NORMALLİYEN İÇİN DEVİRLİ OLMAYAN GRAFLAR

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Özet

Bu çalışmada, p asal bir sayı ve $\beta \geq 4$ olmak üzere $\Gamma_0\left(3^{\beta}p^2\right)$ nin $PSL\left(2,\mathbb{R}\right)$deki normalliyeni için bazı alt yörüngesel graflar incelendi. Ve ayrıca yönlendirilmemiş graflar için orman kavramının devirli olmayan bir graf olduğu vurgulandı.

***Anahtar Kelimeler*:**Normalliyen; Altyörüngesel graf; Yörünge; Devre; Orman; Devir.

NON CYCLES GRAPHS FOR THE NORMALİZER

Abstract

In this paper, we examine some suborbital graphs for the normalizer of $\Gamma_0\left(3^{\beta}p^2\right)$ in $PSL\left(2,\mathbb{R}\right)$ where p is a prime and $\beta \geq 4$. And also it emphasized that the corresponding concept for undirected graphs is a forest, a graph without cycles.

Key Words: Normalizer, Suborbital graph, Orbit, Circuit, Forest, Cycle.[1]

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INTRODUCTION

The main purpose in this study, is to set the foundations of a new method which would help to identify the normalizer of $\Gamma_0(\mathrm{N})$ in $\mathrm{PSL}(2,\square)$ much better, which have been subject to many studies and gaining particular importance since 1970s and to reveal how the producing elements of the normalizer can be gained by this method. With this corresponded which we name as the graph method, the relations between the length of some closed circuits and the orders of the elliptic elements in the normalizer are examined. It is determined that if there are no closed circuits in the suborbital graph, there are no elliptic elements in the normalizer as well.

The main purpose in studies conducted in this field by many scientists is in fact to determine the signature of the normalizer $\mathrm{N}_{\mathrm{PSL}(2,\square)}(\Gamma_0(\mathrm{N}))$ or in other words, to calculate the g-genus which is the missing parameter. We therefore, tried in this study to examine much better the structure of the normalizer. For this, we define a subgroup $\mathrm{N}_0(\mathrm{N})$, and examine the orders of elliptic elements of this group and lengths of suborbital of corresponding graphs. Therefore, in order to find the signature, a new approach is aimed with the help of suborbital graphs. With this new approach, some invariants in the signature of the normalizer is found.

1. PRELIMINARIES

Definition 2.1. Let G is a topological group and X is a topological space. If

$$\begin{array}{rcl} \wedge : \mathrm{G}\times\mathrm{X} & \longrightarrow & \mathrm{X} \\ (g,x) & \longrightarrow & \wedge(g,x) =: g\wedge x \end{array}$$ a continuous transformation and

(i) $g\wedge(h\wedge x) = gh\wedge x$, $g,h\in\mathrm{G}$, $x\in\mathrm{X}$ **(ii)** $e\wedge x = x$, $e\in\mathrm{G}$, $x\in\mathrm{X}$

conditions is provided then $[\mathrm{G},\mathrm{X}]$ is called a topological transformation group. Also G acts on X or G is called an action group on X .

Lemma 2.2. Let $[\mathrm{G},\mathrm{X}]$ is a topological transformation group and x,$y\in \mathrm{X}$. In this case, $x\approx y :\Leftrightarrow \exists g\in \mathrm{G} : gx=y$ is defined as the "$\approx$" relation is an equivalence relation on X .

Definition 2.3. "$\approx$" relation of equivalence classes are called orbits of action. Furthermore, point in the orbit with $x\in \mathrm{X}$ in the orbit of x is called and a set of $\mathrm{G}x := \{gx \mid g\in \mathrm{G}\}$.

Definition 2.4. Let G acts on a set X . If , for each $x,y\in \mathrm{X}$ there exists some $g\in \mathrm{G}$ such that $gx=y$ then we say that G acts transitively on X . According to this definition, if transitif act $\mathrm{G}x=\mathrm{X}$ is obtained for $\forall x\in \mathrm{X}$. In other words there is a single orbit. Orbit is set a transitif acts as of group.

