Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi
Afyon Kocatepe University – Journal of Science and Engineering
https://dergipark.org.tr/tr/pub/akufemubid

Araştırma Makalesi / Research Article
DOI: https://doi.org/10.35414/akufemubid.1404205
AKÜ FEMÜBİD 24 (2024) 051302 (1113-1119)
AKU J. Sci. Eng. 24 (2024) 051302 (1113-1119)

Exact Solutions of Boussinesq Equations by Hirota Direct Method

***Makale Bilgisi / Article Info**
Alındı/Received: 13.12.2023
Kabul/Accepted: 01.07.2024
Yayımlandı/Published: 01.10.2024

Boussinesq Denklemlerinin Hirota Direct Metod ile Tam Çözümleri

Halide GÜMÜŞ*, Abdullah BAYKAL

Department of Mathematics, Dicle University, Diyarbakır, Turkiye

Öz

Boussinesq Denklemleri (BSQ) bu makalenin odak noktasıdır. İlk olarak, nonlineer evolüsyon denklemlere çoklu soliton çözümler oluşturmak için kullanılan Hirota'nın D operatörüne ilişkin temel bir genel bakış sunuyoruz. Daha sonra dördüncü dereceden BSQ ile ilgili bazı detaylar veriliyor ve bir soliton çözüm bulmak için Hirota Direct yöntemini kullanıyoruz. Hirota'nın bilineer yaklaşımı aynı zamanda nonlineer evolüsyon denklem olan altıncı dereceden Boussinesq benzeri denklem sınıfını çözmek için de kullanılır. Sonuçlar, bu yaklaşımın tam integre edilebilirlik gerektirdiğini doğrulamıştır.

Anahtar Kelimeler: *Hirota Direct Metod; Boussinesq Denklemleri; Soliton Çözümler; Mathematica 12.*

Abstract

Boussinesq Equations (BSQ) are the focus of this article. First, we provide a basic overview of Hirota's D operator, which is used to build multi-soliton solutions for equations involving nonlinear evolution. After that, some details regarding fourth-order BSQ are provided, and we use Hirota's direct method to find a one-solution solution. Hirota's bilinear approach is also used to solve a class of sixth-order Boussinesq-like equations with nonlinear evolution. The outcomes verified that this approach requires complete integrability.

Keywords: *Hirota Direct Metod; Boussinesq Denklemleri; Soliton Çözümler; Mathematica 12.*

1. Introduction

Complete integrable and partially integrable nonlinear evolution equations have historically piqued the interest of mathematicians and physicists more than other partial differential equations. The seminal contributions to soliton theory are the discovery made by J. Scott Russell in 1844 and the subsequent method developed by Hirota in 1980. Of all the soliton equations, Korteweg de Vries' equation from 1895 has gained the most notoriety and significance. Zabusky and Kruskal found the other numerically for the KdV equation (1965), which Lax subsequently proved analytically (1968). By using inverse scattering transformation (IST), the multisoliton solutions of KdV were discovered (Gardner et al. 1967 and Kay and Moses 1956). Later on, a broad class of equations was addressed by the application and generalization of this method (Zakharov et al. 1972 and Ablowitz et al. 1974).

Following all of these advancements, in 1971–1972, Hirota created his bilinear approach for creating soliton solutions. A specialized method called the Hirota direct method is applied to soliton equations, integrable systems, and nonlinear partial differential equations in particular. Since its introduction, it has developed into a potent tool for figuring out the precise soliton solutions to integrable equations. The approach is very helpful in comprehending how solitons behave. These solitons, which are able to move without changing their form or energy, are found in a variety of physical systems, including water waves, plasma phenomena, and optical fibers.

