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ON STRONGLY 2-PRIMAL AND 2-PRIMAL RINGS

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ABSTRACT

An associative ring is called *2-primal* if its prime radical contains every nilpotent element of the ring (equivalently, if every minimal prime ideal of the ring is completely prime) and It is called *a strongly 2-primal* if every prime ideal of the ring is completely prime. Some results, old and new ones, connected with astrongly 2-primal rings and 2-primal rings are obtained. Also several new questions related to these rings are discussed.

Mathematics Subject Classification: 16L30, 16N40, 16S36.

Keywords: Strongly 2-primal rings and 2-primal rings.

GÜÇLÜ 2-PRİMAL VE 2-PRİMAL HALKALAR ÜZERİNE

ÖZET

R birleşmeli bir halka olsun. Eğer R nin her prime (asal) radikali halkanın tüm nilpotent elemanlarını kapsıyorsa R halkasına *2-primal* halka adı verilir. Bu çalışmada 2-primal ve güçlü 2-primal ($P(R/I) = N(R/I)$) halkalarla ilgili bazı yeni sonuçlar elde edilmiştir.

Anahtar Kelimeler: Güçlü 2 primal halka ve primal halka

1. INTRODUCTION

Throughout this paper, we assume that R is an associative ring (not necessarily commutative) with unity. The symbols, "$J(R)$" will denote

Jacobson radical, " $P(R)$ " prime radical and " $N(R)$ " the set of all nilpotent elements in R, respectively.

Let R be a ring. Then R is called a *2-primal ring* if $P(R) = N(R)$ (see [2]). All commutative rings, one-sided Artinian local rings and Reduced rings (i.e. if it contains no nonzero nilpotent elements) are 2-primal rings. By [6], R is a 2-primal ring if and only if $R/P(R)$ is a reduced ring. Following [5], a ring R is called *a strongly 2-primal ring* if $P(R/I) = N(R/I)$ for every proper ideal I of R. All simple domains are strongly 2-primal rings. The notions of strongly 2-primal rings and 2-primal rings have been the focus of a number of research papers (see [2,3,4,5,6,7]).

A ring R is called *right duo* if every right ideal of R is two sided ideal. Clearly, right duo rings are strongly 2-primal rings and so 2-primal rings. It is well known that if D is a division ring then the power series ring $D[[x]]$ is duo (every non-zero one-sided ideal is a two-sided ideal of the form (x^n)).

In this paper, we will show that if D is a division ring, then $D[[x]]$ is a strongly 2-primal ring. Among the other results, we will prove that the ring extension of a (strongly) 2-primal ring is again a (strongly) 2-primal ring.

The fundamental definitions and properties used in this paper may be found in [1].

2. THE RESULTS

Clearly, each strongly 2-primal ring is a 2-primal ring.

Theorem 2.1. *Assume that $R/J(R)$ is a semisimple Artinian ring and $J(R)$ is right or left T-nilpotent (i.e., R is an one-sided perfect ring). Then R is a strongly 2-primal ring if and only if R is a 2-primal ring.*

Proof. Let R be a 2-primal ring. By [3, Proposition 3.5], $R/J(R)$ is a finite direct product of division rings. Since R is an one-sided perfect ring, we have $J(R) = P(R)$. By assumption, [2, Proposition 3.3] and [6, Proposition 1.13], the ring R is a strongly 2-primal ring.

Remark: Recall that R is a 2-primal ring if and only if $R/P(R)$ is a subdirect product of reduced rings if and only if $R/P(R)$ is a subdirect product of domains. Hence;

Theorem 2.2. *Let R be a von Neumann regular ring. If R is a strongly 2-primal ring (if and only if R is a 2-primal ring), then R is a subdirect product of division rings.*

Proof. Let R be a von Neumann regular ring. Hence R is a 2-primal ring, and so R is a subdirect product of domains by Remark. Since R is a von Neumann regular ring, R/I is a division ring for minimal prime ideal I of R.

Let R be a ring and X any set of commuting indeterminates over R.

Theorem 2.3. *Let R be a ring and n be a positive integer.*
(1.) *If R is a 2-primal ring, then $R[x]$ is a 2-primal ring.*
(2.) *R is a 2-primal ring if and only if $R[x]/x^n R[x]$ is a 2-primal ring.*
(3.) *R is a strongly 2-primal ring if and only if $R[x]/x^n R[x]$ is a strongly 2-primal ring.*
(4.) *R is a 2-primal ring if and only if $R[[x]]/x^n R[[x]]$ is a 2-primal ring.*
(5.) *R is a strongly 2-primal ring if and only if $R[[x]]/x^n R[[x]]$ is a strongly 2-primal ring.*

Proof. (1.) See [2, Proposition 2.6].
(2.) Note that $xR[x]/x^n R[x]$ is nilpotent and $xR[x]/x^n R[x] \in P(R[x]/x^n R[x]) = (P(R)+xR[x])/x^n R[x]$. Let S denote the set of minimal prime ideals of R. We consider the one to one map $(S+xR[x])/x^n R[x] \to S$. It is easy to see that $R[x]/((S+xR[x])/x^n R[x])$ is isomorphic to $(xR[x]/x^n R[x])/S$. Now, by Remark, proof is obvious.
(3.) We consider the one to one map $(P(R)+xR[x])/x^n R[x] \to P(R)$. Since $(R[x]/x^n R[x])/((P(R)+xR[x])/x^n R[x])$ is isomorphic to $R/P(R)$, the proof is clear by Remark.
(4.) Similar to (2).

(5.) Similar to (3).

In [2, Example 3.13], they shown that polynomial ring over division rings need not be a strongly 2-primal ring.

Theorem 2.4. *Let D be a division ring. Then $D[[x]]$ is a strongly 2-primal ring.*

Proof. Let D be a division ring. Because $D[[x]]$ has any non-zero prime ideal such that $D[[x]]x$, we have two prime factor rings such that $D[[x]]/D[[x]]x$ and $D[[x]]/\{0\}$. By [2, Proposition 3.5] and [6, Proposition 1.13], $D[[x]]$ is a strongly 2-primal ring.

Let R and S be two rings. $T(R,S)$ ring extension is defined by

$$T(R,S)=\left\{\begin{pmatrix} r & s \\ 0 & r \end{pmatrix} : r\in R, s\in S\right\}$$

with the usual operations $(r_1,s_1)(r_2,s_2)=(r_1r_2, f(r_1)s_2+s_1f(r_2))$, where $f:R\to S$ is a ring homomorphism.

Theorem 2.5. (1.) *If R is a 2-primal ring, then $T(R,S)$ is a 2-primal ring.*
(2.) *If R is a strongly 2-primal ring, then $T(R,S)$ is a strongly 2-primal ring.*

Proof. (1.) Let R be a 2-primal ring. Since $T(R,S)/P(T(R,S))$ is isomorphic to $R/P(T(R,S))$, by [2, Proposition 2.2], then $T(R,S)$ is a 2-primal ring.
(2.) Similar to (1).

Questions: 1. Is a subdirect product of 2-primal rings also 2-primal ring ? ([2])
2. Is a subdirect product of strongly 2-primal rings also strongly 2-primal ring ?
3. Assume $R[x]$ is a strongly 2-primal ring. Is $R[x,x^{-1}]$ strongly 2-primal ring?

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