Definition 2.5. $\Gamma := \mathrm{PSL}(2,\mathbb{Z}) = \left\{ z \to \dfrac{az+b}{cz+d} \mid a,b,c,d\in \mathbb{Z} \text{ and } ad-bc=1 \right\}$ subgroup of $\mathrm{PSL}(2,\mathbb{R})$ called Modular group. Γ can be represented by 2x2 integer matrices $\mathrm{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det \mathrm{A}=1$, provided we identify each matrix with its negative, since A and $-\mathrm{A}$ represent the same transformation.

Definition 2.6 N being a positive whole number and of Γ group basic congruence subgroup can be defined as

$\Gamma(\mathrm{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\; ½ a \equiv d \equiv 1 \pmod{\mathrm{N}},\, b \equiv c \equiv 0 \pmod{\mathrm{N}} \right\}$.The most frequent congruence subgroup are $\Gamma_0(\mathrm{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod{\mathrm{N}} \right\}$ group.

Definition 2.7. Let G be a group and $\mathrm{H} < \mathrm{G}$. Normalizer of H in G is called the set of $\mathrm{N_G(H)} := \{ g \in \mathrm{G} \mid g\mathrm{H}g^{-1} = \mathrm{H} \}$.

Theorem 2.8. The elements of the normalizer of $\Gamma_0(\mathrm{N})$ in PSL $(2,\Upsilon)$ are the transformations corresponding to the matrices

$$\mathrm{N}_{\mathrm{PSL}(2,\mathbb{R})}\left(\Gamma_0(\mathrm{N})\right) := \left\{ \begin{pmatrix} ae & b/h \\ c\mathrm{N}/h & de \end{pmatrix} : ade^2 - bc\mathrm{N}/h^2 = e > 0 \right\}$$

where all symbols are integer , $e \,\|\, \mathrm{N}/h^2$ and h is the largest divisor of 24 for which $h^2 \mid \mathrm{N}$ with $h^2 \mid \mathrm{N}$ with the understandings that the determinant of the matrix is $e > 0$, and that $\mathrm{r} \,\|\, \mathrm{s}$ means that $\mathrm{r} \mid \mathrm{s}$ and $\left(\mathrm{r}, \mathrm{s}/\mathrm{r}\right) = 1$ (r is called an exact divisor of s).

Theorem 2.9. Let $\mathrm{N} \in \mathrm{Z}$ and $\mathrm{N} = 2^{\alpha} 3^{\beta} \mathrm{p}_1^{\gamma_1} \dots \mathrm{p}_n^{\gamma_n}$ the prime power decomposition of N. Then the normalizer $\mathrm{N}_{\mathrm{PSL}(2,\mathbb{R})}\left(\Gamma_0(\mathrm{N})\right)$ acts on $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ transitively if and only if

$\alpha \le 7$, $\beta \le 3$ and $\gamma_i \le 1$ where $i = 1, \dots, n$.

Theorem 2.10. The index $\left| \mathrm{N}_{\mathrm{PSL}(2,\mathbb{R})}\left(\Gamma_0(\mathrm{N})\right) : \Gamma_0(\mathrm{N}) \right| = 2^{\rho} \mathrm{h}^2 \tau$, where ρ is the number of distinct prime factors of N/h^2 and $\tau = \left(\frac{3}{2}\right)^{\varepsilon_1} . \left(\frac{4}{3}\right)^{\varepsilon_2}$

$$\varepsilon_1 = \begin{cases} 1 & \text{if } \ 2^2, 2^4, 2^6 \square N \\ 0 & \text{otherwise} \end{cases} \quad , \quad \varepsilon_2 = \begin{cases} 1 & \text{if } \ 9 \square N \\ 0 & \text{otherwise} \end{cases} .$$

Definition 2.11. Let (G,X) be a transitive permutation group and "$\approx$" be an equivalence relation on X. Since $x \approx y$ for $x, y \in$X, if $g(x) \approx g(y)$ is for $\forall g \in$G, then "$\approx$" relation is called a G invariant equivalence relation.