In this research, fourth-order BSQ and a category of nonlinear sixth-order Boussinesq-like equations, referred to as nonintegrable equations, are treated using Hirota's bilinear method. After Hirota (1973,1980,2004) developed this well-known analytical technique, other writers such as Matsuno (1984) and Nakamura (1979) used it to solve nonlinear evolution equations precisely. The inverse dispersion transform (Ablowitz and Segur, 1981) and Whitham's method (1984), which is used to find regular solutions by such function, is the other known techniques. In fluid dynamics, the behavior of indestructible, turbulent flows in a relatively thin layer of fluid is described by the partial differential equation known as the Boussinesq equation. It bears the name Joseph Valentin Boussinesq (1877), a French mathematician and physicist from the 19th century who made a substantial contribution to the field of fluid mechanics research. He created an equation to describe

*Sorumlu Yazar/Corresponding Author: Halide GÜMÜŞ   e-posta/e-mail: halidegumus@hotmail.com

the locations of horizontally stagnant homogenous water. Both the nonlinearity and the wave dispersion in shallower waters are taken into account in the equation. It is particularly useful for modeling wave propagation and wave-structure relationships in geophysics, oceanography, and engineering. Cushman-Roisin (1994) provides a general introduction to geophysical fluid dynamics. It covers a range of geophysical fluid dynamics topics as well as applications of Boussinesq equations. Anderson (2011) concentrates on computational fluid dynamics basics. The book covers foundational concepts and stresses contemporary computational techniques along with subjects like the Boussinesq equation.

2. Hirota's D Operator and Bilinear Form

A novel direct technique for building multi-soliton solutions to integrable nonlinear evolution equations was presented by Hirota in 1971. This approach is predicated on the partial differential equation's Hirota bilinear form, which converts it into a system of bilinear equations. The original partial differential calculation can be precisely solved using bilinear equations. If its aim is constructing soliton solutions it can say that Hirota's direct method is the best way and it can say this method is the fastest in producing results. Hirota introduced the method defined as a formula in which a and b are non-negative integers and which is called Hirota derivative by using the D operator. The D operator is a bilinear operator. The "D" operator in the Hirota method is typically used in integrable systems and plays a specific role in such contexts, often acting as a derivative operator with respect to a time or space variable. On the other hand, the "D" operator in fractional calculus represents an operator for derivatives at non-integer orders. The key difference lies in their respective applications and mathematical properties. Hirota's D operator can be defined as:

$$\mathrm{D}_{\mathrm{X}}^{\mathrm{a}}\mathrm{D}_{\mathrm{t}}^{\mathrm{b}}(\mathrm{f.g}) = \left(\frac{\partial}{\partial \mathrm{x}} - \frac{\partial}{\partial \mathrm{x}'}\right)^{\mathrm{a}} \left(\frac{\partial}{\partial \mathrm{t}} - \frac{\partial}{\partial \mathrm{t}'}\right)^{\mathrm{b}} \mathrm{f(x,t)g(x',t')}|_{\mathrm{x'=x,t'=t}}. \quad (1)$$

In this method, we transform new variables so we can obtain a soliton solution more easily. Multisoliton solutions can be derived by many other methods for example Inverse Scattering Transform (IST) and Darboux Transformation. IST is powerful but more complicated.;

1. For $a = 0$ and $b = 1$,

$$D_t(f.g) = f_t g - f g_t. \quad (2)$$

2. For $a = 1$ and $b = 1$,

$$D_x D_t(f.g) = f_{xt} g - f_x g_t - f_t g_x + f g_{xt}. \quad (3)$$

NOTE:

$\mathrm{D_x D_t(f.g) = D_t D_x(f.g)}$.

and generally

$D_x^a D_t^b(f.g) = D_t^b\, D_x^a(f.g)$.

3. For $a = 1$ and $b = 0$,

$$D_x(f.g) = f_x g - f g_x\,. \quad (4)$$

4. For $a = 2$ and $b = 0$,

$$D_x^2(f.g) = f_{xx} g - 2 f_x g_x + f g_{xx}\,. \quad (5)$$

5. For $a = 3$ and $b = 0$,

$$D_x^3(f.g) = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}\,. \quad (6)$$

6. For $a = 4$ and $b = 0$,

$$D_x^4(f.g) = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx}. \quad (7)$$

Theorem 1:

$D_x^a(f.g) = (-1)^a\, D_x^a(g.f)$.

