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ON THE $\bar{M}$-INTEGRAL CURVES AND $\bar{M}$-GEODESIC SPRAYS

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SUMMARY

Some properties of $\bar{M}$-vector field Z defined on a hypersurface M of $\bar{M}$ were studied by K. Nirmala, R.S. Mishra and Shrikrishna. In this paper $\bar{M}$-integral curve of Z and $\bar{M}$-geodesic spray are defined and it is given the main theorem: The natural lift $\bar{\alpha}$ of the curve α (in $\bar{M}$) is an M-integral curve of the geodesic spray Z iff α is an $\bar{M}$-geodesic.

$\bar{M}$-İNTEGRAL EĞRİLERİ VE $\bar{M}$-GEODEZİK SPRAYLAR ÜZERİNE

ÖZET

M bir Riemann Manifoldu ve M de $\bar{M}$ nin bir hiperyüzeyi olmak üzere, M nin $\bar{M}$-vektör alanlarının bazı özellikleri K.Nirmala, R. S. Mishra ve Shrikrishna tarafından çalışıldı. Bu çalışmada $\bar{M}$-integral eğrileri ve $\bar{M}$-geodezik sprayları tanımlayarak bunlarla ilgili şu esas teoremi veriyoruz: M deki bir $\bar{\alpha}$ eğrisinin α tabii liftinin Z geodezik sprayın bir $\bar{M}$-integral eğrisi olması için gerek ve yeter şart α nın bir $\bar{M}$-geodezik olmasıdır.

I. ON THE $\bar{M}$-INTEGRAL CURVES AND $\bar{M}$-GEODESIC SPRAYS

Let $\bar{M}$ be a Riemannian n-manifold and M be a hypersurface of $\bar{M}$. $\bar{D}$. being the Riemannian connection on $\bar{M}$, S being Weingarten map of M, and N being unit normal vector field of M we have the Gauss' equation given by.

(1) $$\bar{D}_X Y = D_X Y - \langle S(X), Y\rangle N$$

where D is the Riemannian connection on M.

Definition: Let Z be a vector field on $\bar{M}$. Z is called an $\bar{M}$-vector field on M if Z is a mapping which attaches to each point P in M,a vector Z_P in $T_P\bar{M}$, that is,

$$Z:M \to T_P\bar{M}.$$

Any $\overline{M}$-vector field Z can be decomposed into its tangential and normal components given by

$$Z = Z_t + Z_n$$

where Z_t is a tanget vector field on M and Z_n is a vector field of $\overline{M}$ defined on M which is normal to M at every point. We have

(2) $$Z = Z_t + \lambda N$$

where $\lambda \varepsilon C^{\infty}(M,R)$.

Let α be a curve passing through a point P on M and T denote the tangent vector field of α on M. Covariant differentiation of Z in the direction T gives

$$\overline{D}_T Z = \overline{D}_T Z_t + \overline{D}_T \lambda N$$

and then

$$\overline{D}_T Z = D_T Z_t - \langle S(T), Z_t \rangle N + D_T \lambda N - \langle S(T), \lambda N \rangle N.$$

After some calculation we obtain

(3) $$\overline{D}_T Z = \tan \overline{D}_T Z + \text{nor} \overline{D}_T Z,$$

where

(4) $$\tan \overline{D}_T Z = D_T Z_t + \lambda S(T), \quad \text{nor} \overline{D}_T Z = (d\lambda/dt - \langle S(T), Z_t \rangle) N.$$

Definition: The vectors $\overline{D}_T Z$, $\tan\overline{D}_T Z$, and $\text{nor}\overline{D}_T Z$ in (3) are called as the absolute curvature vector, geodesic curvature vector, and normal curvature vector of the $\overline{M}$-vector field Z with respect to α, respectively and the corresponding magnitudes on M as the absolute curvature, geodesic curvature and normal curvature of the $\overline{M}$-vector field Z with respect to α. Hence

(5)
$$\begin{aligned}
\overline{K}_{ZA} &= \|\overline{D}_T Z\| \Leftrightarrow \overline{D}_T Z = \overline{K}_{ZA}\overline{N}_A, \ \overline{N}_A \text{ is a unit vector field on } \overline{M},\\
K_{ZG} &= \|\tan\overline{D}_T Z\| \Leftrightarrow \tan\overline{D}_T Z = K_{ZG} X, X \text{ is a unit vector field on } M,\\
K_{ZN} &= \|\text{nor}\overline{D}_T Z\| \Leftrightarrow \text{nor}\overline{D}_T Z = K_{ZN} N
\end{aligned}$$

where $\overline{K}_{ZA}, K_{ZG}$, and K_{ZN} are absolute curvature, geodesic curvature, and normal curvature, respectively. We have

