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RATE of EQUI-CONVERGENCE of SOME SERIES ASSOCIATED WITH FOURIER SERIES and CERTAIN INTEGRALS

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ABSTRACT

Salem and Zygmund have studied the convergence problem of factored Fourier series and conjugate series the factor being n^γ , $0 < \gamma < 1$. In the present work we study the rate of convergence of the some series for functions belonging to $Lip(w, p)$, $p \geq 1$ class.

AMS subject Classification (2000) : 42A24, 41A25

Keywords: Equi-convergence, monotonic type, $Lip(w, p)$.

FOURIER SERİLERİ VE BELİRLİ İNTEGRALLER İLE ALAKALI BAZI SERİLERİN EŞ-YAKINSAMA ORANLARI

ÖZET

Salem ve Zygmund, carpanlı Fourier serilerinin ve carpanı n^γ , $0 < \gamma < 1$ olan eşlenik serilerin yakınsaklık problemini çalışmışlardır. Şimdiki çalışmada, biz $Lip(w, p)$, $p \geq 1$ sınıfına ait fonksiyonlar için bazı serilerin yakınsaklık oranlarını inceleyeceğiz.

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1. DEFINITIONS

Let f be a 2π - periodic and Lebesgue integrable function over $(-\pi, \pi)$. The Fourier series of f at x is given

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x) \quad (1.1)$$

The series conjugate to (1.1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = \sum_{n=0}^{\infty} B_n(x). \quad (1.2)$$

We write

$$\phi(x, t) = f(x+t) + f(x-t) - 2f(x)$$

$$\psi(x, t) = f(x+t) - f(x-t)$$

$$\tilde{f}(x) = -\frac{1}{\pi} \int_0^{\pi} \psi(x, t) \frac{1}{2} \cot \frac{1}{2} t dt$$

whenever the integral exists in Cauchy sense at the origin. Let w be a modulus of continuity. If

$$\|f(\cdot+t) - f(\cdot)\|_p = \begin{cases} O(w(|t|)) \\ 0(w(|t|)) \end{cases}$$

then respectively we say that [1] $f \in Lip(w, p)$, $p \geq 1$, $f \in lip(w, p)$, $p \geq 1$.

If $w(t) = t^\alpha$, $0 < \alpha \leq 1$ then $Lip(w, p)$, $p \geq 1$ reduces to familiar $Lip(\alpha, p)$, $p \geq 1$ class.

A function f is called of monotonic type ([2], p.33) if for some constant C the function $f(x) + Cx$ is either monotonic increasing or monotonic decreasing.

For $0 < \gamma < 1$ and $0 < \epsilon < \infty$, we write

$$I_\gamma(x; \epsilon) = -\frac{1}{\pi} \Gamma(\gamma+1) \cos \frac{\pi\gamma}{2} \int_\epsilon^\infty \frac{\psi(x, t)}{t^{1+\gamma}} dt$$

$$J_\gamma(x; \epsilon) = -\frac{1}{\pi} \Gamma(\gamma+1) \sin \frac{\pi\gamma}{2} \int_\epsilon^\infty \frac{\phi(x, t)}{t^{1+\gamma}} dt.$$

We define

$$I_\gamma(x) = \lim_{\epsilon \rightarrow 0+} I_\gamma(x; \epsilon) \quad \text{and} \quad J_\gamma(x) = \lim_{\epsilon \rightarrow 0+} J_\gamma(x; \epsilon)$$

whenever the limits exist. Let $C_{2\pi}$ denote the class of 2π -periodic continuous functions.

2. INTRODUCTION

Results on equi-convergence of conjugate series $\sum_{n=1}^{\infty} B_n(x)$ and the conjugate integral $\tilde{f}(x)$ can be found in Salem and Zygmund ([2], p.33) when they have studied the equi-convergence of the series $\sum n^\gamma B_n(x)$ and $\sum n^\gamma A_n(x)$, $0 < \gamma < 1$ respectively with the integrals $I_\gamma(x)$ and $J_\gamma(x)$. Their results read as follows.

Theorem A ([2] p.33). Suppose that $f \in C_{2\pi}$. Let f be a function of bounded variation and of monotonic type. If $f \in \text{lip}\gamma$, $0 < \gamma < 1$, then the difference

$$\alpha_n(x) = I_\gamma\left(x; \frac{1}{n}\right) - \sum_{k=1}^n k^\gamma B_k(x) \quad (2.1)$$

tends to zero uniformly in x as $n \rightarrow \infty$. If $f \in \text{lip}\gamma$, $0 < \gamma < 1$ the difference is uniformly bounded.