Theorem 2:

$D_x^a D_t^b(f.g) = (-1)^{a+b} D_x^a\, D_t^b(g.f)$.

Theorem 3:

$D_x^a(f.1) = \partial_x^a f$.

So $D_x(f.1) = f_x$, $D_x^2(f.1) = f_{xx}$ and $D_x^3(f.1) = f_{xxx}$, $D_x D_t(f.1) = f_{xt}$.

Theorem 4:

$D_x^a(1.g) = (-1)^a \partial_x^a g$.

So $D_x(1.g) = -g_x$, $D_x^2(1.g) = g_{xx}$ and $D_x^3(1.g) = -g_{xxx}$, $D_x D_t(1.g) = g_{xt}$.

7. If $g = f$ and for $a = 1,2,3,4$ we get

$$D_x(f.f) = 0, \quad (8)$$

$$D_x^2(f.f) = 2(f_{xx} f - f_x^2), \quad (9)$$

$$D_x^3(f.f) = 0, \quad (10)$$

$$D_x^4(f.f) = 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2), \quad (11)$$

$$D_x D_t(f.f) = 2(f_{xt} f - f_x f_t). \quad (12)$$

Theorem 5: If a is odd $D_x^a(f.f) = 0$.

Theorem 6: If $a + b$ is odd $D_x^a D_t^b(f.f) = 0$.

Theorem 7: For $\varphi_i = k_i x + w_i t + \gamma_i$ and γ_i is real for $(i = 1,2,3,\dots)$ and $e^{\varphi_1}, e^{\varphi_2}$ are exponential functions. So

$$D_x D_t(e^{\varphi_1}.e^{\varphi_2}) = (k_1 - k_2)(w_1 - w_2)e^{\varphi_1+\varphi_2}\ , \quad (13)$$
$$D_x^a(e^{\varphi_1}.e^{\varphi_2}) = (k_1 - k_2)^a\ . \quad (14)$$

Proof: From result of (3) we can see

$$D_x D_t(e^{\varphi_1}.e^{\varphi_2}) = (k_1 w_1 - k_1 w_2 - k_2 w_1 + k_2 w_2)e^{\varphi_1+\varphi_2},$$
$$= (k_1 - k_2)(w_1 - w_2)e^{\varphi_1+\varphi_2}.$$

easiliy.
From the results (4), (5), (6)

$$D_x(e^{\varphi_1}.e^{\varphi_2}) = (k_1 - k_2)e^{\varphi_1+\varphi_2}, \quad (15)$$
$$D_x^2(e^{\varphi_1}.e^{\varphi_2}) = (k_1^2 - 2k_1k_2 + k_2^2)e^{\varphi_1+\varphi_2},$$
$$= (k_1 - k_2)^2 e^{\varphi_1+\varphi_2}, \quad (16)$$
$$D_x^3(e^{\varphi_1}.e^{\varphi_2}) = (k_1^3 - 2k_1k_2 + k_2^2)e^{\varphi_1+\varphi_2},$$
$$= (k_1 - k_2)^3 e^{\varphi_1+\varphi_2}, \quad (17)$$

and generally;

$$D_x^a(e^{\varphi_1}.e^{\varphi_2}) = (k_1 - k_2)^a e^{\varphi_1+\varphi_2},$$

by using this results we get above equalties;

8. $D_x D_t(e^{\varphi_1}.e^{\varphi_1}) = 0.$

9. $D_x^a(e^{\varphi_1}.e^{\varphi_1}) = 0.$

Definition: Partial differential equation $F(u, u_t, u_x, u_{xx}, \dots) = 0$ should be used to Express it, such that, using the dependent variable transformation $u = u(f)$, the corresponding bilinear form is expressed as $B(f.f) = 0$. Next, we consider perturbation expansion as the equation for a solution, in which ε is an arbitrarily small parameter and f is bounded by x and t.

