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Prospective Mathematics Teachers' Task Modifications Utilizing Their Knowledge of Pattern Generalization

Matematik Öğretmen Adaylarının Örüntü Genellemesi Bilgilerini Kullanarak Yaptıkları Etkinlik Değişiklikleri

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ABSTRACT: The purpose of the study is to evaluate how prospective mathematics teachers (PMTs) modify tasks to facilitate students' learning of pattern generalization through the use of their mathematical knowledge for teaching. Case study, which is a type of qualitative research method, was used to determine the mathematical characteristics that PMTs use when modifying a mathematical task. The knowledge from which PMTs draw to modify the task has also been outlined. Accordingly, data were collected from PMTs' task modifications and reflection reports. When PMTs worked on two or more forms of modification, as compared to just using one type of modification, they modified tasks more effectively and comprehensively. The PMTs who make condition modifications need to utilize specialized content knowledge through the use of models or tables. They aimed to help middle school students understand using these modifications, and thus they also utilized their knowledge of content and students. They also used their knowledge of content and teaching, especially while making modifications to questions and context. Task modification activities can be used to help prospective teachers notice the mathematical and pedagogical affordances and limitations offered by tasks.

Keywords: Task modification, pattern generalization, prospective mathematics teachers, mathematical knowledge for teaching.

ÖZ: Bu çalışmanın amacı, öğrencilerin örüntü genellemelerine yardımcı olmak için matematik öğretmen adaylarının matematik öğretimi bilgilerini kullanarak yaptıkları etkinlik değişikliklerini değerlendirmektir. Öğretmen adaylarının matematiksel bir etkinliği değiştirirken kullandıkları matematiksel özellikleri belirlemek için nitel araştırma yöntemlerinden durum çalışması kullanılmıştır. Ayrıca öğretmen adaylarının etkinliği değiştirirken kullandıkları bilgileri belirlenmiştir. Veriler, öğretmen adaylarının değiştirdiği etkinliklerden ve yansıtma raporlarından toplanmıştır. Öğretmen adayları, etkinlik üzerinde yalnızca bir tür değişiklik yapmaya kıyasla iki veya daha fazla değişiklik türünü birlikte yaptıklarında, etkinliklerini daha anlamlı ve kapsamlı bir şekilde değiştirmişlerdir. Koşul değişiklikleri yapanlar, model veya tablo ekleyerek uzmanlık alan bilgilerini kullanmıştır. Bu değişiklikleri öğrencilerin anlamasına yardımcı olmak amacıyla da yapmışlar ve böylece öğrenci ve alan bilgilerini de kullanmışlardır. Ayrıca öğretmen adayları, alan ve öğretme bilgilerinden özellikle soru ve bağlam değişikliği yaparken yararlanmıştır. Etkinlik değiştirme çalışmaları, öğretmen adaylarının etkinliklerin sunduğu matematiksel ve pedagojik olanakları ve sınırlılıkları fark etmelerine yardımcı olmak için kullanılabilir.

Anahtar kelimeler: Etkinlik değişikliği, örüntü genelleme, matematik öğretmen adayları, matematik öğretim bilgisi.

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Several prior studies have indicated that mathematical tasks play an essential role in the teaching and learning of mathematics (e.g., Ayalon et al., 2021; Chapman, 2013; Doyle, 1983; Henningsen & Stein, 1997; Kusaeri et al., 2022; Thanheiser, 2015). A task is part of a mathematics class activity that helps students learn about a particular mathematical idea and might involve several linked problems or a challenging problem in a class session (Stein & Smith, 1998). Likewise, Doyle (1983) explains that a task is any issue or activity that assists in the development of a concept or ability within the setting of a mathematics classroom.

There is a link between the level of thinking a mathematical task requires and how well students understand mathematics. Stein and Smith (1998) assert that the types of mathematical tasks affect how students learn to think mathematically. When students are asked to follow a memorized procedure in a routine manner, they are provided with one type of thinking opportunity. When students are asked to think conceptually and make connections, they are given a different set of opportunities to think (Stein & Smith, 1998; Stein et al., 2000). Students are expected to take an active role in these tasks, assume responsibility for their outcomes, get experience with various tools and resources, and ultimately complete a product as a consequence of their efforts (Henningsen & Stein, 1997; Swan, 2008).

Stein and Smith (1998) define the Mathematics Task Framework as involving three phases that tasks follow: first, as tasks appear in textbooks, supplementary materials, and so on, second, as they are set up or presented by the teacher, and third, as students in the classroom implement them. Teachers' aims, mathematical knowledge, and knowledge of students' understanding might influence how they design tasks (Henningsen & Stein, 1997). Teachers' content and pedagogical competence can positively affect students' mathematics learning. Teacher education must address teachers' ability to modify and design relevant tasks (Lee et al., 2017; Watson & Mason, 2007). Within the scope of our investigation, our primary focus was on the second phase of the framework by Stein and Smith (1998), which pertains to teachers. Hence, the purpose of this study is to determine the mathematical characteristics that prospective mathematics teachers (PMTs) use while modifying a mathematical task in textbooks. Accordingly, the mathematical knowledge used by PMTs to modify the task is defined.