Theorem B ([2], p.34). Suppose that $f \in C_{2\pi}$. Let f be function of bounded variation and of monotonic type. If $f \in \text{lip}\gamma$, $0 < \gamma < 1$ then the difference

$$\beta_n(x) = J_\gamma\left(x; \frac{1}{n}\right) - \sum_{k=1}^n k^\gamma A_k(x) \quad (2.2)$$

tends to zero uniformly in x as $n \rightarrow \infty$. If $f \in \text{Lip}\gamma$, $0 < \gamma < 1$, the difference is uniformly bounded.

In this present work we study the rate of equi-convergence of the sequence $\alpha_n(x)$ and $\beta_n(x)$.

3. MAIN RESULTS

We prove the following theorems:

Theorem 1. Suppose that $f \in C_{2\pi} \cap \text{Lip}(w, p)$, $p \geq 1$. If further f is of monotonic type, then for $0 < \gamma < 1$.

$$\|\alpha_n(\cdot)\|_p = O(1) \frac{1}{n^{1-\gamma}} \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \quad (3.1)$$

where $\alpha_n(x)$ is defined in (2.1).

Theorem 2. Suppose that $f \in C_{2\pi} \cap Lip(w, p)$, $p \geq 1$. If further f is of monotonic type, then for $0 < \gamma < 1$.

$$\|\beta_n(\cdot)\|_p = O(1) \frac{1}{n^{1-\gamma}} \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \quad (3.2)$$

where $\beta_n(x)$ is defined in (2.2).

Remark. Das, Nath and Ray [1] have obtained the following result on the degree of approximation of functions in $Lip(w, p)$ class by their Fourier series.

Theorem C. If $f \in Lip(w, p)$, $p \geq 1$ and is of monotonic type, then

$$\|s_n(f, x) - f\|_p = O(1) \frac{1}{n^{1/n}} \int_{\pi/n}^{\pi} \frac{w(t)}{t^2} dt$$

where $s_n(f, x)$ is the n th partial sum of (1.1) and w is a modulus of continuity.

4. LEMMAS.

For the proof of Theorem 1 we need the following notations and lemmas.

$$g(x, t, u) = \Psi(x, t+u) + \Psi(x, t-u) - 2\Psi(x, t) \quad (4.1)$$

$$m = \left[\frac{\log n}{\log 2} \right], \eta = \frac{1}{2^m} \quad (4.2)$$

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^n \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin(t/2)} \quad (4.3)$$

We know ([3], vol. I, p.50) that

$$\Psi(x, t) \sim -2 \sum_{k=1}^{\infty} B_k(x) \sin kt \quad (4.4)$$

$$S_n(x, t) = -\sum_{k=1}^n 2 \sin kt B_k(x). \quad (4.5)$$

Lemma1. Let $\psi(x, t)$ and $g(x, t, u)$ be defined as above. If $f \in C_{2\pi} \cap Lip(w, p)$, $p \geq 1$, then

$$\begin{aligned} \text{(i)} \quad & \|g(x, t, u)\|_p = O(w(u)) \\ \text{(ii)} \quad & \|\psi(x, t+u) - \psi(x, t-u)\|_p = O(w(u)) \end{aligned}$$

Proof of (i). By Minkowski's inequality

$$\begin{aligned} \|g(x, t, u)\|_p &= \|\psi(x, t+u) + \psi(x, t-u) - 2\psi(x, t)\|_p \\ &\leq \|f(x+t+u) - f(x+t)\|_p + \|f(x-t) - f(x-t-u)\|_p \\ &\quad + \|f(x+t-u) - f(x+t)\|_p + \|f(x-t) - f(x-t+u)\|_p \\ &= O(w(u)) \end{aligned}$$

which proves Lemma 1 (i).

Proof of (ii). By Minkowski's inequality

$$\begin{aligned} \|\psi(x, t+u) - \psi(x, t-u)\|_p &\leq \|f(x+t+u) - f(x+t)\|_p + \|f(x-t) - f(x-t+u)\|_p \\ &\quad + \|f(x+t) - f(x+t-u)\|_p + \|f(x-t+u) - f(x-t)\|_p \\ &= O(w(u)) \end{aligned}$$

which proves Lemma 1 (ii).

