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REFLECTION GROUPS ON SEMI-EUCLIDEAN SPACES

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ABSTRACT

In this paper,we give a possible construction for subgroups of semi-orthogonal groups generated by reflections in semi-Euclidean space.

ÖZET

Bu çalışmada semi-Öklidyen uzaydaki yansımalar ile üretilen semi-ortogonal grupların altgrupları için mümkün yapıyı vereceğiz.

1. Introduction

Finite reflection groups on Euclidean space equipped with a positive definite inner product are well developed and documented in a long series of papers and books. The first comprehensive treatment of finite reflection groups was given by H. S. M. Coxeter in 1934. In [3], he completely classified the groups and derived several of their properties. Later, he included a discussion of the groups in his book [4]. In 1941, E. Witt presented more algebraic approach in [9]. Another has more recently appeared in N. Bourbaki's chapters on Lie groups and Lie algebras [1].

The main aim of this paper is to give a possible construction of reflection groups on semi-Euclidean spaces. The basic definitions and background material required here may be found in R. W. Carter [2], L. C. Grove and C. T. Benson [5], J. E. Humphreys [6], B. O'Neill [7], D. E. Taylor [8].

2. Reflection Groups on Semi-Euclidean Spaces

Let $\mathbf{R}^n_\nu$ be *semi-Euclidean space* over the real field $\mathbf{R}$ equipped with a *scalar product* $< , >$ which is symmetric, non degenerate bilinear form;

$$< x, y > = -\sum_{i=1}^{\nu} x_i y_i + \sum_{i=\nu+1}^{n} x_i y_i$$

where $x, y \in \mathbf{R}^n$ and ν is an integer with $0 \leq \nu \leq n$.

Now let $V = \mathbf{R}^n_\nu$ and let the *null cone* of the scalar product be the set

$$\Lambda = \{ x \in V \mid < x , x > = 0 \}$$

For $0 \leq \nu \leq n$, the *signature matrix* ε is the diagonal matrix $(\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = \varepsilon_2 = \ldots \varepsilon_\nu = -1$ and $\varepsilon_{\nu+1} = \varepsilon_{\nu+2} = \ldots \varepsilon_n = +1$. By [7], we have $< x, y > = (\epsilon x, y)$, where ϵ is the corresponding transformation to the signature matrix ε and $(,)$ is a positive definite inner product on V.

The *semi – orthogonal group* on V with respect to $< , >$ is

$$O_\nu(n) = \{ S \in GL(n, \mathbf{R}) \mid S^t = \varepsilon S^{-1} \varepsilon \}$$

It is easy to show that $O_\nu(n)$ is isomorphic to

$$O_\nu(V) = \{ \tau \in GL(V) \mid < \tau u , \tau v > = < u , v > \ \textit{for all } u, v \in V \}$$

Then, $\tau \in O_\nu(V)$ is called a *semi – Euclidean reflection* in V if $\tau \neq 1_V$ and $\tau v = v$ for all $v \in H$ for some nondegenerate semi-Euclidean hyperplane H in V.

The following lemma allows us to give a more explicit description of semi-Euclidean reflections in V.

2.1. LEMMA. *Let H be a nondegenerate semi-Euclidean hyperplane in V and let $u \in V - \Lambda$. Then there exists a unique semi-Euclidean reflection $\tau \in O_\nu(V)$ such that*
(i) $\tau h = h$ for all $h \in H$;
(ii) $\tau^2 = 1_V$;
(iii) τ is given by the formula

$$\tau v = v - 2 \frac{< v , u >}{< u , u >} u , \ \textit{for all } v \in V, \ u \in H^\perp$$

Proof. Let $u \in V - \Lambda$ and $H = < u >^{\perp}$. Let τ be a semi-Euclidean reflection which fixes the elements of H. By [8], $V = H^{\perp} \oplus H$, it follows that $\tau^2 = 1_V$. If $v \in V$, let $v = h + a.u$, , $h \in H$, $a \in \mathbf{R}$, then $< v , u > = < h , u > + a < u , u >$, that is, $a = \frac{< v , u >}{< u , u >}$, and so

$$\tau v = v - 2\frac{< v , u >}{< u , u >} u \qquad \blacksquare$$

From now on, this unique semi-Euclidean reflection τ will be denoted by τ_u. Let $a \in \mathbf{R}$ and $u \in V - \Lambda$. We note that

(i) $\tau_u u = -u$;

(ii) $\tau_u = \tau_{au}$;

(iii) $det\ \tau_u = -1$.