Definition 2.12. The equivalence classes of a G invariant equivalence relation are called blocks. Obvious examples of such relations are;

i) the identity relation, $x \approx y$ if and only if $x = y$

ii) the universal relation, $x \approx y$ for all $x,y \in$X.

This relations are called trivial relations. We call (G,X) imprimitive if X admits some G invariant equivalence relation other than (i) and (ii); otherwise, we call (G,X) primitive. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks.

Lemma 2.13. Let (G,X) be a transitive permutation group. (G,X) is primitive if and only if G_x, the stabilizer of $x \in X$, is a maximal subgroup of G for each $x \in X$.

Theorem 2.14. Let (G,X) be a transitive permutation group. There is a well-defined G invariant equivalence relation $\approx$ on X given by, if $G_\alpha \square H \square G$

$$g(\alpha) \approx h(\alpha) \text{ if and only if } g^{-1}h \in H .$$

The number of blocks (equivalence classes) is the index $|G:H|$ and the block containing α is just the orbit $H(\alpha)$.

Definition 2.15. $X \neq \emptyset$ is a set and $\Delta \subset X \times X$ is a relation. $G=(X,\Delta)$ pair is called a graph. Elements of X are vertices of graph and elements of Δ are edges of the graph. If $(a,b) \in \Delta$, this is indicated as $a \to b$. If $(a,b) \in \Delta$ or $(b,a) \in \Delta$, a and b are connected to a edge. In this case, a and b are called neighboring vertices.

Definition 2.16. Let $a = a_0, a_1, \ldots, a_n = b$ be a sequence of G graph vertices. If for $1 \leq i \leq n$, a_{i-1} and a_i are connected with a edge, then this is indicated with the expression from a to b there is a path with the length of n. If $a=b$ and $a_0, a_1, \ldots, a_{n-1}$ vertices are all different, then this is called a n edged circuit. Furthermore, if for the pairs of a_i, a_{i+1} $a_i \to a_{i+1}$ then this is a circuit directed at a circuit. A three edged circuit is called a triangle, four edged circuit is quadrilateral and six edged circuit is called a hexagon.

Definition 2.17. Assuming $n \geq 3$, n edged graph not containing a circuit is called a forest. Here $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is upper half plane and graph is a combination of hyperbolic lines.

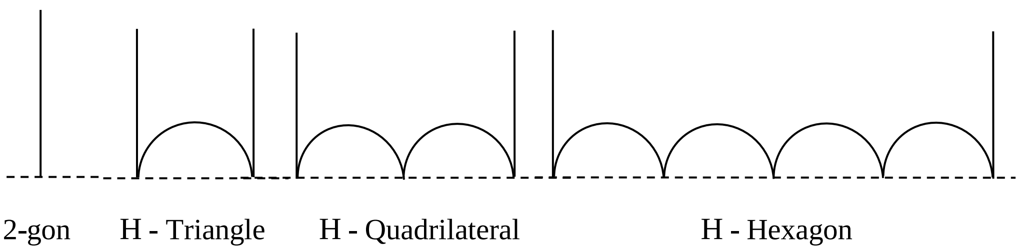

Figure 1. Circuits

Definition 2.18. Let (G,X) be a transitive permutation group. Let us define of G act on X×X as $g:(\alpha,\beta)\rightarrow(\,g(\alpha),\ g(\beta)\,)$, $(\alpha,\beta)\in X\times X$ considering that $g\in$G. Orbits of this act are called of G suborbits. Let us show the suborbit containing (α,β) with $O\,(\alpha,\beta)$.