Lemma 2. If the hypotheses of Theorem 1 holds, then

$$\|S_n(\cdot, t) - \psi(\cdot, t)\|_p = O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du.$$

Proof. Now after simplification

$$\begin{aligned} S_n(x, t) &= -\sum_{k=1}^n 2 \sin kt \left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x, u) \sin kudu \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x, u) \sum_{k=1}^n [\cos k(u-t) - \cos k(u+t)] du \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \int_0^\pi [\Psi(x, t+u) + \Psi(x, t-u)] D_n(u) du \\
 &= \frac{1}{\pi} \int_0^\pi g(x, t, u) D_n(u) du + \Psi(x, t)
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 S_n(x, t) - \Psi(x, t) &= \frac{1}{\pi} \left(\int_0^n + \int_\eta^\pi \right) g(x, t, u) D_n(u) du \\
 &= \frac{1}{\pi} [I(x, t) + J(x, t)]
 \end{aligned} \tag{4.6}$$

For the first integral using Lemma 1 (i), we get

$$\begin{aligned}
 \|I(., t)\|_p &= \int_0^\eta O(w(u)) |D_n(u)| du \\
 &= O(1) w(\eta) n \eta = O(1) w\left(\frac{1}{n}\right).
 \end{aligned} \tag{4.7}$$

Since the function f is of monotonic type there exists a constant C so that $F(x) = f(x) + Cx$ is either never decreasing or never increasing in $(-\infty, \infty)$. Without loss of generality, we may assume that F is increasing as in the case when F is decreasing the problem can be dealt with in a similar manner.

Replacing $f(x)$ by $F(x) - Cx$, we get

$$\begin{aligned}
 g(x, t, u) &= [F(x+t+u) - F(x+t)] - [F(x-t+u) - F(x-t)] \\
 &\quad + [F(x-t) - F(x-t-u)] - [F(x+t) - F(x+t-u)].
 \end{aligned} \tag{4.8}$$

Using (4.8), we get

$$\begin{aligned}
 J(x, t) &= \int_\eta^\pi [F(x+t+u) - F(x+t)] D_n(u) du + \int_\eta^\pi [F(x-t) - F(x-t-u)] D_n(u) du \\
 &\quad - \int_\eta^\pi [F(x-t+u) - F(x-t)] D_n(u) du \\
 &\quad - \int_\eta^\pi [F(x+t) - F(x+t-u)] D_n(u) du \\
 &= J_1(x, t) + J_2(x, t) - J_3(x, t) - J_4(x, t), \text{ (say)}
 \end{aligned} \tag{4.9}$$

We write

$$\begin{aligned}
 J_1(x, t) &= \sum_{j=1}^m \int_{\pi/2^j}^{\pi/2^{j-1}} [F(x+t+u) - F(x+t)] D_n(u) du \\
 &= \sum_{j=1}^m Q_j(x, t)
 \end{aligned} \tag{4.10}$$

where

$$Q_j(x, t) = \int_{\pi/2^j}^{\pi/2^{j-1}} [F(x+t+u) - F(x+t)] \frac{\sin\left(n + \frac{1}{2}\right)u}{2 \sin u/2} du. \tag{4.11}$$

As $F(x+t+u) - F(x+t)$ is increasing in u and $\frac{1}{2\sin(u/2)}$ is decreasing in u ,

by repeated application of Mean-Value Theorem, we get

$$\begin{aligned} Q_j(x, t) &= \frac{1}{2\sin \frac{1}{2^{j+1}}} \int_{1/2^j}^{\xi} [F(x+t+u) - F(x+t)] \sin\left(n + \frac{1}{2}\right) u du \left(\frac{1}{2^j} < \xi < \frac{1}{2^{j-1}}\right) \\ &= \frac{1}{2\sin \frac{1}{2^{j+1}}} [F(x+t+\zeta) - F(x+t)] \int_{\zeta_1}^{\zeta} \sin\left(n + \frac{1}{2}\right) u du \left(\frac{1}{2^j} < \zeta_1 < \zeta\right) \\ &= \frac{F(x+t+\zeta) - F(x+t)}{2\sin \frac{1}{2^{j+1}}} \left[\frac{\cos\left(n + \frac{1}{2}\right) \zeta_1 - \cos\left(n + \frac{1}{2}\right) \zeta}{\left(n + \frac{1}{2}\right)} \right], \end{aligned}$$

from which it follows that for $j = 1, 2, \dots, m$

$$\begin{aligned} \|Q_j(\cdot, t)\|_p &\leq \frac{2}{\left(n + \frac{1}{2}\right)} \frac{\|F(\cdot, t+\zeta) - F(\cdot, t)\|_p}{2 \cdot \frac{1}{\pi 2^{j+1}}} \left(\sin u \geq \frac{2u}{\pi}\right) \\ &= O(1) \frac{2^j}{n} w(\zeta) = O(1) 2^j w\left(\frac{1}{2^{j-1}}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \|J_1(\cdot, t)\|_p &\leq \sum_{j=1}^m \|Q_j(\cdot, t)\|_p \\ &= O(1) \frac{1}{n} \sum_{j=1}^m 2^j w\left(\frac{1}{2^{j-1}}\right) \\ &= O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u) du}{u^2}. \end{aligned} \quad (4.12)$$