$$f = 1 + \sum_{i=1}^{\infty} \varepsilon^i f_i\ . \quad (18)$$

An approximative solution would be provided by this expansion. But when dealing with a bilinear equation $B(f.f) = 0$, the right value of f_1 is selected to truncate the infinite expansion with a finite number of terms, providing an exact solution. It obtain if it write $f.f$ and take $f_0 = 1$.

$$f.f = (1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots)(1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots)$$
$$= 1.1 + \varepsilon(f_1.1 + 1.f_1) + \varepsilon^2(f_2.1 + f_1.f_1 + 1.f_2) + \cdots$$

Converting to $B(f.f) = 0$ and gathering the ε^1 powers we possess,

$$\varepsilon^0 : B(1.1) = 0\ ,$$
$$\varepsilon^1 : B(f_1.1 + 1.f_1) = 0,$$
$$\varepsilon^2 : B(f_2.1 + f_1.f_1 + 1.f_2) = 0,$$
$$\vdots$$
$$\varepsilon^n : B\left(\sum_{k=0}^{n} f_{n-k}.f_k\right) = 0.$$

in which B represents a bilinear operator and $f_0 = 1$ for a certain positive integer n. We can write;

$$f_1 = \sum_{i=1}^{N} e^{\varphi_i}$$

Thus, we obtain a one-soliton solution by using ε^1, a two-soliton solution by using ε^2, and an N-soliton solution by using ε^N.

Soliton solutions are obtained by;

1. Logarithmic Transformation
2. The Rational Conversion.
3. The Arctan Conversion.

in the bilinear Hirota method. Even though the answer isn't always obvious, you can still write an equation in bilinear form using the sum of bilinears. For KdV class equations, logarithmic transformation is typically used.

3. Boussinesq Equation (BSQ)

The most widely used versions for simulating shallow water waves are the fourth-order extension of the classical Boussinesq equation and its variant. An expansion of the classical Boussinesq equation that takes higher-order dispersive effects into account is the fourth-order Boussinesq equation. It is used to more accurately model wave behavior in specific shallow water systems. As it write for the equation;

$$u_{tt} - u_{xx} - 3u_{xx}^2 - u_{xxxx} = 0. \quad (19)$$

We search for the logarithmic transformation of the equation. Boussinesq equation (BSQ) is the name of this equation. Generally, this equation is expressed as a formula;

$$\mathrm{u_{tt}} + \gamma \mathrm{u_{xx}} + \beta \mathrm{u_{xx}^2} + \alpha \mathrm{u_{xxxx}} = 0.$$

When α is positive, it is recognized as the good BSQ; if it is negative, it is classified as the bad BSQ. Here, we use the Hirota Direct method to find the soliton solution of the bad BSQ of (19), in which $u = u(x,t)$ and the boundary condition $u \to 0$ as $|x| \to \infty$. The form contains the solution to the equation. Equation has the solution in the form $u = 2(lnf)_{xx}$ in which $f_x, f_{xx}, f_t, \dots \to 0$ as $|x| \to \infty$. Let $\emptyset = lnf$ then $u = 2\emptyset_{xx}$. If it substitute into (19) it obtain

$$2\emptyset_{xxtt} - 2\emptyset_{xxxx} - 12(\emptyset_{xx}^2)_{xx} - 2\emptyset_{xxxxxx} = 0.$$

After twice integrating this equation, we get;

$$2\emptyset_{tt} - 2\emptyset_{xx} - 12\emptyset_{xx}^2 - 2\emptyset_{xxxx} = 0. \quad (20)$$