2. SUBORBITAL GRAPHS OF $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$ ON $\hat{\mathbb{Q}}$ $(3^{\beta}p^2)$

Let $N=3^{\beta}p^2$ and $T\in N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$. For $\alpha\geq 8$, $p>3$ and p prime then,

$$h=3^{\min\{1,\lfloor\beta/2\rfloor\}}=3 \quad \text{and that} \quad A=\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix},\ e\,\|\,N/h^2,\ \det A=e=1,2^{\alpha-6},\ p^2,\ 2^{\alpha-6}$$

p^2. Where $\lfloor\,\rfloor$ is greatest integer function. That is, the elements of the normalizer are of the below forms:

$$A_1=\begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix},\quad A_2=\begin{pmatrix} 3^{\beta-2}a & b/3 \\ 3^{\beta-1}p^2c & 3^{\beta-2}d \end{pmatrix},$$

$$A_3=\begin{pmatrix} ap^2 & b/3 \\ 3^{\beta-1}p^2c & dp^2 \end{pmatrix},\quad A_4=\begin{pmatrix} 3^{\beta-2}ap^2 & b/3 \\ 3^{\beta-1}p^2c & 3^{\beta-2}dp^2 \end{pmatrix}$$

Lemma 3.1. The orbits of the action of $\Gamma_0(3^{\beta}p^2)$ on $\hat{\mathbb{Q}}$ are

(i) $\begin{pmatrix}1\\1\end{pmatrix};\begin{pmatrix}1\\3\end{pmatrix};\begin{pmatrix}2\\3\end{pmatrix};\begin{pmatrix}1\\3^2\end{pmatrix};\begin{pmatrix}2\\3^2\end{pmatrix};\begin{pmatrix}4\\3^2\end{pmatrix};\begin{pmatrix}5\\3^2\end{pmatrix};\begin{pmatrix}7\\3^2\end{pmatrix};\begin{pmatrix}8\\3^2\end{pmatrix};\begin{pmatrix}1\\p^2\end{pmatrix};\begin{pmatrix}1\\3p^2\end{pmatrix};\begin{pmatrix}2\\3p^2\end{pmatrix};$

$\begin{pmatrix}1\\3^2p^2\end{pmatrix};\begin{pmatrix}2\\3^2p^2\end{pmatrix};\begin{pmatrix}4\\3^2p^2\end{pmatrix};\begin{pmatrix}5\\3^2p^2\end{pmatrix};\begin{pmatrix}7\\3^2p^2\end{pmatrix};\begin{pmatrix}8\\3^2p^2\end{pmatrix}$

(ii) $\begin{pmatrix}1\\p\end{pmatrix},\begin{pmatrix}2\\p\end{pmatrix},\begin{pmatrix}3\\p\end{pmatrix},K,\begin{pmatrix}p-1\\p\end{pmatrix}$

(iii) $\begin{pmatrix} a_1 \\ 3^{\beta}\mathrm{p} \end{pmatrix}, \begin{pmatrix} a_2 \\ 3^{\beta}\mathrm{p} \end{pmatrix}, \ldots, \begin{pmatrix} a_{\mathrm{p}-1} \\ 3^{\beta}\mathrm{p} \end{pmatrix}$ where $a_i \not\equiv a_j \,(\mathrm{mod}\,\mathrm{p})$ for $\begin{pmatrix} a_i \\ 3^{\beta}\mathrm{p} \end{pmatrix}$ and $\begin{pmatrix} a_j \\ 3^{\beta}\mathrm{p} \end{pmatrix}$.

Proof. It is clear from the definition of the orbit.

The number of all orbits are $\sum_{\mathrm{d}|\mathrm{N}} \varphi\left(\left(\mathrm{d}\,,\ \frac{3^{\beta}\mathrm{p}^2}{\mathrm{d}}\right)\right)$. The number of orbits can easily be found based on β number being an odd or even number.