(6) $$\overline{K}_{ZA}{}^2 = K_{ZN}{}^2 + K_{ZG}{}^2 ,$$

(7) $$K_{ZN} = \overline{K}_{ZA} \cos\theta, \cos\theta = < \overline{N}_A, N>.$$

Definition: A vector $X \varepsilon T_p M$ is called as an asymptotic vector of Z if

(8) $$(d\lambda/dt - <S(X),Z_t>) (d\lambda/dt - <S(X),Z_t>) = 0$$

[1]. The curve α in M is called as the asymptotic curve of Z if the tanget vector field of α coincides with the asymptotic vector field of Z, that is,

$$(d\lambda/dt - <S(T),Z_t>) (d\lambda/dt - <S(T),Z_t>) = 0,$$

where $T = d\alpha/dt$.

Definition: X, $Y \varepsilon T_p M$ are called as conjugate vectors of Z if

(9) $$(d\lambda/dt - <S(X),Z_t>) (d\lambda/dt - <S(Y),Z_t>) = 0.$$

X is called self-conjugate vector field of Z if

(10) $$(d\lambda/dt - <S(X),Z_t>) (d\lambda/dt - <S(X),Z_t>) = 0.$$

Using (8) and (10) we obtain the results:

Corollary: Tangent vector field of every asymptotic curve of Z is a self-conjugate vector field of Z.

Corollary: If X is an asymptotic vector field of Z then for the value of λ in (2) we have

(11) $$\lambda = \int <S(X),Z_t> dt.$$

Definition: For an $\overline{M}$-vector field $Z = Z_t + Z_n$, a curve $\alpha \subset M$ is called an $\overline{M}$-integral curve of Z if

(12) $$Z_t(\alpha(t)) = (d\alpha/dt)\,|_{\alpha(t)}.$$

Definition: $\alpha \subset M$ being a differentiable curve, the curve $\overline{\alpha} : I \to TM$ given by

(13) $$\overline{\alpha}(t) = \dot{\alpha}(t)\,|_{\alpha(t)}$$

is called as the natural lift of α on the manifold TM [2].

Definition: An $\overline{M}$-vector field Z is called as an $\overline{M}$-geodesic spray if for $V \varepsilon TM$

(14) $$Z_t(V) = (d\lambda/dt - \langle V, S(V)\rangle)N.$$

This definition is the generalization of the definition of geodesic spray on the manifold TM [2]. Indeed for $\lambda = 0$ we have

(15) $$Z(V) = -\langle V, S(V)\rangle N$$

Theorem: The natural lift $\overline{\alpha}$ of the curve α is an $\overline{M}$-integral curve of the $\overline{M}$-geodesic spray Z iff α is an $\overline{M}$- geodesic on M.

Proof: Let $\overline{\alpha}$ be an $\overline{M}$-integral curve of the $\overline{M}$-geodesic spray Z. Thus

(16) $$Z_t(\overline{\alpha}) = d\overline{\alpha}/dt.$$

Since Z is a geodesic spray on $T\overline{M}$ ($\overline{M}$-geodesic spray) we have

(17) $$Z_t(\overline{\alpha}) = (d\lambda/dt - \langle\overline{\alpha}, S(\overline{\alpha})\rangle)N.$$

Joining (13), (16), and (17) we obtain that

$$d\overline{\alpha}/dt = (d\lambda/dt - \langle\dot{\alpha}, S(\dot{\alpha})\rangle)N.$$

On the other hand, since

$$d\overline{\alpha}/dt = d\dot{\alpha}/dt = \overline{D}_{\dot{\alpha}}\dot{\alpha}\ ,$$

Using (3) we have

$$D_{\dot{\alpha}}\dot{\alpha} + \lambda S(\dot{\alpha}) = 0.$$

This shows that α is an $\overline{M}$-geodesic on M which is to be shown.

Conversely, if α is an $\overline{M}$-geodesic on M then it is obvious that the natural lift $\bar{\alpha}$ is an $\overline{M}$-integral curve of the $\overline{M}$-geodesic spray Z.

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