For $\emptyset = lnf$;

$$\emptyset_{tt} = \frac{f_{tt}f - f_t^2}{f^2},$$

$$\emptyset_{xx} = \frac{f_{xx}f - f_x^2}{f^2},$$

$$\emptyset_{xxxx} = \frac{f_{xxxx}}{f} - 4\frac{f_x f_{xxx}}{f^2} + 12\frac{f_x^2 f_{xx}}{f^3} - 3\frac{f_{xx}^2}{f^2} - 6\frac{f_x^4}{f^4}.$$

Upon substituting into (20), we arrive at;

$$(f_{tt}f - f_t^2) - (f_{xx}f - f_x^2) - (f_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) = 0 \quad (21)$$

By using (8), we can obtain bilinear form;

$$(D_t^2 - D_x^2 - D_x^4)(f.f) = 0. \quad (22)$$

To find a one-soliton solution, let's take $B = D_t^2 - D_x^2 - D_x^4$ and find ε^1 by perturbation expansion. Let's look at the BSQ $B(f.f) = 0$ in bilinear form. We are now attempting to solve (22) in terms of (18). By replacing (18) with (22) and matching the coefficients of powers ε, we can obtain;

$$\varepsilon^1 = f_{1,tt} - f_{1,xx} - f_{1,xxxx} = 0. \quad (23)$$

For $N = 1, f_1 = e^{\varphi_1}$ where $\varphi_1 = k_1 x + w_1 t + \gamma_1$. If we substitute $f_1 = e^{\varphi_1}$ into (23) we get

$$(w_1^2 - k_1^2 - k_1^4)e^{\varphi_1} = 0, \; w_1 = \sqrt{k_1^2 + k_1^4}$$

$(k_1 \neq 0)$. So we can write

$f = 1 + e^{k_1 x + \sqrt{k_1^2 + k_1^4} + \gamma_1}$. Then for

$f = 1 + e^{\varphi_1}$, $f_x = k_1 e^{\varphi_1}$ and $f_{xx} = k_1^2 e^{\varphi_1}$.

In perturbation expansion by getting ε^2 we may choose $f_2 = 0$ and we may choose $f_i = 0 \; i \geq 2$. We can set $\varepsilon = 1$ without loss of generality. Thus we have

$$u(x,t) = 2k_1^2 \frac{e^{\varphi_1}}{(1 + e^{\varphi_1})^2} = \frac{1}{2}k_1^2(\text{sech}\frac{\varphi_1}{2})^2$$

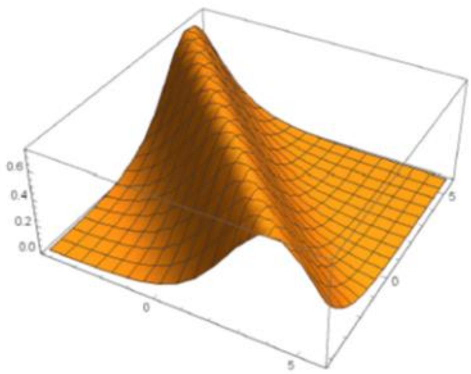

Figure 1: Figure of the BSQ equation's one-soliton solution for $\gamma = 0.11, k = -0.86$ and $w = -0.92$.

This is the one-soliton solution for the BSQ equation. In this way we can get multisoliton solutions.

4.Sixth Order Boussinesq Equation

Nonlinear sixth-order generalized Boussinesq equation is known as not completely integrable. Hirota's Direct Method is a well- known method which can help one to obtain exact solutions of completely integrable equations. If a nonlinear partial differantial equation can be expanded to the simple bilinear form $B(D_x, D_t)(f.g) = 0$ where B is a exponential or polynomial function and D is the Hirota's bilinear differantial operator then we can obtain N-soliton solution for this nonlinear equation.

In this work we look for the equation which is called sixth-order Boussinesq equation;

$$u_{tt} - u_{xx} - 3u_{xx}^2 - u_{xxxx} - \mu u_{xxxxxx} = 0, \quad (24)$$

where μ is a positive parameter. For $\mu = 0$, we can easily see that it is a bad Boussinesq equation.