In case where β number is odd, the divisors of $\mathrm{N}=3^{\beta}\mathrm{p}^2$ are

I) $\sum_{\mathrm{d}|3^{\beta}} \varphi\left(\left(\mathrm{d}, \mathrm{N}/\mathrm{d}\right)\right) = 2^{\left\lfloor \frac{\beta}{2} \right\rfloor + 2} - 2$ for $1, 3, 3^2, 3^3, \ldots, 3^{\beta}$

II) $\sum_{\mathrm{d}|3^{\beta}} \varphi\left(\left(\mathrm{d}, \mathrm{N}/\mathrm{d}\right)\right) = \left(2^{\left\lfloor \frac{\beta}{2} \right\rfloor + 2} - 2\right)(\mathrm{p}-1)$ for $\mathrm{p}, 3\mathrm{p}, 3^2\mathrm{p}, 3^3\mathrm{p}, \ldots, 3^{\beta}\mathrm{p}$

III) $\sum_{\mathrm{d}|3^{\beta}} \varphi\left(\left(\mathrm{d}, \mathrm{N}/\mathrm{d}\right)\right) = 2^{\left\lfloor \frac{\beta}{2} \right\rfloor + 2} - 2$ for $\mathrm{p}^2, 3\mathrm{p}^2, 3^2\mathrm{p}^2, 3^3\mathrm{p}^2, \ldots, 3^{\beta}\mathrm{p}^2$.

The total number of orbits, in that case is: $\left(2^{\left\lfloor \frac{\beta}{2} \right\rfloor + 2} - 2\right)(\mathrm{p}+1)$. Where $\lfloor . \rfloor$ is greatest integer function. In case where β number is even, the divisors of $\mathrm{N}=3^{\beta}\mathrm{p}^2$ are

I) $\sum_{\mathrm{d}|3^{\beta}} \varphi\left(\left(\mathrm{d}, \mathrm{N}/\mathrm{d}\right)\right) = 3.2^{\frac{\beta}{2}} - 2$ for $1, 3, 3^2, 3^3, \ldots, 3^{\beta}$

II) $\sum_{\mathrm{d}|3^{\beta}} \varphi\left(\left(\mathrm{d}, \mathrm{N}/\mathrm{d}\right)\right) = \left(3.2^{\frac{\beta}{2}} - 2\right)(\mathrm{p}-1)$ for $\mathrm{p}, 3\mathrm{p}, 3^2\mathrm{p}, 3^3\mathrm{p}, \ldots, 3^{\beta}\mathrm{p}$

III) $\sum_{d|3^{\beta}} \varphi\left(\left(d, N/d\right)\right) = 3.2^{\frac{\beta}{2}} - 2$ for $p^2, 3p^2, 3^2p^2, 3^3p^2, ..., 3^{\beta}p^2$.

Therefore the total number of orbits, in that case is: $\left(3.2^{\frac{\beta}{2}} - 2\right)(p+1)$.

Lemma 3.2. The action of the normalizer $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$ on $\hat{\mathbb{Q}}$ is not transitive.

Proof. Theorem 2.9. gives the result.

Theorem 3.3. $\hat{\mathbb{Q}}(3^{\beta}p^2) := \begin{pmatrix}1\\1\end{pmatrix} \cup \begin{pmatrix}1\\3\end{pmatrix} \cup ... \cup \begin{pmatrix}8\\3^2\end{pmatrix} \cup \begin{pmatrix}a_1\\3^{\beta-k}\end{pmatrix} \cup \begin{pmatrix}a_2\\3^{\beta-k}\end{pmatrix} \cup ... \cup \begin{pmatrix}1\\p^2\end{pmatrix}$

$\cup \begin{pmatrix}1\\3p^2\end{pmatrix} \cup \begin{pmatrix}2\\3p^2\end{pmatrix} \cup \begin{pmatrix}1\\3^2p^2\end{pmatrix} \cup ... \cup \begin{pmatrix}8\\3^2p^2\end{pmatrix} \cup \begin{pmatrix}a_1\\3^{\beta-k}p^2\end{pmatrix} \cup \begin{pmatrix}a_2\\3^{\beta-k}p^2\end{pmatrix} \cup ... \cup \begin{pmatrix}a_{\varphi\left(3^{\beta-k},3^k\right)}\\3^{\beta-k}p^2\end{pmatrix}$,

$k \in \{0,1,2,...,\beta-3\}$ is a maximal subset of $\hat{\mathbb{Q}}$ on which the normalizer $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$ acts transitively.