Similarly we prove that

$$\|J_k(\cdot, t)\|_p = O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u) du}{u^2} \quad \text{for } k = 2, 3, 4 \quad (4.13)$$

Using (4.12) and (4.13), we obtain

$$\|J(\cdot, t)\|_p \leq \sum_{k=1}^4 \|J_k(\cdot, t)\|_p = O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u) du}{u^2} \quad (4.14)$$

Using (4.3) and (4.14) in (4.6), we get

$$\|S_n(\cdot, t) - \Psi(\cdot, t)\|_p = O(1) w\left(\frac{1}{n}\right) + O(1) \int_{\pi/n}^{\pi} \frac{w(u) du}{u^2}$$

$$= O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u) du}{u^2}$$

since by the monotonicity of w ,

$$\int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \geq w(\pi/n) \int_{\pi/n}^{\pi} \frac{du}{u^2} = w(\pi/n) \left(\frac{n-1}{\pi} \right).$$

Lemma 3. Let the hypotheses of Theorem 1 hold. Then

$$\|S'_n(.,t)\|_p = \left\| \frac{\partial}{\partial t} (S_n(.,t)) \right\|_p = O(1) \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du.$$

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial t} (S_n(x,t)) &= - \sum_{k=1}^n 2k \cos kt B_k(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(x,u) \sum_{k=1}^n k [\sin k(u+t) - \sin k(u-t)] du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(x,u) \frac{\partial}{\partial u} D_n(u+t) du - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(x,u) \frac{\partial}{\partial u} D_n(u-t) du \\ &= -\frac{1}{\pi} \left[\int_0^{\eta} + \int_{\eta}^{\pi} \right] [\Psi(x,t+u) + \Psi(x,t-u)] D'_n(u) du \\ &= -\frac{1}{\pi} [L(x,t) + K(x,t)]. \end{aligned} \quad (4.15)$$

Using Lemma 1 (ii), we get

$$\begin{aligned} \|L(.,t)\|_p &\leq \int_0^{\eta} \|\Psi(x,t+u) - \Psi(x,t-u)\|_p |D'_n(u)| du \\ &= O(1) w(u) \int_0^{\eta} n^2 du = O(1) w(\eta) n^2 \eta \\ &= O(1) n w\left(\frac{1}{n}\right) \end{aligned} \quad (4.16)$$

Replacing $f(x)$ by $Fx - Cx$, we get, after rearrangement,

$$\begin{aligned} K(x,t) &= \int_{\eta}^{\pi} [F(x+t+u) - F(x-t-u) + F(x-t+u) - F(x+t-u) - 4Cu] D'_n(u) du \\ &= \int_{\eta}^{\pi} [F(x+t+u) - F(x+t)] D'_n(u) du \\ &\quad + \int_{\eta}^{\pi} [F(x-t+u) - F(x-t)] D'_n(u) du + \int_{\eta}^{\pi} [F(x-t) - F(x-t-u)] D'_n(u) du \\ &\quad + \int_{\eta}^{\pi} [F(x+t) - F(x+t-u)] D'_n(u) du - 4C \int_{\eta}^{\pi} u D'_n(u) du \\ &= \sum_{i=1}^4 K_i(x,t) - 4C \int_{\eta}^{\pi} u D'_n(u) du. \end{aligned} \quad (4.17)$$

We write

$$K_1(x, t) = \sum_{j=1}^m P_j(x, t)$$

where

$$P_j(x, t) = \int_{\pi/2^j}^{\pi/2^{j-1}} [F(x+t+u) - F(x+t)] D'_n(u) du. \quad (4.18)$$