2.2. Lemma. *Let $u \in V - \Lambda$ and $\sigma \in O_\nu(V)$. Then $\sigma\tau_u\sigma^{-1} = \tau_{\sigma u}$.*

Proof. Let $v \in V$. Then

$$\begin{aligned} \sigma\tau_u\sigma^{-1} v &= \sigma\left(\sigma^{-1} v - 2 \frac{< \sigma^{-1}v , u >}{< u , u >} u\right) \\ &= v - 2 \frac{< \sigma^{-1}v . u >}{< u , u >} \sigma u \end{aligned}$$

Since $\sigma \in O_\nu(V)$, we have $< \sigma u , \sigma u > = < u , u >$. But $< \sigma^{-1}v , u >= (\epsilon\sigma^{-1}v , u)$ and since $\sigma^{-1} = \varepsilon\sigma^t\varepsilon$ we have

$$< \sigma^{-1}v , u > = (\sigma^t\epsilon v , u) = (\epsilon v , \sigma u) = < v , \sigma u >$$

Then $\sigma\tau_u\sigma^{-1}v = \tau_{\sigma u}v$ $\blacksquare$

Now let $\mathcal{G}$ be a subgroup of $O_\nu(V)$ generated by τ_u, $u \in V - \Lambda$. Then we have the following definition.

2.3. Definition. Let $u \in V - \Lambda$. The two unit vectors $\pm u$ are called *semi − Euclidean roots* of $\mathcal{G}$ associated with $\tau_u \in \mathcal{G}$.

2.4. Lemma. *Let W be a semi-Euclidean hyperplane in V and let $T \in O_\nu(V)$. Then $(TW)^{\perp} = TW^{\perp}$. If $TW = W$, then $TW^{\perp} = W^{\perp}$.*

Proof. If $y \in TW^{\perp}$, then there exists $x \in W^{\perp}$ such that $y = Tx$. So $< x , z >= 0$, for all $z \in W$. Since $T \in O_\nu(V)$, we have $0 =< x , z >=< Tx , Tz >$ for all $z \in W$. Then $< y , Tz >= 0$, for all $z \in W$, that is, $y \in (TW)^{\perp}$.

Conversely, $y \in (TW)^{\perp}$, then $< y , x > = 0$ for all $x \in TW$. Then we have

$$< y , Tu > = < T^t y , u > = 0 \; for \; all \; u \in W.$$

Since $T \in O_\nu(V)$, we have

$$\begin{aligned} 0 &= < T^t y , u > \\ &= < \epsilon T^{-1} \epsilon y , u > \\ &= (\epsilon \epsilon T^{-1} \epsilon y , u) \\ &= (T^{-1} \epsilon y , u) \\ &= (\epsilon y , (T^{-1})^t u) \\ &= < y , (T^{-1})^t u > \\ &= < T^{-1} y , u > , \; for \; all \; u \in W. \end{aligned}$$

Then $T^{-1}y \in W^{\perp}$, that is, $y \in TW^{\perp}$, so $(TW)^{\perp} = TW^{\perp}$.

If $TW = W$, then $(TW)^{\perp} = W^{\perp}$ and $TW^{\perp} = W^{\perp}$.

Now, we can give the following lemma.

2.5. LEMMA. *If α is a semi-Euclidean root of $\mathcal{G}$ and if $T \in \mathcal{G}$, then also $T\alpha$ is a semi-Euclidean root of $\mathcal{G}$.*

Proof. Set $H = \alpha^{\perp}$, $H' = TH$ and $T\alpha = x$. Then H' is a semi-Euclidean hyperplane and by the preceding lemma $H' = (T\alpha)^{\perp} = x^{\perp}$. If $y = Tz \in H'$, with $z \in H$, then by Lemma 2.2. we have $T\tau_\alpha T^{-1}y = T\tau_\alpha z = Tz = y$. Also $T\tau_\alpha T^{-1}x = T\tau_\alpha \alpha = -T\alpha = -x$. Hence, $T\alpha$ is a semi-Euclidean root of $\mathcal{G}$. ■

If $W_1, \ldots, W_k$ are subspaces of V, then it can be easily seen that $(W_1 + \ldots + W_k)^{\perp} = W_1^{\perp} \cap \ldots \cap W_k^{\perp}$.