Sixth-order Boussinesq equation is expressed in the form of

$$u_{tt} - \alpha u_{xx} - \beta u_{xx}^2 - \gamma u_{xxxx} - K^2 u_{xxxxxx} = 0.$$

by Daripa(2002).

There, we take the bilinear transformation of (24) and simplify it into a bilinear equation and a residual equation. The sixth-order term prevents a solon-type solution for the residual part. The sixth-order term, the dispersion term, enhances the basic structural instability of the Boussinesq equation. However, this term causes integrability to be lost. Here we use (24) in which $u = u(x,t)$ and the boundary condition $u \to 0$ as $|x| \to \infty$.

Equation has the solution in the form $u = 2(lnf)_{xx}$ where $f_x, f_{xx}, f_t, \ldots \to 0$ as $|x| \to \infty$. Let $\emptyset = lnf$ then $u = 2\emptyset_{xx}$. If we substitute into (24) we obtain

$$2\emptyset_{xxtt} - 2\emptyset_{xxxx} - 12(\emptyset_{xx}^2)_{xx} - 2\emptyset_{xxxxxx} - 2\mu\emptyset_{xxxxxxxx} = 0.$$

If we integrate this equation two times we obtain

$$\emptyset_{tt} - \emptyset_{xx} - 6\emptyset_{xx}^2 - \emptyset_{xxxx} - \mu\emptyset_{xxxxxx} = 0. \quad (25)$$

For $\emptyset = lnf$;

$$\emptyset_{tt} = \frac{f_{tt}f - f_t^2}{f^2},$$

$$\emptyset_{xx} = \frac{f_{xx}f - f_x^2}{f^2},$$

$$\emptyset_{xxxx} = \frac{f_{xxxx}}{f} - 4\frac{f_x f_{xxx}}{f^2} + 12\frac{f_x^2 f_{xx}}{f^3} - 3\frac{f_{xx}^2}{f^2} - 6\frac{f_x^4}{f^4},$$

$$\emptyset_{xxxxxx} = \frac{f_{xxxxxx}}{f} - 6\frac{f_x f_{xxxxx}}{f^2} - 15\frac{f_{xx} f_{xxxx}}{f^2} + 30\frac{f_x^2 f_{xxxx}}{f^3} + 120\frac{f_x f_{xx} f_{xxx}}{f^3} - 10\frac{f_{xxx}^2}{f^2} + 30\frac{f_{xx}^3}{f^3} - 120\frac{f_x^3 f_{xxx}}{f^4} + 360\frac{f_x^4 f_{xx}}{f^5} - 270\frac{f_x^2 f_{xx}^2}{f^4} - 120\frac{f_x^6}{f^6}.$$

If we substitute into (25) we obtain

$$(f_{tt}f - f_t^2) - (f_{xx}f - f_x^2) - (f_{xxxx}f - 4f_x f_{xxx} + 3f_{xx}^2) - (f_{xxxxxx}f - 6f_x f_{xxxxx} + 15 f_{xx} f_{xxxx} - 10 f_{xxx}^2) - f^2(f_{xxxx}f - 4f_x f_{xxx} + 3f_{xx}^2) - 4(f_{xx}f - f_x^2)^2 - 2\frac{f^4}{5} = 0. \quad (26)$$