The Background of the Study

Teachers' Competencies for Task Modification

Arbaugh and Brown (2005) state that tasks have an effect on how mathematics is learned. Therefore, it is important for teachers to understand how tasks work. Accordingly, Ball (2000) states, "Acquiring the ability to think with precision about mathematical tasks and their use in class can equip teachers with more developed skills in the ways they select, modify, and enact mathematical tasks with their students" (p. xii). For example, Zaslavsky (2008) observed changes in mathematics teaching and learning after task modification. The study's findings suggest that being aware of learning opportunities while analyzing and modifying tasks enhances mathematics teaching and learning. Likewise, teachers improve their mathematics knowledge and their ability to construct lessons that use mathematics by working with, modifying, and appropriating tasks (Pepin, 2015).

According to a growing body of literature, teachers need to understand the characteristics of mathematical tasks and choose tasks that are well-suited to the learning goals (e.g., Arbaugh & Brown, 2005; Ball, 2000; Liljedahl et al., 2007). Therefore, professional development based on task analysis assists teachers in identifying the affordances and limitations of tasks (Johnson et al., 2016; Son & Kim, 2015). For example, Stephens (2006) and Papatistodemou et al. (2014) studied pre-service teachers' perceptions about the potential for mathematical concept growth in tasks. However, research indicates that pre-service teachers are ignorant of the capacity of activities to increase mathematical engagement and the growth of mathematical concepts. Teachers must comprehend the affordances and limitations of tasks in relation to student inquiry to recognize the need for task modification (Lee et al., 2019). Similarly, Liljedahl et al. (2007) noted that tasks may be adapted more effectively if both mathematical and pedagogical factors are considered. Sullivan and Mousley (2001) believe that teacher professional development should help teachers comprehend the dynamics of classroom decision-making about tasks.

Task analysis and modification are linked, and teachers can change mathematical tasks by understanding pedagogical and mathematical affordances and limits (Lee et al., 2019). Lee et al. (2016) aimed to discover tendencies that Korean prospective mathematics teachers adopt while modifying textbook tasks. Their study categorized prospective teachers' task modifications as context, condition, and question. Context adjustment involves making tasks more student-friendly (familiar contexts to students' experiences) or differentiated. Adding, eliminating, or changing task conditions is condition modification. Prestage and Perks (2007) suggested that changing the conditions of closed problems can improve students' mathematical thinking. When conditions in problems are removed, students can create particular criteria according to their own thinking. When circumstances are included, students may practice problem-solving. It is similar to Brown and Walter's (1990) "what-if-not" techniques, which involve manipulating a problem's circumstances to pose a new one. Changing what students must respond to is called question modification. Crespo (2003) mentions that prospective teachers initially focus on the students' enthusiasm and offer simple problems in her study but gradually turn their focus to the students' misunderstandings and pose problems with increased cognitive demands. Consequently, she suggests transforming the task into an open-form or investigative one with question types.

Different tasks offer various learning opportunities; thus, textbook tasks must be modified or new tasks designed to satisfy desired goals or curricular requirements (Lee et al., 2019). In addition, task modification activities support prospective teachers' learning and teaching skills. Lee et al. (2017) showed how important it was to change textbook tasks in order to support prospective teachers' creativity. Thompson (2012) and Kaur and Lam (2012) also proposed modifying tasks to incorporate reasoning and communication skills, and they used conjectures to evoke mathematical exploration. For example, Lee et al. (2019) implied that pre-service teachers' growing ability to notice student thinking is related to their growing comprehension of the mathematical and pedagogical components of tasks, and this growth has an impact on how they modify problems.

The tasks are put into action in the classroom by the teachers as well as the students through their interpretations and performances. The teacher shapes the task and directs students' efforts so they can participate meaningfully in mathematics. There are a number of variables that might affect this process, such as teachers' content knowledge, their knowledge of students, task objectives, instructional disposition, and beliefs. The teacher's mathematical-task knowledge for instruction will determine how they handle tasks (Chapman, 2013). In a similar vein, Henningsen and Stein (1997) studied classroom characteristics that encourage or discourage high-level mathematical thinking and reasoning. According to their framework, teachers' aims, topic content expertise, and knowledge of students influence task arrangement. Accordingly, a teacher can enhance or reduce a task's cognitive demand. For example, Sullivan et al. (2010) studied the relationship between task, teacher, and student learning. Two teachers who lacked confidence in their mathematical skills made the problems basic and discouraged students' diverse answers, reducing the learning potential of the activities. Boston (2013) found that knowing task cognitive demands might help teachers enhance their knowledge and instructional practices, which have characteristics connected to increased student learning. Guberman and Leikin (2013) asserted that the experience with multiple solution tasks would likely make prospective teachers aware of the nature and significance of such tasks in their instruction. This awareness can be achieved by assessing the level of interest and difficulty of the mathematical problems.

Pattern Generalization

In algebra, the generalization of patterns for the transition from arithmetic to algebra is critical (English & Warren, 1998). Students can begin to develop algebraic thinking as early as their elementary school years (Doerfler, 2008; Radford, 2008). One of the ways to strengthen students' algebraic thinking is by generalizing different types of patterns in successions of figures or numbers or a combination of both (English & Warren, 1998). According to Radford (2008), generalization requires recognizing a pattern, expanding that pattern to include all sequence terms, and establishing a rule that can be used to identify any term of the pattern.