2.6. LEMMA. *Let $\mathcal{G}$ be a subgroup of $O_\nu(V)$ generated by semi-Euclidean reflections along semi-Euclidean roots $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then $\mathcal{G}$ is effective if and only if $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ contains a basis for V.*

Proof. Let $W = \bigcap_{i=1}^{k} \alpha_i^{\perp}$. Since the semi-Euclidean reflection along α_i acts as the identity transformation on $\alpha_i^{\perp}$ and each $T \in \mathcal{G}$ is a product of the generating semi-Euclidean reflections, we have $T|_W = 1_W$, for all $T \in \mathcal{G}$. If $V_0(\mathcal{G}) = \bigcap_{T \in \mathcal{G}} V_T$, where V_T is the subspace $\{ x \in V \mid Tx = x \}$, then $W \subseteq V_0(\mathcal{G})$. On the other hand, if $x \in V_0(\mathcal{G})$, then in particular, each generating semi-Euclidean reflection leaves x invariant, so $x \in \alpha_i{}^\perp$, for each $1 \leq i \leq k$. Thus $x \in W$ and $W = V_0(\mathcal{G})$. Consequently, $\mathcal{G}$ is effective if and only if $W = 0$ or $W^\perp = V$. But $W^\perp = (\bigcap_{i=1}^{k} \alpha_i{}^\perp)^\perp = \sum_{i=1}^{k} \alpha_i{}^{\perp\perp}$. In other words, the set $\{\alpha_1 , \dots , \alpha_k\}$ spans $W^\perp$, since $\alpha_i{}^{\perp\perp}$ is the subspace spanned by α_i. Then $\mathcal{G}$ is effective if and only if $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ spans V. ∎

2.7. DEFINITION. Let $\mathcal{G}$ be a subgroup of $O_\nu(V)$ generated by a finite set of semi-Euclidean reflections. Let Φ be the set of all semi-Euclidean roots corresponding to the generating semi-Euclidean reflections, together with all images of these semi-Euclidean roots under all transformations in $\mathcal{G}$. The set Φ is called a *semi – Euclidean root system* for $\mathcal{G}$.

2.8. LEMMA. *Let $\mathcal{G}$ be a subgroup of $O_\nu(V)$ generated by a finite set of semi-Euclidean reflections and let $\mathcal{G}$ be effective. If the semi-Euclidean root system Φ is finite, then $\mathcal{G}$ is finite.*

Proof. By the definition of semi-Euclidean root system we have $T\Phi = \Phi$, for all $T \in \mathcal{G}$. Thus by restricting each $T \in \mathcal{G}$ to Φ, we may consider $\mathcal{G}$ as a permutation group on Φ. By the preceding lemma, since $\mathcal{G}$ is effective Φ contains a basis for V; so if $T|\Phi$ is the identity map on Φ then $T = 1_{\mathcal{G}}$, that is, $\mathcal{G}$ is faithful on Φ, so $\mathcal{G}$ is finite if Φ is finite. ∎

2.9. DEFINITION. A finite effective subgroup $\mathcal{G}$ of $O_\nu(V)$ generated by a set of semi-Euclidean reflections is called a *semi – Euclidean reflection group.*

From now on, we assume that $\mathcal{G}$ is a semi-Euclidean reflection group, with semi-Euclidean root system Φ.

It can be easily seen that there is a vector $t \in V - \Lambda$ such that $< t , \alpha > \neq 0$ for every root α of $\mathcal{G}$. Then the root system Φ is partitioned into two subsets;

$$\Phi_t{}^+ = \{ x \in V \mid < x , t > > 0 \} \text{ and } \Phi_t{}^- = \{ x \in V \mid < x , t > < 0 \}$$

Geometrically, $\Phi_t{}^+$ and $\Phi_t{}^-$ are the subsets of Φ lying on the two sides of the hyperplane $t^\perp$. If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $< t , -\alpha > = - < t , \alpha >$. Thus $\alpha \in \Phi_t{}^+$ if and only if $-\alpha \in \Phi_t{}^-$ and so $| \Phi_t{}^+ | = | \Phi_t{}^- |$.