By using (8) we can obtain bilinear form

$$(D_t^2 - D_x^2 - D_x^4 - D_x^6)(f.f) = 0. \quad (27)$$

Additionally, a residual term

$$f^2 D_x^4(f.f) - 4(D_x^2(f.f))^2 - 2\frac{f^4}{5} = 0.$$

Let take $B = D_t^2 - D_x^2 - D_x^4 - D_x^6$ and find ε^1 by perturbation expansion to find one-soliton solution. Let us consider the bilinear form $B(f.f) = 0$. Now we try to find a solution of (27) in the form of (18). If we substitute (18) into (27) and equate coefficients of powers ε we obtain

$$\varepsilon^1 = f_{1,tt} - f_{1,xx} - f_{1,xxxx} - f_{1,xxxxxx} = 0.$$

For $N = 1, f_1 = e^{\varphi_1}$ where $\varphi_1 = k_1 x + w_1 t + \gamma_1$. If we substitute $f_1 = e^{\varphi_1}$ into (28) we get

$$(w_1^2 - k_1^2 - k_1^4 - k_1^6)e^{\varphi_1} = 0,$$

$w_1 = \sqrt{k_1^2 + k_1^4 + k_1^6}$ $(k_1 \neq 0)$. So we can write

$f = 1 + e^{k_1 x + \sqrt{k_1^2 + k_1^4 + k_1^6} + \gamma_1}$. Then for

$f = 1 + e^{\varphi_1}$, $f_x = k_1 e^{\varphi_1}$ and $f_{xx} = k_1^2 e^{\varphi_1}$.

In perturbation expansion by getting ε^2 we may choose $f_2 = 0$ and we may choose $f_i = 0$ $i \geq 2$. We can set $\varepsilon = 1$ without loss of generality. Thus we have

$$u(x,t) = \frac{e^{\varphi}(-2k^6 e^{2\varphi}(-26\cosh\varphi + \cosh 2\varphi + 33))}{(e^{\varphi}+1)^6} - \frac{e^{\varphi}k^4(e^{\varphi}+1)^4 - e^{\varphi}k^2(e^{\varphi}+1)^4}{(e^{\varphi}+1)^6} + \frac{e^{\varphi}w^2(e^{\varphi}+1)^4}{(e^{\varphi}+1)^6}.$$

one soliton solution by Mathematica 12.

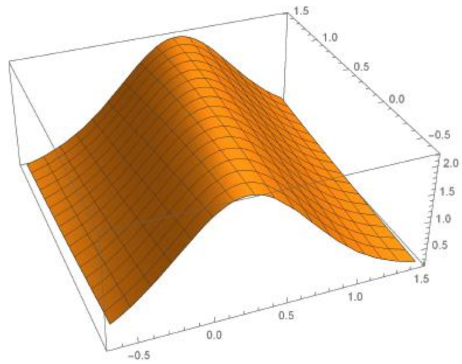

Figure 2: Figure representing the one-soliton solution for $\gamma = -1, k = -0.5$ and $w = 3$ of sixth order BSQ equation.

5.Conclusions

The classical Boussinesq equation and a class of nonlinear, incompletely integrable sixth-order Boussinesq-like equations are both subjected to the bilinear transformation. This study shows that some non integrable partial differential equations can be solved using this approach. If the linear terms of the independent functions are the same and there is a residual term in the bilinear form, the balance terms become possible. The figures are plotted to display the dinamical features of the solutions. Mathematica has been used for presenting figures of solutions. It is possible to observe soliton solutions for appropriate values of γ, k and w. In mathematica, a dynamic drawing program, it is possible to visualize the travelling waves of the solution for different values of γ.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement
Author-1: Conceptualization, investigation, methodology and software, visualization and writing – original draft, Writing – review and editing
Author-2: Investigation, Resources, Supervision, Project administration

Declaration of Competing Interest
The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement
All data generated or analyzed during this study are included in this published article.

6.References

Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H. 1973. Nonlinear evolution equations and inverse scattering. *Physical Review Letters*, **31(2)**, 125-127. https://doi.org/10.1103/PhysRevLett.31.125

Ablowitz, M. J., Kaup, D. J., Newell, A. C., & Segur, H. 1974. The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems. *Studies in Applied Mathematics,* **53(4)**, 249-315. https://doi.org/10.1002/sapm1974534249

Ablowitz, M.J., Segur, H. 1981. Solitons and the inverse scattering transform. SIAM Studies in Applied Mathematics, 425 pages, SIAM, Philadelphia, PA.