Proof. Let us analyze the act of $\begin{pmatrix}1\\1\end{pmatrix}$ orbit with $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$.

If the element of $A_1 = \begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix}$, with $ad - 3^{\beta-2}bp^2c = 1$ is taken into consideration, the rational number of

$\begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} a+\frac{b}{3} \\ 3^{\beta-1}p^2c+d \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3\left(3^{\beta-1}p^2c+d\right) \end{pmatrix}$ is obtained. In this case,

(i) if $3 \nmid d$, then $\dfrac{3a+b}{3\left(3^{\beta-1}p^2c+d\right)} \in \begin{pmatrix}1\\3\end{pmatrix}$ or $\begin{pmatrix}2\\3\end{pmatrix}$

(ii) if $3 \| d$, then $\dfrac{3a+b}{3^2\left(3^{\beta-2}p^2c+d_0\right)} \in \begin{pmatrix}1\\3^2\end{pmatrix}$ or $\begin{pmatrix}1\\3^2\end{pmatrix},\begin{pmatrix}2\\3^2\end{pmatrix},\begin{pmatrix}4\\3^2\end{pmatrix},\ldots,\begin{pmatrix}8\\3^2\end{pmatrix}$

(iii) if $3^2 \| d$, then $\dfrac{3a+b}{3^3\left(3^{\beta-3}p^2c+d_1\right)} \in \begin{pmatrix}1\\3^3\end{pmatrix}$ or $\begin{pmatrix}a_l\\3^3\end{pmatrix}$ and $\left(a_l,3^3\right)=1.$

We can prove other cases similarly.

Lemma 3.4 . The stabilizer of a point in $\hat{\square}\,(3^\beta p^2)$ is an infinite cyclic group.

Proof. Because of the transitive action, stabilizers of any two points are conjugate.So it is enough to look at just $\infty \in \begin{pmatrix}1\\3^\beta p^2\end{pmatrix}$. As $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}ae & b/3\\ 3^{\beta-1}p^2c & de\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \dfrac{ae}{3^{\beta-1}p^2c} = \begin{pmatrix}1\\0\end{pmatrix}$,

then $c=0$ and $e=1$. From the determinant equality, $T=\begin{pmatrix}1 & b/3\\ 0 & 1\end{pmatrix}$. Consequently take

$N_{PSL(2,\square)}\left(\Gamma_0\left(3^\beta p^2\right)\right)_\infty = \left\langle \begin{pmatrix}1 & 1/3\\ 0 & 1\end{pmatrix} \right\rangle$ in place of G_∞.Now we consider the imprimitive action on $\hat{\square}\,(3^\beta p^2)$ of $\Gamma_0\left(3^\beta p^2\right)$ we define the group $N_0(3^\beta p^2) := \left\langle \Gamma_0(3^\beta p^2), A_1, A_2 \right\rangle$ and denote the stabilizer of ∞ by $N_{PSL(2,\square)}\left(\Gamma_0\left(3^\beta p^2\right)\right)_\infty$. It is clear that $N_{PSL(2,\square)}\left(\Gamma_0\left(3^\beta p^2\right)\right)_\infty \underset{\neq}{<} N_0(3^\beta p^2) \underset{\neq}{<} N_{PSL(2,\square)}\left(\Gamma_0\left(3^\beta p^2\right)\right)$.Let $a+d=1$. The element of A_1 is taken into consideration therefore,

$A_1^2 = \begin{pmatrix} a-1 & b/3 \\ 3^{\beta-1}p^2c & -a \end{pmatrix}$, $A_1^3 = -I$ and also $A_2^2 = \begin{pmatrix} 3^{\beta-2}a-1 & b/3 \\ 3^{\beta-1}p^2c & 3^{\beta-2}d-1 \end{pmatrix}$,