Callejo and Zapatera (2017) examined pre-service teachers' descriptions and interpretations of students' responses to pattern generalization questions to evaluate their ability to notice students' understanding of pattern generalization. They proposed three mathematical elements related to the pattern generalization procedure: in the first element, it is assumed that the students continued the pattern (near generalization, Radford, 2011) but were unable to associate the numerical and figural features; the second element is related to the students' ability to make connections with numerical and figural features and generalize the relationship verbally or algebraically (functional relationship) (Smith, 2008); and the third element is the reverse operation, which determines the position of the pattern. For the first element, students use recursive thinking to understand the relationship in the pattern by concentrating on the difference between subsequent output values. Then, they are required to apply the recursive strategy to the explicit rule (Healy & Hoyles, 1999; Lannin et al., 2006). To identify the explicit rule, it is important to define the functional relationship between a figure's location and the number of elements it contains (Rivera, 2010; Warren, 2000). In addition, generalization is facilitated by students' ability to identify links between input-output values of patterns through the use of visual aids such as diagrams, tables, spreadsheets, and figures (figural patterns) (Steele & Johanning, 2004; Warren &

Cooper, 2008). Students employ figural and numerical reasoning in addition to these representations to generalize patterns algebraically (Walkowiak, 2014). Students' use of the numerical method relies on numerical information obtained from several examples of the pattern. In the figural method, learning is centered on the students' ability to recognize and articulate the underlying patterns and relationships they observe in the given figures (Lee & Lee, 2021).

Mathematical Knowledge for Teaching (MKT)

In particular to mathematics teaching, the mathematical knowledge for teaching (MKT) model by Ball et al. (2008) is utilized in this study. They defined MKT as "the mathematical knowledge needed to carry out the task of teaching (p. 395)". The categories of MKT include Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). One of the components of SMK, Common Content Knowledge (CCK), refers to the mathematical knowledge utilized by individuals working with mathematics. With this understanding, teachers are able to appropriately solve problems and apply mathematical language and notations. Specialized Content Knowledge (SCK) refers to the mathematics-specific knowledge that mathematics instructors should possess. It is beyond conceptual understanding. This knowledge is utilized by educators for instructional objectives. The teacher must grasp both the conceptual structure and the visual features of the mathematical topic for the learner to comprehend it. The Knowledge of Content and Student Knowledge (KCS) component of PCK relates to the teachers' planning of mathematics-specific lessons that take into consideration the students' thinking, interest, level, difficulties, misconceptions, and prior knowledge. The second component of PCK, Knowledge of Content and Teaching Knowledge (KCT), needs instructors to be able to make teaching decisions, arrange subjects for instruction, select examples, and evaluate the efficacy of models and representations. The third component of PCK, Knowledge of Content and Curriculum (KCC), is concerned with organizing subjects in accordance with the curriculum, including activities and explanations given by the curriculum.

Significance

The current research aims to contribute to mathematics education through PMTs' task modification skills in two significant ways. First, it appears that providing opportunities for prospective teachers in teacher training to make critical analyses and revisions to the tasks supplied in textbooks is extremely significant (Cheng et al., 2021; Lee et al., 2016). One of the most important sources of instructional tasks that teachers use to shape student learning is mathematics textbooks (Cheng et al., 2021; Haggarty & Pepin, 2002; Kaur & Lam, 2012; Thomson & Fleming, 2004). Although inquiry-based education is specifically promoted in mathematics curriculums, many tasks in school mathematics textbooks allow students to achieve accurate answers by applying processes or algorithms (Basyal et al., 2023; Hidayah & Forgasz, 2020; Ubuz et al., 2010). Therefore, teachers must be able to utilize appropriate instructional tasks by designing new tasks or modifying old tasks in consideration of curricular requirements (Lee et al., 2019). Thus, this study intended to provide opportunities for recognizing the importance of tasks in mathematics teaching to PMTs through task modification.

Additionally, understanding how prospective teachers modify tasks and apply their knowledge may aid teacher educators in developing tools that assist prospective teachers in creating tasks (Lee et al., 2016). Because it is important for prospective teachers to put their knowledge into practice (Llinares & Krainer, 2006), one of the ways that is done is by engaging with tasks (Callejo & Zapatera, 2017). Recent studies have shown that teacher knowledge may have either a positive or negative impact on students' learning throughout the task design and modification phases (e.g., Sullivan et al., 2009; Stein & Smith, 1998; Swan, 2008). In particular, the design of a task by a teacher is influenced by the teacher's goal, knowledge of the topic to be addressed, and knowledge of the student (Sullivan et al., 2009). Thus, for the second contribution to mathematics education literature, it is expected that this study will reveal how PMTs apply their knowledge of algebra to modifying tasks.

In light of the fact that teachers build the students' algebraic understanding in the early grades, they play a crucial role in the teaching of algebra (Malara & Navarra, 2009). Students start to utilize algebraic symbols and notations as they learn about generalizing patterns in school algebra. As a result, pattern generalization is crucial since it marks the start of formal algebra and can help students comprehend the idea of variables. Generalizing patterns also introduces the concept of functional thinking with input-output linkages (Hoyles et al., 1999). Hence, the purpose of the study is to evaluate how teachers modify tasks to facilitate students' learning of pattern generalization through the use of their knowledge. Thus, the following research question is addressed: What characteristics of the task are changed by PMTs utilizing their knowledge to help students' pattern generalization?

