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Research Article

Rings Whose Units Commute with Zero Divisor Elements

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Abstract

This paper is about rings *R* for which units commute with zero divisors. These rings are called uni-zero divisor rings. It is deduced that uni-zero divisor ring is not abelian and also give several examples of some important classes of rings (a kind of integral matrices, matrix ring over any stably finite ring) that are not uni-zero divisor ring. Here many characterizations of uni-zero divisor rings are compared with well-known classes of rings. Moreover, the uni-zero divisor rings property is studied under some algebraic constructions.

Keywords: Units of rings, the set of zero divisor elements of ring, abelian ring.

Halkaların Birimleri ile Sıfır Bölen Elemanlarının Değişmeli Olması

Öz

Bu makale, R halkasının birimleri ile sıfır bölenlerinin değişmeli olması hakkındadır. Bu halkalara tek-sıfır bölen halkalar denir. Tek-sıfır bölen halkaların değişmeli olmadıklarını gösterildi, öte yandan halkaların bazı önemli sınıflarının (integral matrislerinin bir çeşidi, düzgün sonlu halkalar üzerine tanımlanmış matris halkaları) tek-sıfır bölen halka olamayacağına örnekler verildi. Bu çalışmada tek-sıfır bölen halkalarının birçok karakterizasyonu, iyi bilinen halka sınıflarıyla karşılaştırıldı. Ayrıca tek-sıfır bölen halkaların bazı cebir inşaaları üzerine çalışmalar yapıldı.

Anahtar Kelimeler: Birimli halkalar, halkalanın sıfır bölücü elemanları kümesi, değişmeli halkalar.

1. Introduction

The focus of this paper is on the rings R which is called uni-zero divisor ring if units commute with zero-divisors. In Example 12.7, Lam (1995), it is shown that rings whose idempotents commute with nilpotents or idempotents commute with units turn out to be precisely the so called Abelian rings. In Chebotar et al. (2009) and Khurana et al. (2010), authors considered rings with commuting units and rings with commuting nilpotents. The first were examined by many authors as rings with Abelian group of units. Commutativity of rings with abelian or solvable units is also study in (Nicholson and Springer, 1976). Recently, rings in which units commute with nilpotents were studied, under the name of uni rings (see Calugareanu (2018)). This study leads to our motivation of the class of rings in which units commute with zero divisors. In this study we investigate classes of rings in which elements of the set of units and elements of the set of zero divisors commute.

In section 2, we determine that ring with commuting zero divisors which is not Abelian and Dedekind finite rings may not have only commuting zero-divisors. At the end of the section it is shown by an example that commuting zero-divisors are not uni-zero divisor ring. In section 3, some basic properties of uni-zero divisor rings are proved. Also, it is deduced that the corner rings of a uni-zero divisor ring are again uni-zero divisor. In Proposition 3.5, the uni-zero divisor property of the Dorroh extension is investigated and this section ends with the trivial extention *T* (*R*, *M*), we prove that the trivial extension *T* (*R*, *M*) of *R* by *M* is a uni-zero divisor ring if and only if *R* is uni-zero divisor and *um* = *mu* for all $u \in U(R)$ and $m \in M$. For the last section, various characterizations of uni-zero divisor rings are obtained. In Proposition 4.1 we proved that a ring *R* is uni-zero divisor if and only if the localization of *R* at *S* is uni-zero divisor. By taking into consideration of this proposition, it is also obtained a sufficient and necessary condition for the Laurent polynomial ring for a ring *R* to be uni-zero divisor ring. Moreover, some new equivalent conditions of uni-zero divisor ring are presented.

Throughout, *R* is an associative ring with unity. The Jacobson radical and the center of *R* are denoted by *J*(*R*) and *C*(*R*). The group of units, the set of idempotents, the set of nilpotents and the set of zero divisors of *R* are denoted by *U*(*R*), *Id*(*R*), *N*(*R*) and *Z*(*R*), respectively.

2. Examples

By taking into consideration of the Example 2.5 in Khurana et al. (2010), we first show that uni-zero divisor ring or ring with commuting zero divisors which is not Abelian. Let R = $\begin{pmatrix} F_2 & V \\ 0 & F_2 \end{pmatrix}$ where V is any nonzero F_2-vector space. Then R is a semiprimary ring with commuting units which is not Abelian and unit central.