2.10. DEFINITION. Let π be a minimal subset of $\Phi_t{}^+$ such that every $\alpha \in \Phi_t{}^+$ is a linear combination, with all coefficients non-negative, of elements of π.

Then π is called a $t - base$ for Φ.

2.11. DEFINITION. Let $\pi = \{ \alpha_1 , \dots , \alpha_m \}$ be a fixed t-base for Φ. A vector $x \in V$ is called $t - positive$ if it is possible to write x as a linear combination of $\alpha_1 , \dots , \alpha_m$ with all coefficients non-negative. Similarly, $x \in V$ is called $t - negative$ if it is a nonpositive linear combination of $\alpha_1 , \dots , \alpha_m$.

From now on, we shall say positive rather than t-positive and negative rather than t-negative.

2.12. LEMMA.*Let $\alpha_i , \alpha_j \in \pi$, with $i \neq j$ and λ_i, λ_j are positive real numbers, then the vector $\alpha = \lambda_i\alpha_i - \lambda_j\alpha_j$ is neither positive nor negative.*

Proof. Suppose that α is positive. Then we have

$$\alpha = \lambda_i\alpha_i - \lambda_j\alpha_j = \sum_{k=1}^{m} \mu_k\alpha_k , \ with \ all \ \mu_k \geq 0$$

If $\lambda_i < \mu_i$, then

$$0 = (\mu_i - \lambda_i)\alpha_i + (\mu_j + \lambda_j)\alpha_j + \sum\{\mu_k\alpha_k \ : \ k \neq i , j\}$$

But

$$0 =< t , (\mu_i - \lambda_i) \alpha_i + (\mu_j + \lambda_j) \alpha_j + \sum\{\mu_k\alpha_k \ : \ k \neq i , j\} >$$

and so $0 \geq \lambda_j < \alpha_j , t > > 0$. This is a contradiction. If $\lambda_i > \mu_i$, then

$$(\lambda_i - \mu_i)\alpha_i = (\lambda_j + \mu_j)\alpha_j + \sum\{\mu_k\alpha_k \ : \ k \neq i , j\}$$

Since $\lambda_i - \mu_i \neq 0$, we may divide by $\lambda_i - \mu_i$ and express α_i as a non-negative linear combination of the elements of $\pi \setminus \{\alpha_i\}$, contradicting the minimality of π. Thus α is not positive. On the other hand, if α were negative, then $-\alpha$ would be positive, which is impossible by the above argument with i and j interchanged. ∎

2.13. LEMMA. *Let $\alpha_i , \alpha_j \in \pi$,with $i \neq j$and let τ_i denote the semi-Euclidean reflection along α_i. If α_i is timelike (spacelike) and $< \alpha_i , \alpha_j > \geq 0$ ($< \alpha_i , \alpha_j > \leq 0$), then $\tau_i(\alpha_j) \in {\Phi_t}^+$.*

Proof. By Lemma 2.5 $\tau_i(\alpha_j) \in \Phi$, we know that $\tau_i(\alpha_j) \in \Phi$ is either positive or negative. But

$$\tau_i(\alpha_j) = \alpha_j - 2\frac{< \alpha_i , \alpha_j >}{< \alpha_i , \alpha_i >} \alpha_i$$

with one coefficient positive. If α_i is timelike (spacelike), by the preceding lemma, both coefficients must be non-negative, so $< \alpha_i , \alpha_j > \geq 0$ ($< \alpha_i , \alpha_j > \leq 0$) and $\tau_i(\alpha_j) \in \Phi_t{}^+$.