Anderson, J. D., 2011. Computational Fluid Dynamics: The Basics with Applications, John D. Anderson (alan editörü), McGraw-Hill Education, New York. 1-121.

Boussinesq, J. 1877. Théorie de l'écoulement tourbillonnant des liquides. *Mémoires présentés par divers savants à l'Académie des Sciences*,**23**, 1-680.

Cushman-Roisin, B. 1994. Introduction to Geophysical Fluid Dynamics, Bernard Cushman-Roisin (alan editörü), Prentice-Hall, New Jersey.

Daripa, P., Rajan, K., D., 2002. Analytical and Numerical Studies of a singularly perturbed Boussinesq equation, *Applied Mathematics and Computation*, **126**, 1-30. https://doi.org/10.1016/S0096-3003(01)00166-7.

Gardner, C.S., Greene, J. M., Kruskal, M.D., Miura, R.M. 1967. *Physical Review Letters*, **19**(19):1095-1097. https://doi.org/10.1103/PhysRevLett.19.1095.

Hirota, R. 1971. Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys. Rev. Lett*,**27**,1192. https://doi.org/10.1103/PhysRevLett.27.1192.

Hirota, R. 1972. Exact Solution of the Modified Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Journal of the Physical Society of Japan*, **33(5)**:1456-1458. https://doi.org/10.1143/JPSJ.33.1456

Hirota, R. 1973. Exact Envelope-Soliton Solutions of a Nonlinear Wave Equation. *Journal of Mathematical Physics*,**14(7)**:805-809. https://doi.org/10.1063/1.1666399

Hirota, R. 1973. Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, *Journal Mathematical Physics*, **14**: 810- 4. https://doi.org/10.1063/1.1666400

Hirota, R. 1980. Direct methods in soliton thoery. Solitons. Topics in Current Physics, vol. 17. Berlin: Springer-Verlag. https://doi.org/10.1007/978-3-642-81448-8_5.

Hirota, R. 2004. The Direct Method in Soliton Theory. New York: Cambridge University Press, 12-58.

Kay,I., Moses, H.E. 1956. Reflectionless Transmission through Dielectrics and Scattering Potentials. *Journal of Applied Physics*. **27**,1503-1508. https://doi.org/10.1063/1.1722296.

Korteweg, D. J. 1895. On the Change of Form of Long Waves advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves. *Philosophical Magazine*, **39(5)**, 422-443.

Lax, P. D. 1968. Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Communications on Pure and Applied Mathematics*, **21(5)**, 467-490. https://doi.org/10.1002/cpa.3160210503.

Matsuno, Y. 1984. Bilinear Transformation Method. Academic Press, New York.

Nakamura, A. 1979. A Direct Method of Calculating Periodic Wave Solutions to Nonlinear Evolution Equations. I. Exact Two-Periodic Wave Solution. *Journal of the Physical Society of Japan,* **47**, pp. 1701-1705. https://doi.org/10.1143/JPSJ.47.1701.

Russell, J. S. 1844. Report on Waves. Report of the 14th Meeting of the British Association for the Advancement of Science, York, London, 311. https://doi.org/10.4236/jamp.2014.24003.

Whitham, G.B. 1984. Comments on periodic waves and solitons. *IMA Journal of Applied Mathematics,* **32 ,**353-366. https://doi.org/10.1093/imamat/32.1-3.353

Zabusky, N., Kruskal,M.D. 1965. Physical *Review Letters*, **15**,240.

http://dx.doi.org/10.1103/PhysRevLett.15.240.

Zakharov, V. E., Shabat, A. B. 1972. Exact Theory of Two-dimensional Self-focusing and One dimensional Self modulation of Wave in Nonlinear Media. *Soviet Journal of Experimental and Theoretical Physics*,**34**,62-69.