Proof. Let u_1, u_2 be two vectors in V, then $\begin{pmatrix} 1 & u_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & u_1 \\ 0 & 0 \end{pmatrix}$.

However, R is a uni-zero divisor ring, $\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}$ so with commuting zero divisors.

By this example it is shown that commuting units which is not unit central.

Now let R = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: a ≡ d (mod 2), b ≡ c ≡ 0 (mod 2), be an integral matrices. Since R has only trivial idempotents, R is Abelian. But $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in$ Z(R), $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \in$ U(R) do not commute so R is not uni-zero divisor ring. Also $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in$ Z(R), do not commute.

Any matrix ring over any stably finite ring is Dedekind finite but may not have only commuting zero-divisors because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Commuting zero-divisors are not uni-zero divisor ring. For 2×2 upper triangular matrices over a reduced ring, T_2(R). Then

Z(T_2(R)) = $\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ is a commuting zero-divisors. But it is not uni-zero divisor, since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. Units with commuting zero-divisors

A ring with identity R is called uni-zero divisor if $zu = uz$ for every $u \in U(R)$ and every $z \in Z(R)$. In such rings $U(R)Z(R) = Z(R)U(R) = Z(R)$. The followings hold by the definitions.

Proposition 3.1. (i) Commutative rings are uni-zero divisor.

(ii) Matrix rings are not uni-zero divisor. Therefore uni-zero divisor does not hold for Morita invariant feature.

(iii) Products of uni-zero divisor rings are uni-zero divisor.

Remarks 3.2. Let R be a ring such that $R - U(R) = Z(R)$. Then

(i) if R is uni-zero divisor, then R is zero-divisor central;

(ii) if R has commuting units then R is unit-central.

Proposition 3.3. Let R be a ring and e ∈ R an idempotent. If R is uni-zero divisor, then the corner ring eRe is uni-zero divisor.

Proof. Suppose R is uni-zero divisor with ere ∈ U(eRe) and eze ∈ Z(eRe). Then ere + (1 − e) is contained in U(R) (indeed, if (ere)v = v(ere) = e with v ∈ eRe, then (ere+e)(v+e) = (v+e)(ere+e) = e + e = 1), so it commutes with eze ∈ Z(R), and hence ere and eze commute.

Example 3.4. Let R =$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is any field with at least three elements and $0 \neq u_1 \neq 1$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is zero-divisor, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} = \begin{pmatrix} 0 & u_1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, R is not uni-zero divisor. For an ideal I=$\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ of R. So R/I is commutative and so uni-zero divisor. Hence for an ideal I of R, even R/I is uni-zero divisor, R may not be uni-zero divisor.

The Dorroh Extension. Given a ring R and a ring without identity I, we will say that I is an R ring without identity if it is an (R, R)-bimodule, for which the actions of R are compatible with the multiplication in I (i.e. r(ij) = (ri)j, i(rj) = (ir)j and (ij)r = i(jr), for every r ∈ R and i, j ∈ I). If R is a ring with identity and I is a ring without identity, then one can turn the abelian group R ⊕ I into a ring, by defining the multiplication by

$$(r, i).(p, j) = (r p, i p + r j + i j),$$

for $r, p \in R$ and $i, j \in I$. Such a ring is called an ideal extension (it is also called the Dorroh extension), and denoted by $E(R, I)$ - see (Mesyan, 2010)

Proposition 3.5. A ring R is uni-zero divisor if and only if the Dorroh extension E(R, I) of R is uni-zero divisor.

Proof. By Alwis (1994), we have $Z(E(R,I))=\{(0,k):k\in I\} \cup \{(r,-r): r\in R\}$ and $(u,0)\in U(E(R,I))$ for $u \in U(R)$. Hence the claim follows.