Method

In this study, the qualitative research method was used to determine the mathematical characteristics that PMTs use when modifying a mathematical task from a textbook. In light of this, the knowledge that PMTs draw from in order to modify the task has been outlined. Case study, one of the qualitative research approaches, was applied in particular. Case studies are used to figure out the details of a situation, come up with possible explanations for a situation, and look deeply at a situation to determine the what, how, and why of the study's subject matter (Yıldırım & Şimşek, 2013). In the present study, the specific context/situation was PMTs' modification of the task associated with pattern generalization, and it was investigated through the case study how the interactions were between PMTs' modified tasks and knowledge utilization.

Participants

This study was conducted with 36 (29 females and seven males) fourth-grade PMTs who attended the "Task Design in Mathematics Education" course at a state university as part of the Elementary School Mathematics Teacher Training Program in Turkey. Participation in this study was voluntary. We also obtained consent forms from PMTs. The participants studied design principles for mathematical tasks, the implementation of tasks in the classroom, and the evaluation of students' thinking process in task-based instruction as part of the course. The participants also attended the majority of teaching-related courses (e.g., algebra teaching, mathematics textbook evaluation) in previous semesters. Within the algebra teaching course, participants were exposed to teaching methods and strategies pertaining to pattern generalization in the field of algebra learning. In order to obtain precise information from PMTs, it was taken into consideration that they had specifically taken this course. Thus, the participants were selected for a specific reason using purposeful sampling (Merriam, 2009).

Data Collection and Analysis

The data gathering instrument is a task for pattern generalization. When provided to PMTs, the numbers of the pattern in the textbook were merely altered (see Figure 1). To encourage students' study of pattern generalization, the participants were asked to modify the task. In addition, they were required to write a reflection report that explains how and why they modified this task.

Figure 1

The Task

"Let's find the general rule of the 6, 10, 14, 18... pattern."

- Modify the above task to promote student exploration and engagement.
- Provide justifications for each of the modifications you have made.

Based on the study by Lee et al. (2016), the task modifications of PMTs were divided into three different categories: context, condition, and question. The descriptions for Lee et al.'s (2016) classifications were adapted specifically to pattern generalization for this study. Thus, the descriptions in this study are as follows: Condition modification is including multiple representations, such as figures and/or tables, to help students understand the task. Context modification is incorporating materials or contexts relevant to students' real-world experiences. Question modification is the act of posing or adding new questions to the task. PMTs used only one modification type (only question and only context) or two to three modification types together. Thus, three combined categories emerged: condition and question modifications, context and question modifications, and context-condition-question modifications. For example, PMT1 added a figure and a table that presented the relationship of the pattern in modifying the task (condition modification), and she also added questions that prompted students to identify a general rule (question modification) (see Figure 3). Thus, she used two types of modifications, and we categorized her attempts as condition and question modifications.

In order to code the textual assertions of PMTs, the data from the reflection reports were classified and grouped into idea units. The data unit analysis included a significant comment, explanation, paragraph, or example (Strauss & Corbin, 1998). The extracted units were then categorized according to Ball et al.'s (2008) Mathematical Knowledge for Teaching model. 'We developed definitions for the categories (SMK, KCS, and KCT) based on Ball et al.'s (2008) descriptions and also the literature-related pattern generalization (e.g., Callejo & Zapatera, 2017; English & Warren, 1998; Healy & Hoyles, 1999; Radford, 2008; Smith, 2008; Steele & Johanning, 2004; Walkowiak, 2014). Table 1 shows the definitions and examples:

Table 1

The Categories with Definitions for Pattern Generalization and Examples

Categories	Definitions	Examples
SCK	The knowledge to use a figure and a table to represent the relationship of the pattern	using squares within a figural pattern; utilizing table with position number, term, and the relationship
KCS	The knowledge of students' understanding, misconceptions, and difficulties	assisting students in understanding the concept of pattern generalization; student's thinking successively between terms; student's getting a general rule algebraically
KCT	The knowledge to organize the questions for making students think inductively or need a general rule; The knowledge of context-based instruction	asking the values of far terms to allow finding a general rule; using the numbers of pages in the context of reading books from students' daily lives

In this study, the researcher and an expert in mathematics education research individually coded the data for cross-checking. There is a 91% match in our independent coding. We discussed the controversial codes and their meanings until we reached a complete agreement.

Ethical Procedures

The fact that the research does not pose an ethical problem has been confirmed by the ethics committee report issued 367596 and dated 30.11.2022 received from the Human Research Ethics Committee of Trakya University.

Before starting the implementation, the participants were informed about the research. They participated in the study voluntarily. The names of participants were reported using codes in accordance with ethical rules.

Findings

The PMTs' task modifications are divided into three main categories: context modification, condition modification, and question modification. It was observed that most PMTs made two modifications at the same time (see Figure 2). The questions were also changed, particularly when the context or condition was modified. In addition, there were PMTs who performed all three modifications.