2.14. LEMMA. $\alpha_1 , \dots , \alpha_m \in V - \Lambda$. *Let* $U = Sp\{\alpha_1 , \dots , \alpha_\nu\}$ *be a subspace of* V *such that the scalar product is negative definite on* U *and let* $W = Sp\{\alpha_{\nu+1} , \dots , \alpha_m\}$ *be a subspace of* V *such that the scalar product is positive definite on* W. *Suppose that* $< \alpha , \alpha_i > > 0$. $1 \leq i \leq m$, *for some* $\alpha \in V$. *If*

$$< \alpha_i , \alpha_j > \geq 0 , 1 \leq i , j \leq \nu , i \neq j$$

$$< \alpha_i , \alpha_j > \leq 0 , \nu + 1 \leq i , j \leq m , i \neq j$$

$$< \alpha_i , \alpha_j > = 0 , 1 \leq i \leq \nu , \nu + 1 \leq j \leq m$$

then $\{\alpha_1 , \dots , \alpha_m\}$ *is a linearly independent set.*

Proof. Suppose that $\{\alpha_1 , \dots , \alpha_m\}$ is a linearly dependent set. Then there is a dependence relation of the form

$$\sum_{i=1}^{k} \lambda_i \alpha_i = \sum_{j=k+1}^{m} \mu_j \alpha_j , \textit{ with all } \lambda_i \geq 0 , \textit{ all } \mu_j \geq 0 \textit{ and some } \lambda_i > 0$$

This will proceed in two steps.

(1) Let $1 \leq k \leq \nu$. Then we have

$$\sum_{i=1}^{k} \lambda_i \alpha_i = \sum_{j=k+1}^{\nu} \mu_j \alpha_j + \sum_{j=\nu+1}^{m} \mu_j \alpha_j , \lambda_i \geq 0 , \mu_j \geq 0$$

Since the scalar product is negative definite on U,

$$0 \geq < \sum_{i=1}^{k} \lambda_i \alpha_i , \sum_{i=1}^{k} \lambda_i \alpha_i > = \sum_{i=1}^{k} \sum_{j=k+1}^{\nu} \lambda_i \mu_j < \alpha_i , \alpha_j > + \sum_{i=1}^{k} \sum_{j=\nu+1}^{m} \lambda_i \mu_j < \alpha_i , \alpha_j >$$

Since $< \alpha_i , \alpha_j > = 0 , 1 \leq i \leq \nu , \nu + 1 \leq j \leq m$, then

$$0 \geq < \sum_{i=1}^{k} \lambda_i \alpha_i , \sum_{i=1}^{k} \lambda_i \alpha_i > = \sum_{i=1}^{k} \sum_{j=k+1}^{\nu} \lambda_i \mu_j < \alpha_i , \alpha_j >$$

Since $< \alpha_i , \alpha_j > \geq 0 , 1 \leq i , j \leq \nu , i \neq j$, we have

$$0 \geq < \sum_{i=1}^{k} \lambda_i \alpha_i , \sum_{i=1}^{k} \lambda_i \alpha_i > \geq 0$$

and so

$$< \sum_{i=1}^{k} \lambda_i \alpha_i \ , \ \sum_{i=1}^{k} \lambda_i \alpha_i \ > = \ 0$$

Then $\sum_{i=1}^{k} \lambda_i \alpha_i = 0$. By our assumption $< \ \alpha_i \ , \ \alpha \ > \ > \ 0 \ , \quad 1 \leq i \leq m$, for some $\alpha \in V$ and so

$$0 = \sum_{i=1}^{k} \lambda_i < \ \alpha_i \ , \ \alpha \ > \ > \ 0$$

This is a contradiction.

(2) Let $\nu + 1 \leq k \leq m$. Then we have

$$\sum_{i=1}^{\nu} \lambda_i \alpha_i + \sum_{i=\nu+1}^{k} \lambda_i \alpha_i = \sum_{j=k+1}^{m} \mu_j \alpha_j \ , \ \lambda_i \geq 0 \ , \ \mu_j \geq 0$$

and so

$$\sum_{i=1}^{\nu} \lambda_i \alpha_i = \sum_{j=\nu+1}^{k} (-\lambda_j) \alpha_j \sum_{j=k+1}^{m} \mu_j \alpha_j \ , \ \lambda_i \geq 0 \ , \ \mu_j \geq 0$$