The Trivial extension. Let R be a ring and M a bimodule over R. The trivial extension of R and M is

$$T(R, M) = \{(r, m) : r \in R \text{ and } m \in M\}$$

with an addition defined componentwise and a multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The trivial extension $T(R, M)$ is isomorphic to the subring

$\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M\}$ of the formal 2 × 2 matrıx rıng $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and also

$T(R,R) \equiv R[x]/(x^2)$. We also note that the set of units of trivial extension $T(R, M)$ is

$$U(T(R, M)) = T(U(R), M)$$

by [Anderson et al. (2017), Proposition 4.9 (2)] and

$$J(T(R, M)) = T(J(R), M)$$

by [Anderson et al. (2017), Corollary 4.8 (2)].

Let R be a uni-zero divisor ring and M an (R, R)-bimodule. Then the trivial extension $T(R, M)$ of R by M need not be a uni-zero divisor ring.

Proposition 3.6. *Let R be a ring and M an (R, R)-bimodule. The trivial extension T (R, M) of R by M is a uni-zero divisor ring if and only if R is uni-zero divisor and um = mu for all u ∈ U(R) and m ∈ M.*

Proof. Clear.

4. Some Extensions of uni-zero divisor rings

This section considers on some ring extensions of uni-zero divisor rings. Let S denote a multiplicatively closed subset of a ring R consisting of central nonzero-divisors. Let $S^{-1}R = \{ab - 1; a \in R,\ b \in S\}$ be the localization of R at S. Also every non-zero element $\frac{a}{s}$ is a unit in $S^{-1}R$, with inverse $\frac{a}{s}$.

Proposition 4.1. A ring R is uni-zero divisor if and only if $S^{-1}R$ is uni-zero divisor.

Proof. Suppose that $ab^{-1} \in S^{-1}R$ is a zero divisor in $S^{-1}R$ where $a, b \in R$. So there exists $0 \neq cd^{-1} \in S^{-1}R$ such that $ab^{-1}cd^{-1} = 0$. Since bd is invertible, $ac = 0$. As R is uni-zero divisor ring, zero divisor element a of R commutes with units of R such that $au = ua$ for $u \in U(R)$. Then we have $ab^{-1}ud^{-1} = ud^{-1}ab^{-1}$ for $ud^{-1} \in U(S^{-1}R)$. Hence $S^{-1}R$ is uni-zero divisor.

For the converse, let $a \in R$ be a zero divisor element of R so there exists $0 \neq b \in R$ such that $ab = 0$. So $0 = (a1^{-1})(b1^{-1}) \in S^{-1}R$. Since $S^{-1}R$ is uni-zero divisor ring, zero divisor element $(a1^{-1})$ of $S^{-1}R$ commutes with units of $S^{-1}R$ such that $(a1^{-1})(u1^{-1}) = (u1^{-1})(a1^{-1})$ for $u1^{-1} \in U(S^{-1}R)$ and so $au = ua$. Hence R is uni-zero divisor ring.

We write $R[x]$ and $R[x; x^{-1}]$ for the polynomial ring and the Laurent polynomial ring for a ring R, respectively.

Proposition 4.2. Let R be a ring. R[x] is uni-zero divisor if and only if $R[x; x^{-1}]$ is uni-zero divisor ring.

Proof. Let S $\{1, x, x^2, \cdot\cdot\cdot\}$, so S is a multiplicatively closed subset of R[x]. since

$R[x; x^{-1}] = S^{-1}R[x]$, we can observe that $R[x]$ is uni-zero divisor ring if and only if $R[x; x^{-1}]$ is uni-zero ring by the Proposition 4.1.

Proposition 4.2. Let R be a ring. If R is unit fussible, then it is commutative.

Proof. Let $a \in R$, since R is unit fussible ring there exists a unit u and a zero-divisor z exists in R such that $a = u + z$. Since commuting units implies uni-zero divisors and commuting zero-divisors $ab = ba$ for any $a, b \in R$.

An element a in any ring R is said to be von Neumann regular if there exists some $b \in R$ such that $a = aba$. We call a ring R von Neumann regular if every element of R is von Neumann regular.

Theorem 4.4. The following are equivalent for a ring R:

- (i) R is strongly (von Neumann) regular;
- (ii) R is (von Neumann) regular and reduced;
- (iii) R is von Neumann regular and Abelian; (iv) every principal left ideal of R is generated by a central idempotent;
- (iv) R is (von Neumann) regular and uni-zero divisor.

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