Figure 2

Frequency of Task Modification Types

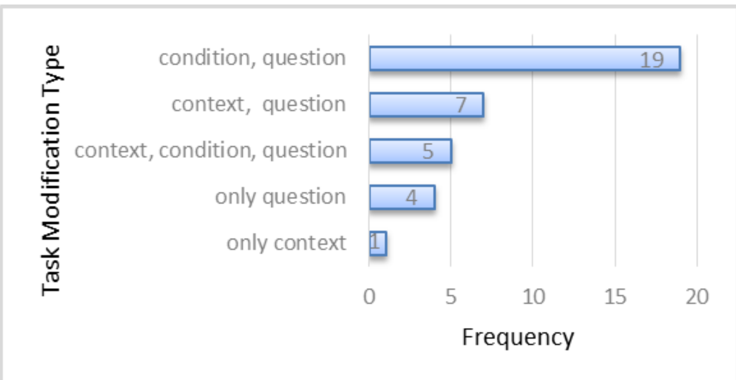


Figure 2 shows the frequencies for the task modification types of PMTs. As shown in Figure 2, about half of the PMTs (53%) used condition and question modifications mostly. Then, 7 (19%) PMTs used context and question modifications, 5 PMTs (14%) used context-condition-question modifications, 4 PMTs (11%) used only question modification, and 1 (3%) PMT used only context modification, respectively. The following sections provide examples for each type of modification as well as how PMTs apply their knowledge while making modifications.

Condition and Question Modifications

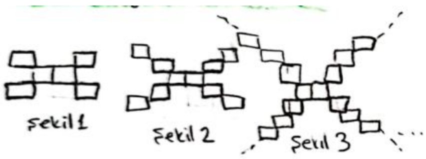
Out of the 36 tasks, 19 of them involved modifications to both the conditions and questions. The majority of PMTs modified the conditions to eliminate any misconceptions students may have regarding generalizations or to assist those who were having difficulty obtaining generalizations in the task. Most PMTs chose to include multiple representations, such as figures and/or tables, to help students understand the task.

Use of Figures in the Pattern

The majority of PMTs who chose to model the pattern with figures or material were able to do it accurately (9 out of 11). For instance, PMT1 stated that students struggled to identify the general rule in pattern generalization and express it algebraically. By illustrating this circumstance, she emphasized that students tend to find a rule by merely focusing on the difference between terms in numerical patterns that were constantly changing. She illustrated this argument by stating that students might first determine the difference as 4 increments and then determine the rule as n+4. Or, she asserted that they were more likely to continue the pattern and discover the next term (the fifth term) than to discover a general rule. To address these challenges and misunderstandings in determining the general rule, she recommended employing comprehensive questions (1-8) as opposed to merely asking for the general rule (see Figure 3). She intended to help students gain an awareness of generalization by adding new conditions to the third and fifth questions that would enable them to adopt a figural pattern and use figural reasoning. She also asked that they transmit the numerical relationships illustrating the fixed and expanding squares in the figures. Then, she posed the seventh question so that the students would feel compelled to develop a general rule, as they could not achieve the n^{th} term by writing down each step. In the final question, she drew attention to the functional relationship between the terms and the square numbers and asked the students to associate this relationship with the general rule. PMT1 modified the task conditions by assigning students tasks on the figure and table as well as step-by-step questions to determine the pattern generalization. She explained that this would facilitate a more meaningful learning experience and make it easier for students to develop mathematical concepts.

Figure 3

PMT1's Modified Task



1.How many squares are in figure 1?
2.What is the relationship between figures?
3.How much increase do squares multiplying from vertices cause each time?
4.How can we express figure 1?
5.Do the 2 squares in the middle exist anyway?
6. How can we express figure 2?
7. Continue the below table. So can we reach the nth term in this way?

	Toplam kare sayısı	
Şekil1	2+4	= 2+4
Şekil2	2+4+4	= 2+4.2
Şekil3	2+4+4+4	= 2+4.3
⋮	⋮	⋮
Şekiln	2+n+n+..+n	= 2+4.n

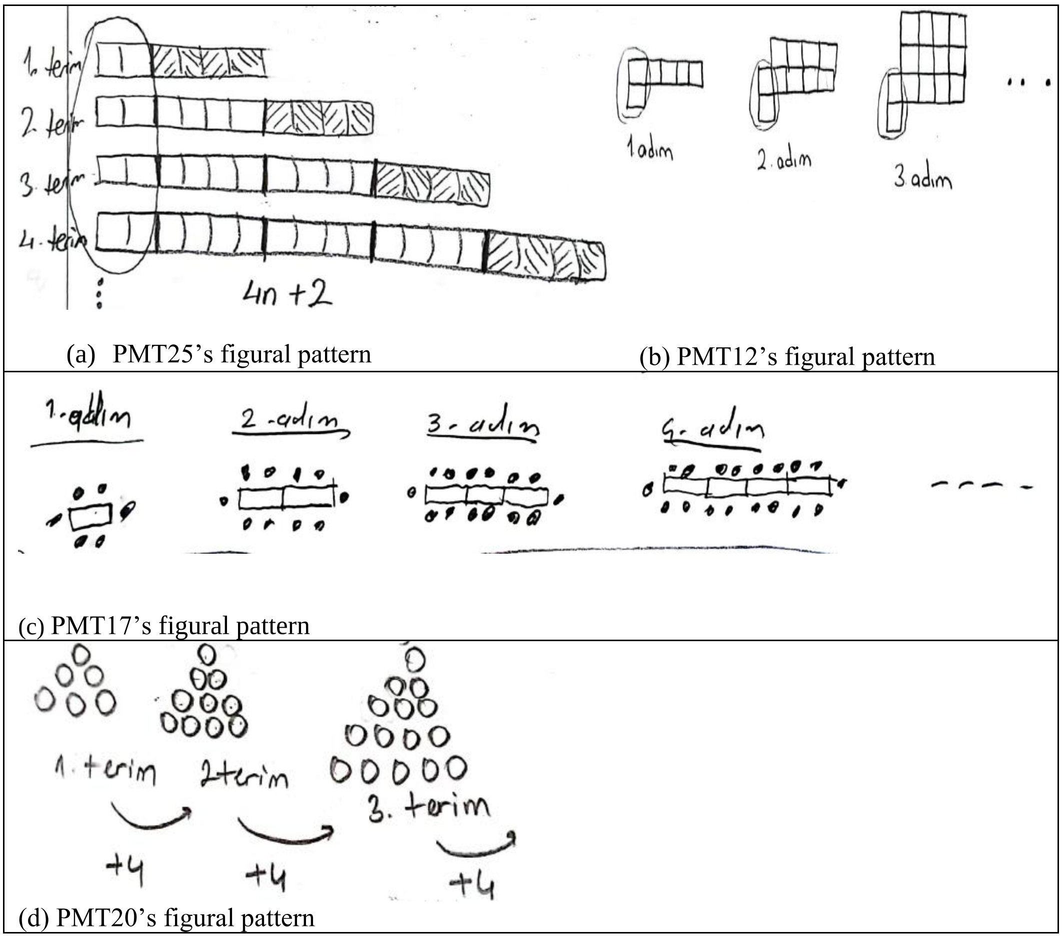
8.What is the relationship between the terms and the resulting square numbers?

