
Further if $(x_1, x_2, \ldots, x_{n+1}) \in P_2 \cap \nu_f(p)$ then $(x_1, x_2, \ldots, x_n) \in \nu_g(p)$ and $q(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n+1})$. So $q : \nu_g(p) \to P_2 \cap \nu_f(p)$ is one to one and onto, and $\upsilon_f(p)$ is the closure of $\{\lambda x \mid x \in P_2 \cap \nu_f(p), \lambda \in R\}$ in R^{n+1}.

Next note that $\theta(\Gamma_g(p)) = P_2 \cap \nu_f(p)$ since if $(y_1, y_2, \ldots, y_n)$ satisfies equation (4) and (5) then $\theta(y_1, y_2, \ldots, y_n)$ satisfies equation (2) and (3). Further the closure of $\{\lambda\,\theta(y) \mid y \in \Gamma_g(p), \lambda \in R\}$ in R^{n+1} is contained in $\Gamma_f(p)$ as in Figure 2.

From Equation (2), (3), (4), (5) and if $y \in \Gamma_g(p)$ with weight k then $\theta(y) \in \Gamma_f(p)$ with weight k, $\forall \lambda \neq 0$.

Case 1. Suppose that $\Gamma_g(p)$ does not have a sheet of focal points "at infinity", i.e. there exists a normal line L, through $g(p)$ in $\upsilon_g(p)$ with total weight of $\Gamma_g(p) \cap L$ equal to m. Then $\Gamma_f(p)$ is equal to the closure of $\{\lambda\,\theta(y) \mid y \in \Gamma_g(p), \lambda \in R\}$ in R^{n+1}, since $\left(\frac{1 - f_{n+1}(p)}{2}\right)\theta(L)$ is the line through $f(p)$, and the total weight of $\Gamma_f(p) \cap \left(\frac{1 - f_{n+1}(p)}{2}\right)\theta(L)$ equal to m.

Case 2. Suppose that $\Gamma_g(p)$ has a sheet of focal points "at infinity", with weight k. So the maximum total weight of focal points in $\Gamma_g(p)$ on a normal line through $g(p)$ in $\upsilon_g(p)$ is $m - k$, and therefore the maximum weight of focal points in $\Gamma_f(p)$ on a normal line through $f(p)$ in $P_2 \cap \nu_f(p)$ is $m - k$. As $f(M) \subset S^n$, $\Gamma_f(p)$ cannot have sheet of focal points "at infinity" and therefore P_0 must be a sheet of focal points contained in $\Gamma_f(p)$, with weight k.

More precisely if $x = (x_1, x_2, \ldots, x_{n+1})$ and $\sum_{i=1}^{n} x_i f_i - (1 - f_{n+1})x_{n+1} = k$ then

$$x = \frac{k}{2}\left(\frac{2x_1}{k}, \frac{2x_2}{k}, \ldots, \frac{2x_n}{k}, \frac{\left(\sum_{i=1}^{n} \frac{2x_i f_i}{k}\right) 2}{(1 - f_{n+1})} \frac{\partial^2 f_{n+1}}{\partial p_r \partial p_s}\right) = \frac{k}{2}\theta(y)$$

where $y = \left(\frac{2x_1}{k}, \frac{2x_2}{k}, \ldots, \frac{2x_n}{k}\right)$.

So $\Gamma_f(p) = (P_O \cap \nu_f(p)) \cup \{\lambda\,\theta(y) \mid y \in \Gamma_g(p), \lambda \in R\}$.

Hence in either case 1 or case 2, $G_f(p)$ is a union of hyperplanes in $u_g(p)$ if and only if $G_f(p)$ is a union of hyperplanes in $\upsilon_f(p)$.

Finally we prove of the following theorem.

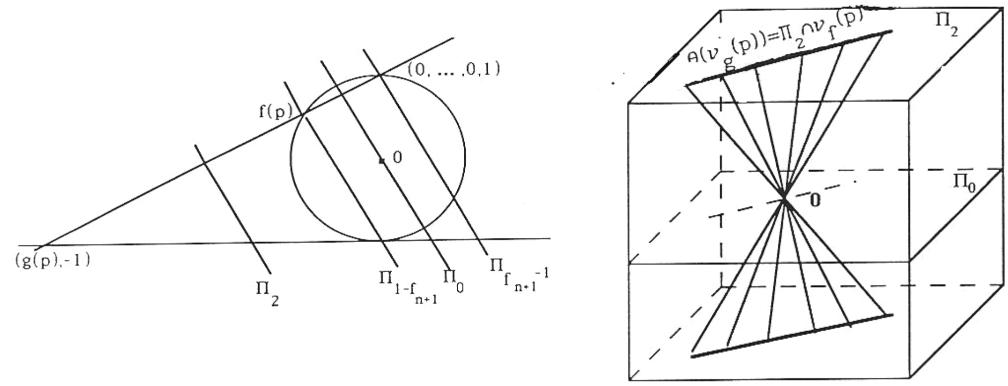

Figure 1.

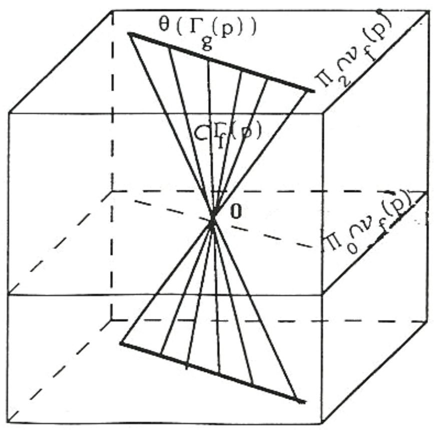

Figure 2.

Theorem.3 Let $f : M^m \to S^n \subset R^{n+1}$ be an immersion with arbitrary codimension and an immersion $g : M^m \to R^n$ is defined by

$$g(p) = \left(\frac{2f_1(p)}{1 - f_{n+1}(p)}, \frac{2f_2(p)}{1 - f_{n+1}(p)}, \ldots, \frac{2f_n(p)}{1 - f_{n+1}(p)} \right),$$

where $f(p) = \left(f_1(p), f_2(p), \ldots, f_{n+1}(p)\right)$ and $f_{n+1}(p) \neq 1$ for all $p \in M$. (i.e. g is obtained by composition f with a stereographic projection from S^n to R^n). Then g has TRFS if and only if f has TRFS.

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TAMAMEN İNDİRGENEBİLEN FOCAL CÜMLE İLE STEREOGRAFIK PROJEKSIYON

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Özet.

S. Carter ve yazar, $f: M \rightarrow R^n$ tamamen indirgenebilen focal cümleye (TRFS) sahib immersiyon tanımını [1] de verdi. Bu immersiyon her $p \in M$ için, p ye bağlı focal cümle, $f(p)$ de $f(M)$ ye normal düzlemdeki hiperyüzeylerin bir birleşimidir. Bu çalışmada bu özelliği sağlayan TRFS ye sahip immersiyonlar, stereografik projeksiyonun ters görüntüsü altında korunduğu gösterildi.

Anahtar Kelimeler: Focal Cümle, Tamamen İndirgenebilen Focal Cümle, Stereografik Projeksiyon

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