As $F(x+t+u) - F(x+t)$ is increasing in u , by using Mean-Value theorem, we have, for $\pi/2^j < \zeta < \pi/2^{j-1}$

$$\begin{aligned} P_j(x, t) &= \int_{\pi/2^j}^{\zeta} [F(x+t+u) - F(x+t)] D'_n(u) du \\ &= \left[F\left(x+t+\frac{1}{2^{j-1}}\right) - F(x+t) \right] \left(D_n(\zeta) - D_n\left(\frac{\pi}{2^j}\right) \right) \end{aligned}$$

which ensures that

$$\begin{aligned} \|P_j(., t)\|_p &= O(1) w\left(\frac{1}{2^{j-1}}\right) \left| D_n(\zeta) - D_n\left(\frac{\pi}{2^j}\right) \right| \\ &= O(1) 2^j w\left(\frac{1}{2^{j-1}}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \|K_1(., t)\|_p &\leq \sum_{j=1}^m \|P_j(., t)\|_p = O(1) \sum_{j=1}^m 2^j w\left(\frac{1}{2^{j-1}}\right) \\ &= O(1) \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \end{aligned} \quad (4.19)$$

Similarly we can prove that, for $i=2,3,4$

$$\|K_i(., t)\|_p = O(1) \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du. \quad (4.20)$$

Integrating by parts, we get

$$\begin{aligned} 4C \int_{\eta}^{\pi} u D'_n(u) du &= 4C \left[\pi (D_n(\pi) - \eta D_n(\eta)) \right] - \int_{\eta}^{\pi} D_n(u) du \\ &= O(1) \end{aligned} \quad (4.21)$$

Using (4.20) and (4.21) in (4.17), we get

$$\|K(., t)\|_p = O(1) \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \quad (4.22)$$

Using (4.16) and (4.22) in (4.15), we get

$$\left\| \frac{\partial}{\partial t} S_n(., t) \right\|_p = O(1) \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du$$

which proves Lemma 3.

Lemma 4. $([2], p.31)$.

$$\int_0^\infty \frac{\sin kt}{t^{1+\alpha}} dt = \frac{\pi k^\alpha}{2 \cos(\pi\alpha/2) \Gamma(\alpha+1)}$$

5. PROOF OF THEOREM 1.

We have from (4.5) and Lemma 4,

$$\begin{aligned} \int_0^\infty \frac{S_n(x, t)}{t^{1+\gamma}} dt &= - \sum_{k=1}^n 2B_k(x) \int_0^\infty \frac{\sin kt}{t^{1+\gamma}} dt \\ &= - \frac{\pi}{\cos \frac{\pi\gamma}{2} \Gamma(\gamma+1)} \sum_{k=1}^n k^\gamma B_k(x) \end{aligned} \quad (5.1)$$

Now from (2.1) and (5.1)

$$\begin{aligned} I_\gamma\left(x, \frac{1}{n}\right) - \sum_{k=1}^n k^\gamma B_k(x) &= \frac{-\Gamma(\gamma+1) \cos \frac{\pi\gamma}{2}}{\pi} \left[\int_{1/n}^\infty \frac{\Psi(x, t)}{t^{1+\gamma}} dt - \int_0^\infty \frac{S_n(x, t)}{t^{1+\gamma}} dt \right] \\ &= \frac{\Gamma(\gamma+1) \cos \frac{\pi\gamma}{2}}{\pi} \left[\int_{1/n}^\pi \frac{S_n(x, t) - \Psi(x, t)}{t^{1+\gamma}} dt + \int_0^{1/n} \frac{S_n(x, t)}{t^{1+\gamma}} dt \right] \end{aligned} \quad (5.2)$$

Using Lemma 2, we get

$$\begin{aligned} \left\| \int_{1/n}^\infty \frac{S_n(., t) - \Psi(., t)}{t^{1+\gamma}} dt \right\|_p &\leq \int_{1/n}^\infty \frac{\|S_n(., t) - \Psi(., t)\|}{t^{1+\gamma}} dt \\ &= O(1) \frac{1}{n} \int_{\pi/n}^\pi \frac{w(u) du}{u^2} \int_{1/n}^\infty \frac{dt}{t^{1+\gamma}} \\ &= O(1) \frac{1}{n^{1-\gamma}} \int_{\pi/n}^\pi \frac{w(u) du}{u^2}. \end{aligned} \quad (5.3)$$

As $S_n(x, 0) = 0$, we get $S_n(x, t) = S_n(x, t) - S_n(x, 0) = t S'_n(x, \theta)$, $0 < \theta < t$.