In addition, several PMTs utilized various shapes to support the figural pattern to facilitate the discovery of the general rule (see Figure 4). It was observed that the items utilizing the form positioned the squares in a different location so that students could observe the fixed 2 (PMT25 in Figure 4a and PMT12 in Figure 4b) or the points/chairs were put on both ends of the rectangle table and the number of points did not increase (PMT17 in Figure 4c). Students can deduct from these figures that the general rule has a constant of 2 and that other units increase by a factor of 4. In order to eliminate the difficulties in identifying the general rule and to prevent the misunderstanding of creating a general rule by focusing solely on the difference between terms, these PMTs modified their activities by adding figures to the number pattern by modifying the condition. This modification demonstrates that these PMTs' SCK is sufficient.

However, the PMTs with insufficient SCK were unable to utilize the figural pattern effectively. For instance, PMT20's figural pattern (see Figure 4d) did not correspond to the expected number pattern, yet she was unable to detect this. Even though the first two terms are 6 and 10, the third term contains 15 circles. Moreover, according to her method, the difference in the numbers of circles rises by one with each step, although it should remain constant at four. This demonstrates that PMT20 was unable to focus on figurative thinking or believing that just the first two terms fit, while the others did not. Consequently, this shows that her SCK is insufficient.

Figure 4

PMTs' Representations with Figures of the Pattern



(a) PMT25's figural pattern (b) PMT12's figural pattern

(c) PMT17's figural pattern

(d) PMT20's figural pattern

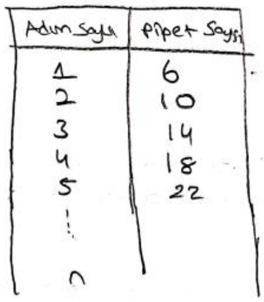
Use of Table Representation

About half of the PMTs utilizing the table were unable to use it properly (5 out of 11). In Figure 5a, for instance, PMT24 wrote the position number in one column and the terms in the other column in the exact same order as the numerical pattern. This representation is insufficient to facilitate the functional reasoning of students. Or the use of some PMTs may result in more misconceptions. PMT3 demonstrated, for instance, in Figure 5b, that the second term is generated by adding 4 to the first term, and the third term is produced by adding 4 to the second term. This presentation may cause students to focus exclusively on the difference. Consequently, this notation may impede functional thinking between the number of steps and the term. In order to facilitate students' understanding, term order, term, and representations indicating the relationship should each be presented in a distinct column. The table created by PMT31 is a good illustration of this point (see Figure 5c).

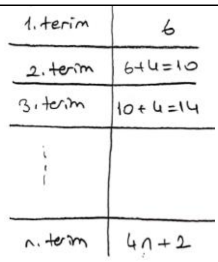
In conclusion, when the PMTs had sufficient knowledge of the difficulties and misconceptions that students might have regarding pattern generalization (KCS), they attempted to change the condition of the task with the figure/model or table by employing their SCK to facilitate students' comprehension. However, PMTs that were unable to employ table representations effectively lacked appropriate SCK. In addition, PMTs inserted sub-questions to the task to allow for inductive reasoning considering the cognitive demand levels of the students. The use of these questions to stimulate student thinking shows the adequate KCTs of the PMTs.

Figure 5

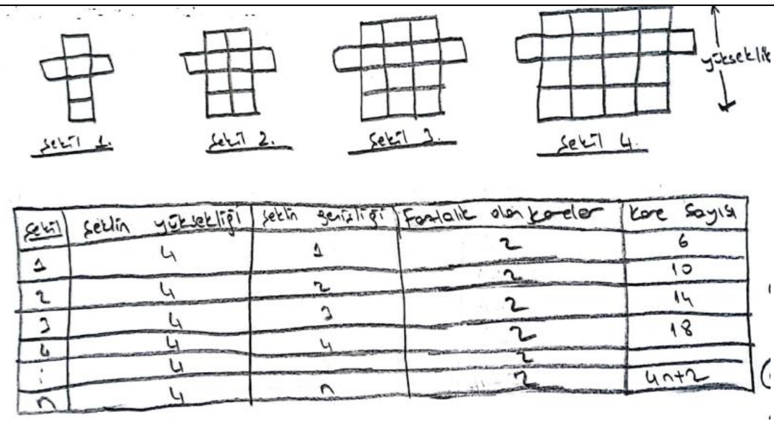
PMTs' Table Representations of the Pattern



(a) PMT24's table representation



(b) PMT3's table representation



c) PMT31's table representation

Context and Question Modifications

There are seven out of thirty-six tasks in which both the context and questions are modified. The PMTs who modified the context claimed that there were no contexts in the given task that might attract students' attention. Therefore, they proposed incorporating materials or contexts relevant to students' real-world experiences into the tasks. For instance, PMT10 believed that a problem based on reading as a context could help students create connections between mathematics and real life. The modified context was presented by PMT10 in Figure 6.