Since the scalar product is negative definite on U and positive definite on W,

$$0 \geq < \sum_{i=1}^{\nu} \lambda_i \alpha_i \, , \sum_{i=1}^{\nu} \lambda_i \alpha_i > = < \sum_{j=\nu+1}^{k} (-\lambda_j) \alpha_j \sum_{j=k+1}^{m} \mu_j \alpha_j \, , \ \sum_{j=\nu+1}^{k} (-\lambda_j) \alpha_j \sum_{j=k+1}^{m} \mu_j \alpha_j > \geq 0$$

and so

$$< \sum_{i=1}^{\nu} \lambda_i \alpha_i \ , \ \sum_{i=1}^{\nu} \lambda_i \alpha_i \ > = \ 0$$

and

$$< \sum_{j=\nu+1}^{k} (-\lambda_j) \alpha_j \sum_{j=k+1}^{m} \mu_j \alpha_j \ , \ \sum_{j=\nu+1}^{k} (-\lambda_j) \alpha_j \sum_{j=k+1}^{m} \mu_j \alpha_j \ > = \ 0 \tag{2.1}$$

If $1 \leq \ell \leq \nu \leq k$, since the scalar product is negative definite on U, we have

$$\sum_{i=1}^{\nu} \lambda_i \alpha_i = 0 \ , \ 1 \leq \ell \leq \nu \leq k \ , \ \lambda_\ell > 0$$

and

$$0 = \sum_{i=1}^{\nu} \lambda_i < \ \alpha_i \ , \ \alpha \ > \ > \ 0$$

This is a contradiction. By (2.1) and since the scalar product is positive definite on W we have

$$\sum_{j=\nu+1}^{k}(-\lambda_j)\alpha_j + \sum_{j=k+1}^{m}\mu_j\alpha_j = 0\ ,\ \nu+1 \le \ell \le k\ ,\ \lambda_\ell\ >\ 0$$

or

$$\sum_{j=\nu+1}^{k}(\lambda_j)\alpha_j\ =\ \sum_{j=k+1}^{m}\mu_j\alpha_j,\ \nu+1 \le \ell \le k\ ,\ \lambda_\ell\ >\ 0$$

Hence by our assumption $< \alpha_i\ ,\ \alpha_j\ >\le 0$ and the scalar product is positive definite on W, we have

$$0 \le < \sum_{i=\nu+1}^{k}\lambda_i\alpha_i\ ,\ \sum_{i=\nu+1}^{k}\lambda_i\alpha_i\ >= \sum_{i=\nu+1}^{k}\sum_{j=k+1}^{m}\lambda_i\mu_j < \alpha_i\ ,\ \alpha_j\ >\le 0$$

and so

$$\sum_{i=\nu+1}^{k}\lambda_i\alpha_i = 0$$

and by our assumption $<\ \alpha\ ,\ \alpha_i\ >\ >\ 0$ and so

$$0 =< \sum_{i=\nu+1}^{k}\lambda_i\langle\alpha_i\ ,\ \alpha\ >\ >\ 0$$

This is a contradiction, so $\{\alpha_1\ ,\ \dots\ ,\ \alpha_m\}$ is a linearly independent set. ■

2.15. THEOREM. *If π is a t-base for Φ_t, then π is a basis for V.*

Proof. Since $\mathcal{G}$ is effective, by Lemma 2.6 Φ_t spans V. Since every $\alpha \in \Phi_t$ is a linear combination of roots in π, V is spanned by π. By Lemma 2.13 and Lemma 2.14, π is linearly independent, so π is a basis for V. ■

2.16. LEMMA. *There is only one t-base for Φ_t.*

Proof. It follows from Proposition 4.1.8 [5]. ■

In order to illustrate the concepts discussed so far, we give the following example:

2.17. EXAMPLE. The seven reflections in $\mathbf{R}^3_1$ generate $\mathcal{H}^6_2\ +\ \mathcal{A}_1$, where $\mathcal{H}^6_2$ is the dihedral group of order 12 and $\mathcal{A}_1$ is a cyclic group of order 2. The root system

$$\Phi = \{\ \pm(2,1,0), \pm(1,2,0), \pm(-1,-2,1), \pm(1,2,1), \pm(0,0,2), \pm(-\frac{1}{2},-1,\frac{3}{2}), \pm(\frac{1}{2},1,\frac{3}{2})\ \}$$