$A_2^6 = \begin{pmatrix} * & * \\ 3^{\beta-1}p^2c\left(3^{2\beta-4}-2.3^{\beta-2}+3\right) & * \end{pmatrix} \in \Gamma_0\left(3^{\beta}p^2\right)$ is obtained. Additionally, in case of $a+d=-1$, $A_2^6 \in \Gamma_0\left(3^{\beta}p^2\right)$ is achieved again. Thus, $A_1^3=-I$ and $A_2^6 \in \Gamma_0\left(3^{\beta}p^2\right) \Leftrightarrow a+d=\pm 1$. As a consequence $\left\{I, A_1, A_1^2\right\} x \left\{I, A_2, A_2^2, A_2^3, A_2^4, A_2^5\right\}$ $= \left\{I, A_2, A_2^2, A_2^3, A_2^4, A_2^5, A_1, A_1A_2, A_1A_2^2, ..., A_1^2A_2^5\right\}$ is achieved as a representatives set of cosets. In this case the number of the blocks is 2 because of the index $\left|N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right) : \Gamma_0(3^{\beta}p^2)\right| = 2^2.3^2.1 = 36$, that is

$$\left|N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right) : N_0(3^{\beta}p^2)\right| . \left|N_0(3^{\beta}p^2) : \Gamma_0(3^{\beta}p^2)\right| = 2.18 = 36$$

It is clear that, $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right) = N_0(3^{\beta}p^2) \cup \begin{pmatrix} ap^2 & b/3 \\ 3^{\beta-1}p^2c & dp^2 \end{pmatrix} N_0\left(3^{\beta}p^2\right)$.

The purpose is to gain to the normalizer by the elements of $N_0(3^{\beta}p^2)$ group, to examine its structure and to characterize the elements of $N_0(3^{\beta}p^2)$ by graphs. We denote by "≈" the invariant equivalence relation reduced by $N_0\left(3^{\beta}p^2\right)$ on $\hat{\mathbb{Q}}(3^{\beta}p^2)$. Let $F\left(\infty, \frac{u}{N}\right)$ denote the subgraph of $G\left(\infty, \frac{u}{N}\right)$ whose vertices form the block $[\infty]:=$ $\left\{\frac{x}{y} \in \mathbb{Q} \cup \{\infty\} : y \equiv 0 \pmod N\right\}$. Shortly, instead of $F\left(\infty, \frac{u}{N}\right)$, we will write $F_{u,N}$. Blocks formed as a result of the imprimitive action are given below:

$$[\infty]=\binom{1}{p^2}\cup\binom{1}{3p^2}\cup\binom{2}{3p^2}\cup\binom{1}{3^2p^2}\cup\binom{2}{3^2p^2}\cup\binom{4}{3^2p^2}\cup\binom{5}{3^2p^2}\cup$$

$$\binom{7}{3^2p^2}\cup\binom{8}{3^2p^2}\cup\binom{a_1}{3^{\beta-k}p^2}\cup\binom{a_2}{3^{\beta-k}p^2}\cup\ldots\cup\binom{a_{\varphi\left(3^{\beta-k},3^k\right)}}{3^{\beta-k}p^2}$$ and

$$[0]=\binom{1}{1}\cup\binom{1}{3}\cup\binom{2}{3}\cup\binom{1}{3^2}\cup\binom{2}{3^2}\cup\binom{4}{3^2}\cup\binom{5}{3^2}\cup\binom{7}{3^2}\cup\binom{8}{3^2}\cup$$

$$\binom{a_1}{3^{\beta-k}}\cup\binom{a_2}{3^{\beta-k}}\cup\binom{a_3}{3^{\beta-k}}\cup\ldots\cup\binom{a_{\varphi\left(3^{\beta-k},3^k\right)}}{3^{\beta-k}}.$$