According to PMT10, the task shown in Figure 1 lacked an experiential opportunity for students to investigate real-life pattern generalization. He explained that this is due to the fact that this task requires students to create a rule using only numbers, implying abstract mathematics consisting of operations and rules. Consequently, he employed the context of reading books from the classroom or the students' daily lives. In Question 1 of Figure 6, he encouraged students to consider the functional relationship between the number of book pages and the number of days. The Questions (3-4-5) in Figure 5, PMT10 also asked students to apply the found algebraic relation. In Questions 3 and 5, he substituted 17 for n. In Question 4, which demands calculating the equation

102=4n+2 and provides inverse relationship thinking, he substituted 17 for n. In addition, there were items with contexts such as putting money in a piggy bank (PMT3), collecting stamps (PMT26), collecting butterflies (PMT36), and arranging the woods (PMT29 in Figure 7). These examples, taken from the context-modified tasks, were based on the experiences of the PMTs, who believed that the contexts in mathematical tasks should be associated with real-life situations and, consequently, the students' experiences or situations that could be encountered. These PMTs favored context-based instruction and utilized their KCT.

Figure 6

PMT10's Modified Task

Ali read 6 pages on the day he bought the book. Ali regularly reads 4 pages a day.

1) Express the relationship between the number of pages of the book and the number of days.
2) What is this relationship algebraically for the number of pages in the book that Ali reads from day to day?
3) According to the relationship, how many pages did Ali read in total on the 17th day?
4) On what day did Ali reach page 102?
5) Since Ali finished the book on the 57th day, how many pages is the book?

Context, Condition, and Question Modifications

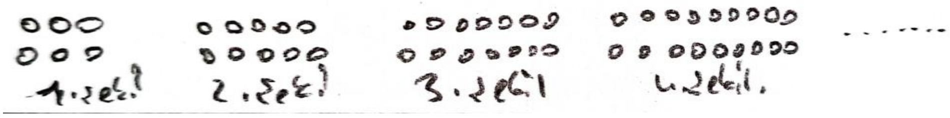
Five PMTs out of 36 opted to modify all three conditions simultaneously. When they modified questions, they tended to modify the conditions and context of the tasks along with the questions. For instance, PMT29 modified a mathematical context to a real-life environment by employing the wood-stacking context (see Figure 7). By inviting students to reflect on the background he employed, he also included questions that could stimulate class discussion. He guided the students step-by-step so that they would feel it necessary to discover the general rule. With the fourth and fifth questions, he was attempting to demonstrate that it was challenging to discover far terms by drawing or counting. He prompted them to consider if the seventh question's generalization might be applicable to all forms and explain why this was the case. Students were thus given the opportunity to consider the role of the general rule. In its assessment, PMT29 stated that this question gave students the opportunity to reflect on their own views by evaluating the solution's reasoning and providing an explanation. In addition, he highlighted the importance of asking for justification in the questions in order to allow students to use mathematical language and to allow for questioning. In addition to context and question modifications, PMT29 approximated the wood-stacking rule using a form and a pattern (see Figure 7). He explained that the purpose of this approach was to aid the student's figural reasoning and visualization.

In conclusion, we can state that the PMTs who need to make these three modifications utilize their KCT because of the order of the questions, and these questions contribute to the class discussion. They also use their SCK and KCS in the use of models or tables to aid students' understanding and must make these modifications.

Figure 7

PMT29's Modified Task

Mehmet cut 6 equal pieces of wood and arranged them as in the first figure. He continued to cut 4 equal woods every day after. He continued to arrange these woods, as shown in the figures.



1) Is there a relationship between the number of days and the number of woods?
2) What could the relationship be?
3) How many woods will there be in the 5th figure?
4) How many woods will there be in the 50th figure?
5) Is there any other way we can use instead of counting or drawing?
6) Can you express the relationship you found in (2) algebraically?
7) Does this relationship valid for all shapes? Please explain with justification.

Only Question or Only Context Modification

It may be noteworthy to evaluate the tasks and see that PMTs modified only one aspect. These tasks were not as effective for use in teaching the tasks with the combination of two or three modifications. For example, PMTs who made modifications solely to the questions posed them in general expressions. PMT8, for instance, emphasized the process that follows the discovery of the general rule, asking questions such as "Can you check the rule you found?" or "What does the general rule do?" Consequently, these questions would not assist students in the process of discovering the general rule. On the other hand, there was one PMT that modified the context only, PMT27. She posed a question with a contradiction: "Ali has 6 marbles. Each friend has four marbles. Ali will receive his marbles if he defeats another classmate in the race. How many students Ali has to beat to have 30 marbles in total?" Nonetheless, this problem is more about finding a result, or a value, than a general rule.