Choosing $t = (-1, 1, 6)$, we have

$$\Phi_t^+ = \{\ (2,1,0),(1,2,0),(-1,-2,1),(1,2,1),(0,0,2),(-\frac{1}{2},-1,\frac{3}{2}),(\frac{1}{2},1,\frac{3}{2})\ \}$$

$$\pi = \{\ (2,1,0),(1,2,0),(-1,-2,1)\ \}$$

2.18. LEMMA. *Let τ_i be the semi-Euclidean reflection along $\alpha_i \in \pi = \{\alpha_1, ..., \alpha_n\}$. If $\alpha \in \Phi_t^+$, with $\alpha \neq \alpha_i$, then $\tau_i\alpha \in \Phi_t^+$.*

Proof. If $\alpha \in \pi$, by Lemma 2.13 $\tau_i\alpha \in \Phi_t^+$. If $\alpha \notin \pi$, then $\alpha = \sum_{j=1}^{n} \lambda_j\alpha_j$ and at least two of the coefficients λ_j are positive; so we can assume that $\alpha_i \neq \alpha_1$ and that $\lambda_1 > 0$. Thus

$$\begin{aligned} \tau_i\alpha &= \sum_{j=1}^{n} \lambda_j\tau_i(\alpha_j) \\ &= \lambda_1\alpha_1 + \sum_{j=2}^{n} \lambda_j\alpha_j - 2\ (\ \textstyle\sum_{j=1}^{n} \lambda_j\ <\ \alpha_i, \alpha_j\ >\ )\ \alpha_i \end{aligned}$$

Since $\tau_i\alpha \in \Phi$, $\tau_i\alpha$ is either positive or negative. But it has at least one positive coefficient λ_1, we conclude that all coefficients are non-negative and so that $\tau_i\alpha \in \Phi^+$. ∎

2.19. DEFINITION. The semi-Euclidean roots $\alpha_1, ..., \alpha_n$ in the base π are called *simple semi – Euclidean roots*. The semi-Euclidean reflections $\tau_1, \tau_2, \ldots, \tau_n$ along the simple semi-Euclidean roots are called *simple semi–Euclidean'reflections* of $\mathcal{G}$.

We denote by $\mathcal{G}_t$ the subgroup $< \tau_i\ :\ 1 \leq i \leq n >$ of $\mathcal{G}$. It will be shown (Theorem 2.22) that $\mathcal{G}_t = \mathcal{G}$, that is, $\mathcal{G}$ is generated by its simple semi-Euclidean reflections.

2.20. LEMMA. *If $\alpha \in V$, there is a transformation $T \in \mathcal{G}_t$ such that $< T\alpha\ ,\ \alpha_i\ > \geq 0$ for all $\alpha_i \in \pi$.*

Proof. Let $\alpha_0 = \frac{1}{2} \sum_{\alpha\in\Phi^+} \alpha$. Since $\mathcal{G}_t$ is a finite group, it is possible to choose $T \in \mathcal{G}_t$ such that $< T\alpha\ ,\ \alpha_0\ >$ is maximal. If τ_i is the semi-Euclidean reflection along α_i, then by the preceding lemma we have:

$$\begin{aligned}\tau_i\alpha_0 &= \tau_i(\tfrac{1}{2}\alpha_i + \tfrac{1}{2}\sum\{\alpha \in \Phi^+ \ : \ \alpha \neq \alpha_i\}) \\ &= -\tfrac{1}{2}\alpha_i + \tfrac{1}{2}\sum\{\alpha \in \Phi^+ \ : \ \alpha \neq \alpha_i\} \\ &= \tfrac{1}{2}\sum\{\alpha \ : \ \alpha \in \Phi^+\} - \alpha_i \\ &= \alpha_0 - \alpha_i\end{aligned}$$