Therefore, by Lemma 3,

$$\begin{aligned} \left\| \int_0^{1/n} \frac{S_n(., t)}{t^{1+\gamma}} dt \right\|_p &\leq \int_0^{1/n} \frac{\|S_n(., t)\|}{t^{1+\gamma}} dt \\ &= O(1) \left(\int_{\pi/n}^\pi \frac{w(u)}{u^2} du \right) \int_0^{1/n} \frac{dt}{t^\gamma} \\ &= O(1) \frac{1}{n^{1-\gamma}} \int_{\pi/n}^\pi \frac{w(u)}{u^2} du. \end{aligned} \quad (5.4)$$

Using (5.3) and (5.4) in (5.2), we get

$$I_\gamma\left(x, \frac{1}{n}\right) - \sum_{k=1}^n k^\gamma B_k(x) = O(1) \frac{1}{n^{1-\gamma}} \int_{\pi/n}^\pi \frac{w(u)}{u^2} du$$

which completes the proof of Theorem 1.

By taking $w(t) = t^\alpha, 0 < \alpha \leq 1$, in Theorem 1, we obtain the following corollaries:

Corollary 1. If $f \in C_{2\pi} \cap Lip(\alpha, p), p \geq 1, 0 < \alpha \leq 1$ and f is of monotonic type then

$$\|\alpha_n(\cdot)\|_p = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma}}, 0 < \gamma \leq \alpha < 1 \\ \frac{\log n}{n^{1-\gamma}}, 0 < \gamma < \alpha = 1 \end{cases}$$

Putting $p = \infty$, in Corollary 1, we get

Corollary 2. If $f \in C_{2\pi} \cap Lip\alpha, 0 < \alpha \leq 1$ and f is of monotonic type then

$$\|\alpha_n(\cdot)\|_c = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma}}, 0 < \gamma \leq \alpha < 1 \\ \frac{\log n}{n^{1-\gamma}}, 0 < \gamma < \alpha = 1 \end{cases}$$

Note: In Corollary 2 in the special case when $\gamma = \alpha$ the result reduces to second part of Theorem A due to Salem and Zygmund.

6. ADDITIONAL NOTATIONS AND LEMMAS FOR THEOREM 2

We Write

$$\begin{aligned} \phi^*(x, t) &= f(x+t) + f(x-t) \\ \phi(x, t) &= \phi^*(x, t) - 2f(x) \\ h(x, t, u) &= \phi(x, t+u) + \phi(x, t-u) - 2\phi(x, t) \\ m &= \left\lceil \frac{\log n}{\log 2} \right\rceil, \eta = \frac{1}{2^m} \end{aligned}$$

We know ([4] Vol. I, p. 50) that

$$\phi^*(x, t) \sim 2 \sum_{n=1}^{\infty} A_n(x) \cos nt \quad (6.1)$$

$$\tilde{S}_n(x, t) = -2 \sum_{k=1}^n (1 - \cos kt) A_k(x). \quad (6.2)$$

We need the following additional lemmas for the proof of Theorem 2.

Lemma 5. Let $\phi(x, t)$ and $h(x, t, u)$ be defined as above, If

$$f \in C_{2\pi} \cap Lip(w, p), \quad p \geq 1$$

then

$$\begin{aligned} \text{(i)} \quad & \|h(x, t, u)\|_p = O(w(u)) \\ \text{(ii)} \quad & \|\phi(x, t+u) - \phi(x, t-u)\|_p = O(w(u)). \end{aligned}$$

Proof:

(i). By Minikowski's inequality

$$\|h(x, t, u)\|_p = \|\phi(x, t+u) + \phi(x, t-u) - 2\phi(x, t)\|_p = O(w(u))$$

proceeding as in Lemma 1 (i)

(ii). By Minkowski's inequality

$$\begin{aligned} \|\phi(x, t+u) + \phi(x, t-u)\|_p &= \|f(x+t+u) + f(x-t-u) - 2f(x) - f(x+t-u) - f(x-t+u) + 2f(x)\|_p \\ &= O(w(u)). \end{aligned}$$

proceeding as in Lemma 1 (ii)

Lemma 6. Under the hypotheses of Theorem 2

$$\|\tilde{S}_n(., t) - \phi(., t)\|_p = O(1)\omega(\pi/n)$$

where $\tilde{S}_n(x, t)$ is defined in (6.2).