Discussion and Conclusions

In this study, the forms of task modifications performed by PMTs, as well as the types of teacher knowledge engaged throughout the modification process, are investigated. When PMTs worked on two or more forms of task modification, as compared to just one, they modified tasks more properly and comprehensively in a relevant manner, as stated in Lee et al.'s (2016) study. Accordingly, they frequently combined condition and question modification forms.

During the modification process, SCK and KCS were the two forms of knowledge that PMTs utilized most frequently. When PMTs had sufficient knowledge of the difficulties and misconceptions that students could have with pattern generalization (KCS), they attempted to modify the condition of the task using the figure/model or table by employing their SCK to facilitate students' comprehension. The PMTs indicated that students may have difficulties discovering and expressing the general rule algebraically. Prior research revealed that pupils had difficulty progressing beyond seeing and characterizing patterns to generalizing them and discovering function rules or algebraic representations (e.g., English & Warren, 1998; MacGregor & Stacey, 1995). In addition, the PMTs reported that students might employ a recursive strategy by focusing on the difference rather than the relationship between the position of terms and the term value. Warren (2000) discovered that students tended to prioritize recursive strategies over functional relations. Consequently, the KCS of PMTs is frequent enough. PMTs with this KCS were able to employ figures or models that give figural reasoning, demonstrating SCK to assist students with pattern generalization (Wilkie, 2014). Numerous research studies advocate employing figural reasoning to enhance students' comprehension of the link between evaluating the differences between figures (e.g., Barbosa & Vale, 2015; Becker & Rivera, 2005; Markworth, 2010; Walkowiak, 2014; Warren & Cooper, 2008). The use of figural reasoning, such as questioning the students about how the units in the figures get together and what the connection is based on, actually helps to understand the rule of the pattern conceptually (Thornton, 2001). Similarly, the use of tables indicates PMTs' SCK of pattern generalization (Wilkie, 2014). Students will benefit from the usage of diagrams, tables, spreadsheets, and figures (figural patterns) throughout the process of generalization (Lannin et al., 2006; Steele & Johanning, 2004). However, there are certain PTs who are unable to use the table to comprehend the functional relationship, which might indicate their lack of appropriate SCK knowledge. Warren and Cooper (2008) noted that establishing the connection between the position number and the corresponding word in the table rows can lead to an effective tabular representation of patterns.

PMTs who relied on KCS as their pedagogical content knowledge transformed the context, with a particular emphasis placed on those who considered the interests of students. Consequently, utilizing their KCT, they offered problems based on real-world or familiar contexts. In addition, they added questions to have students generalize patterns within a problem-solving process (Prestage & Perks, 2007). Thus, the PMTs modified the context and question by efficiently employing their KCT. Consequently, the PMTs that lacked appropriate KCT to instruct pattern generalization affected just the context or the question. However, as indicated by Lee et al. (2016), these single change types did not provide important opportunities to acquire pattern generalization.

In sum, we can state that teachers who need to make the three modifications also utilize SCK and KCS to assist students understand through the use of models or tables. Kaput (1999) also promoted a multi-representational approach, which entails giving students real-world experience in contexts they are familiar with and presenting issues using diagrams, tables of values, language, equations, and graphs to help students understand them. In particular, the PMTs who used figures or models built figural patterns and represented the constant within figures. Moss et al. (2008) suggested using these methods to support students' functional thinking and represent the general rule algebraically. In addition, the PMTs organized the questions in a way that encouraged inductive reasoning. At this time, we may assert that the KCT of the PMTs is adequate since the questions used to conduct the exercise allow the student to both generalize and contribute to the class discussion by promoting student thinking (Smith et al., 2008), PMTs also used their KCT in presenting the task within a context that required problem-solving as a teaching technique.

Teachers need to have a strong conceptual grasp of mathematics as well as an awareness of students' thinking to teach pattern generalization effectively (Girit Yildiz & Akyuz, 2020). According to Liljedahl et al. (2007), tasks can be changed more efficiently if both mathematical and pedagogical elements are addressed. According to Magiera et al. (2013), prospective teachers have a limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students because this ability is mainly based on their algebraic thinking. Similarly, content knowledge (Bartell et al., 2013) and particularly SCK allow teachers to analyze student thinking, aiding in the identification of student misunderstandings (Mosvold et al., 2014). However, content knowledge by itself is insufficient; prospective teachers must also acquire abilities such as task development and gain experience in this respect (Bartell et al., 2013; Callejo & Zapatera, 2021). When teachers lacked the necessary experience to design activities for teaching pattern generalization, they were unable to properly teach functional reasoning using tables or input-output values (Wilkie, 2014). According to Guberman and Leikin (2013), prior experience with multiple solution tasks allows prospective teachers to evaluate the interest and complexity of mathematics problems. In this sense, task modification activities can be a good way for future teachers to gain experience while learning about the role of mathematical tasks in teaching and learning mathematics (Lee et al., 2019).

Finally, this study has several limitations. The most significant limitation of this study was that PMTs did not perform their modified tasks with actual students. Thus, the work has implications for future task modification research. With the modification of tasks, the phases that include implementing the tasks and assessing students' understanding might be included. It is also important to examine what teachers notice about student thinking following implementation and what they would change as a consequence. Additionally, the data for this study are restricted to modified tasks and PMTs' written reflection reports. Future researchers can interview the participants to support the study's findings.

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