By the maximality of $< T\alpha \ , \ \alpha_0 \ >$ we have $< T\alpha \ , \ \alpha_0 \ > \geq < \tau_i T\alpha \ , \ \alpha_0 \ >$. On the other hand

$$\begin{aligned}< \tau_i T\alpha \ , \ \alpha_0 \ > &= (\ \epsilon\tau_i T\alpha \ , \ \alpha_0 \ ) \\ &= (\ \tau_i T\alpha, \epsilon\alpha_0 \ ) \\ &= (\ T\alpha \ , \ \tau_i{}^t\epsilon\alpha_0 \ ) \\ &= (\ T\alpha \ , \ \epsilon\tau_i\epsilon\epsilon\alpha_0 \ ) \\ &= (\ T\alpha, \epsilon\tau_i\alpha_0 \ ) \\ &= (\ \epsilon T\alpha \ , \ \tau_i\alpha_0 \ ) \\ &= < T\alpha \ , \ \tau_i\alpha_0 \ >\end{aligned}$$

Then we have

$$< T\alpha \ , \ \alpha_0 > \geq < T\alpha \ , \ \tau_i\alpha_0 > = < T\alpha \ , \ \alpha_0 - \alpha_i \ > = < T\alpha \ , \ \alpha_0 > \ - \ < T\alpha \ , \ \alpha_i >$$

2.21. LEMMA. *If $\alpha \in \Phi^+$, then $T\alpha \in \pi$ for some $T \in \mathcal{G}_t$.*

Proof. If $\alpha \in \pi$, we can choose $T = 1_{\mathcal{G}_t}$. If $\alpha \notin \pi$, then it follows Lemma 2.13, Lemma 2.14 and Theorem 2.15 that $< \alpha_{i_1} \ , \ \alpha \ > < 0$ or $< \alpha_{i_1} \ , \ \alpha \ > > 0$ for some semi-Euclidean root $\alpha_{i_1} \in \pi$; otherwise, $\pi \cup \{\alpha\}$ would be linearly independent. Let $a_1 = \tau_{i_1}\alpha = \alpha - 2\,\frac{< \alpha_{i_1} \ , \ \alpha \ >}{< \alpha_{i_1} \ , \ \alpha_{i_1} \ >}\alpha_{i_1}$. By Lemma 2.18 $a_1 \in \Phi^+$, and

$$< a_1 \ , \ t \ > = < \alpha \ , \ t \ > - 2\,\frac{< \alpha_{i_1} \ , \ \alpha \ >}{< \alpha_{i_1} \ , \ \alpha_{i_1} \ >} < \alpha_{i_1} \ , \ t \ >$$

If $a_1 \in \pi$, set $T = \tau_{i_1} \in \mathcal{G}_t$. If $a_1 \notin \pi$, apply the above process to a_1, obtaining $\alpha_{i_2} \in \pi$, and $a_2 = \tau_{i_2}(a_1) = \tau_{i_2}\tau_{i_1}\alpha \in \Phi^+ < a_2 \ , \ t \ > < < a_1 \ , \ t \ >$. If $a_2 \in \pi$, set $T = \tau_{i_2}\tau_{i_1} \in \mathcal{G}_t$; if $a_2 \notin \pi$, the process is continued. Since Φ^+ is finite, the process must terminate with some $a_k \in \pi$. Then $a_k = \tau_{i_k}...\tau_{i_1}\alpha$ and if we set $T = \tau_{i_k}...\tau_{i_1} \in \mathcal{G}_t$, then lemma is proved. ■.

2.22. THEOREM. *The simple semi-Euclidean reflections $\tau_1, \tau_2, \ldots, \tau_n$ generate $\mathcal{G}$, that is, $\mathcal{G}_t = \mathcal{G}$.*

Proof. Since $\mathcal{G} = < \tau_\alpha \ : \ \alpha \in \Phi \ >$ and since $\tau_{-\alpha} = \tau_\alpha$, it will bee sufficient to prove that if $\alpha \in \Phi^+$, then $\tau_\alpha \in \mathcal{G}_t$. Let $\alpha \in \Phi^+$. By the preceding lemma there is a transformation $T \in \mathcal{G}_t$ such that $T\alpha \in \pi$, say $T\alpha = \alpha_i$. By Lemma 2.2 we have $\tau_\alpha = T^{-1}\tau_i T \in \mathcal{G}_t$. ∎

We note that a reflection τ is a semi-Euclidean reflection if and only if $\tau\epsilon = \epsilon\tau$, that is, a reflection is not a semi-Euclidean reflection in general.

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