Proof. We have

$$\begin{aligned} \tilde{S}_n(x, t) &= -2 \sum_{k=1}^n (1 - \cos kt) \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^*(x, u) \cos ku \, du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^*(x, u) \sum_{k=1}^n \{\cos k(u-t) + \cos k(u+t) - 2\cos ku\} \, du \\ &= \frac{1}{\pi} \int_0^{\pi} [\phi^*(x, t+u) + \phi^*(x, t-u) - 2\phi^*(x, u)] D_n(u) \, du \\ &= \frac{1}{\pi} \int_0^{\pi} h(x, t, u) D_n(u) \, du - \frac{2}{\pi} \int_0^{\pi} \phi(x, u) D_n(u) \, du + \frac{2}{\pi} \phi(x, t) \int_0^{\pi} D_n(u) \, du \quad (6.3) \end{aligned}$$

As

$$\frac{2}{\pi} \int_0^{\pi} D_n(u) \, du = 1$$

from (6.3), it follows that

$$\tilde{S}_n(x, t) - \phi(x, t) = \frac{1}{\pi} \int_0^{\pi} h(x, t, u) D_n(u) \, du - \frac{2}{\pi} \int_0^{\pi} \phi(x, u) D_n(u) \, du.$$

Proceeding in the lines of proof adopted in the proof of Lemma 1, $h(x, t, u)$ takes the place of $g(x, t, u)$, it can be shown that

$$\left\| \frac{1}{\pi} \int_0^\pi h(x, t, u) D_n(u) du \right\|_p = O(1) \omega(\pi/n).$$

As $\|\phi(x, u)\|_p$ and $\|g(x, t, u)\|_p$ have the same estimate, adopting the argument used in the proof of Lemma 1 it can be shown that

$$\left\| \frac{2}{\pi} \int_0^\pi \phi(x, u) D_n(u) du \right\|_p = O\omega(\pi/n).$$

This completes the proof of Lemma 6.

Lemma 7. Under the hypotheses of Theorem 2

$$\left\| \frac{\partial}{\partial t} \left(\tilde{S}_n(x, t) \right) \right\|_p = O(1) n \omega(\pi/n).$$

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{S}_n(x, t) &= - \sum_{k=1}^n 2k \sin kt A_k(x) \\ &= - \sum_{k=1}^n 2k \sin kt \left(\frac{1}{2\pi} \int_{-\pi}^\pi \phi(x, u) \cos k u du \right) \\ &= - \frac{1}{2\pi} \sum_{k=1}^n k \int_{-\pi}^\pi \phi(x, u) [\sin k(u+t) - \sin k(u-t)] du \\ &= \frac{1}{2\pi} \int_{-\pi}^\pi [\phi(x, t-u) - \phi(x, t+u)] D'_n(u) du \\ &= \frac{1}{\pi} \left(\int_0^\eta + \int_\eta^\pi \right) [\phi(x, t-u) - \phi(x, t+u)] D'_n(u) du. \end{aligned} \quad (6.4)$$

Using Lemma 5(ii), we get

$$\begin{aligned} &\left\| \int_0^\eta [\phi(x, t-u) - \phi(x, t+u)] D'_n(u) du \right\|_p \\ &\leq \int_0^\eta \|\phi(., t-u) - \phi(., t+u)\|_p |D'_n(u)| du \\ &= O(n^2) \int_0^\eta \omega(u) du = O(1) n \omega(\pi/n) \end{aligned} \quad (6.5)$$

Writing $F(x) - Cx$ in place of $f(x)$, we get

$$\begin{aligned} &\int_\eta^\pi [\phi(x, t-u) - \phi(x, t+u)] D'_n(u) du \\ &= - \int_\eta^\pi [F(x+t+u) - F(x+t-u)] D'_n(u) du + \int_\eta^\pi \{F(x-t+u) - F(x-t-u)\} D'_n(u) du. \end{aligned}$$

As the expressions in the curly bracket are increasing functions of u , proceeding as in Lemma 3, it can be shown that

$$\left\| \int_{\eta}^{\pi} [\phi(x, t-u) - \phi(x, t+u)] D'_n(u) du \right\|_p = O(1) \int_{\pi/n}^{\pi} \frac{\omega(u)}{u^2} du. \quad (6.6)$$

This completes the proof of Lemma 7.

Lemma 8. ([2], p.32)

$$\int_0^{\infty} \frac{\sin^2 kt/2}{t^{1+\alpha}} dt = \frac{\pi k^{\alpha}}{4 \sin(\pi\alpha/2) \Gamma(\alpha+1)}.$$

7. PROOF of THEOREM 2

Now for $0 < \gamma < 1$

$$\begin{aligned} \int_0^{\infty} \frac{\tilde{S}_n(x, t)}{t^{1+\gamma}} dt &= -4 \int_0^{\infty} \frac{\sum_{k=1}^n \sin^2 \frac{kt}{2} A_k(x)}{t^{1+\gamma}} dt \\ &= -4 \sum_{k=1}^n A_k(x) \int_0^{\infty} \frac{\sin^2 \frac{kt}{2}}{t^{1+\gamma}} dt. \end{aligned} \quad (7.1)$$

Using (7.1) and Lemma 8 we get

$$\int_0^{\infty} \frac{\tilde{S}_n(x, t)}{t^{1+\gamma}} dt = \frac{-\pi}{\Gamma(\gamma+1) \sin \frac{\pi\gamma}{2}} \sum_{k=1}^n k^{\gamma} A_k(x). \quad (7.2)$$

From (7.2) and (2.2) we get

$$\begin{aligned} \beta_n(x) &\equiv J_{\gamma} \left(x, \frac{1}{n} \right) - \sum_{k=1}^n k^{\gamma} A_k(x) \\ &= \frac{\Gamma(\gamma+1) \sin \frac{\pi\gamma}{2}}{\pi} \left[- \int_{1/n}^{\infty} \frac{\phi(x, t)}{t^{1+\gamma}} dt + \int_0^{\infty} \frac{\tilde{S}_n(x, t)}{t^{1+\gamma}} dt \right] \\ &= \frac{\Gamma(\gamma+1) \sin \frac{\pi\gamma}{2}}{2} \left[\int_0^{1/n} \frac{\tilde{S}_n(x, t)}{t^{1+\gamma}} dt + \int_{1/n}^{\infty} \frac{\tilde{S}_n(x, t) - \phi(x, t)}{t^{1+\gamma}} dt \right] \\ &= \frac{\Gamma(\gamma+1) \sin \frac{\pi\gamma}{2}}{2} [\Delta_1(x) + \Delta_2(x)]. \end{aligned} \quad (7.3)$$

By Lemma 6

$$\|\Delta_2(x)\|_p \leq \int_{1/n}^{\pi} \frac{\|\tilde{S}_n(x,t) - \phi(x,t)\|_p}{t^{1+\gamma}} dt \quad (7.4)$$

As $\tilde{S}_n(x,0) = 0$, we have

$$\tilde{S}_n(x,t) = \tilde{S}_n(x,t) - \tilde{S}_n(x,0) = t \left[\frac{\partial}{\partial t} \tilde{S}_n(x,t) \right]_{t=0}, \quad 0 < \theta < t.$$

Hence by Lemma 7,

$$\begin{aligned} \|\tilde{S}_n(.,t)\|_p &= O(t) \left\| \left[\frac{\partial}{\partial t} \tilde{S}_n(.,t) \right]_{t=0} \right\|_p \\ &= O(1) t n \omega(\pi/n). \end{aligned} \quad (7.5)$$

Using (7.5), we get

$$\begin{aligned} \|\Delta_1(x)\|_p &= O(1) \int_0^{1/n} \frac{\|\tilde{S}_n(x,t)\|_p}{t^{1+\gamma}} dt \\ &= O(1) n \omega(\pi/n) \int_0^{1/n} t^{-\gamma} dt \\ &= O(1) n^\gamma \omega(\pi/n). \end{aligned} \quad (7.6)$$

Using (7.5), (7.6) in (7.3), we get

$$\|\beta_n(x)\|_p = O(1) n^\gamma \omega(\pi/n).$$

and this completes the proof of Theorem 2.

By taking $\omega(t) = t^\alpha, 0 < \alpha \leq 1$, in Theorem 2, we obtain the following corollaries:

Corollary 1. If $f \in C_{2\pi} \cap Lip(\alpha, p), p \geq 1, 0 < \alpha \leq 1$ and f is of monotonic type then

$$\|\beta_n(.)\|_p = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma}}, 0 < \gamma \leq \alpha < 1 \\ \frac{\log n}{n^{1-\gamma}}, 0 < \gamma < \alpha = 1 \end{cases}$$

Putting $p = \infty$, in Corollary 1, we get

Corollary 2. If $f \in C_{2\pi} \cap Lip(\alpha), 0 < \alpha \leq 1$ and f is of monotonic type then

$$\|\beta_n(\cdot)\|_c = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma}}, & 0 < \gamma < \alpha < 1 \\ \frac{\log n}{n^{1-\gamma}}, & 0 < \gamma < \alpha = 1 \end{cases}$$

Note: In Corollary 2 in the special case when $\gamma = \alpha$ the result reduces to second part of Theorem B due to Salem and Zygmund.

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