$\left(N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right),\hat{\mathbb{Q}}\left(3^{\beta}p^2\right)\right)$ being a transitive permutation group, the suborbital graph number of $F_{u,3^{\beta}p^2}$ is $\varphi\left(3^{\beta}p^2\right)$ dir. Therefore, the suborbital graph, vertices of which are on the [∞] block is in the form of $F_{u,3^{\lambda}p^2}$. Considering that $\lambda=0,1,2,3,\ldots,\beta-1,\beta$, total number of graphs on [∞] block is $\varphi\left(p^2\right)+\varphi\left(3p^2\right)+\varphi\left(3^2p^2\right)+\ldots+\varphi\left(3^{\beta-1}p^2\right)+\varphi\left(3^{\beta}p^2\right)$. Taking $F_{u\ ,^{\beta}3^{2}}$ here, the following results are obtained:

Theorem 3.5. Let $\frac{r}{s}$ and $\frac{x}{y}$ be in the block [∞]. Then there is an edge $\frac{r}{s}\rightarrow\frac{x}{y}$ in $F_{u,3^{\beta}p^2}$ if and only if

(i) If $3^{\beta}p^2\|s$ or $3^{\beta-1}p^2\|s$ then $x\equiv\pm ur\left(\bmod 3^{\beta-1}p^2\right)$, $y\equiv\pm us\left(\bmod 3^{\beta}p^2\right)$ and

$ry-sx=\pm\,3^{\beta}p^2$

(ii) For $2\le k\le\beta-1$ and $k\in\mathbb{N}$, if $3^{\beta-k}p^2\|s$ then

$$x \equiv \pm ur\left(\bmod 3p^2\right),\quad y \equiv \pm us\left(\bmod 3^{\beta}p^2\right) \text{ and } ry - sx = \pm\, 3^{2\beta-2}p^2$$

(iii) If $p^2 \| s$ then $x \equiv \pm 3ur\left(\bmod 3p^2\right), y \equiv \pm 3us\left(\bmod 3^{\beta}p^2\right)$ and $ry - sx = \pm\, 3^{2\beta-2}p^2$.

Proof. "$\Rightarrow$": Suppose that $\frac{r}{s} \to \frac{x}{y}$ is an edge in $F_{u,3^{\beta}p^2}$. Therefore there exists some T in the normalizer $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$ such that T sends the pair $\left(\infty, \frac{u}{3^{\beta}p^2}\right)$ to the pair $\left(\frac{r}{s}, \frac{x}{y}\right)$, that is $T(\infty) = \frac{r}{s}$ and $T\left(\frac{u}{3^{\beta}p^2}\right) = \frac{x}{y}$.

(i) Since $3^{\beta}p^2 \| s$ or $3^{\beta-1}p^2 \| s$, T must be of the form $\begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix}$ and detT $= ad - 3^{\beta-2}bp^2c = 1$. If $T(\infty) = \frac{a}{3^{\beta-1}p^2c} = \frac{r}{s}$ then $r = (-1)^i a$ and $s = (-1)^i 3^{\beta-1}p^2c$ for $i = 0,1$ and $c = 3c_0$, $c_0 \in \mathbb{Z}$. If $3 \nmid a$ then

$$T\begin{pmatrix} u \\ 3^{\beta}p^2 \end{pmatrix} = \begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix}\begin{pmatrix} u \\ 3^{\beta}p^2 \end{pmatrix} = \frac{au + 3^{\beta-1}bp^2}{3^{\beta-1}p^2cu + 3^{\beta}dp^2} = \frac{x}{y}.$$

Hence $x = (-1)^j\left(au + 3^{\beta-1}bp^2\right)$ and $y = (-1)^j\left(3^{\beta-1}p^2cu + 3^{\beta}dp^2\right)$ for $j = 0,1$. That is $x \equiv \pm ur\left(\bmod 3^{\beta-1}p^2\right)$, $y \equiv \pm us\left(\bmod 3^{\beta}p^2\right)$.

Moreover $\begin{pmatrix} a & b/3 \\ 3^{\beta-1}p^2c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 3^{\beta}p^2 \end{pmatrix} = \begin{pmatrix} (-1)^i r & (-1)^j x \\ (-1)^i s & (-1)^j y \end{pmatrix}$ $i,j = 0,1$ we get $ry - sx = \pm 3^{\beta}p^2$. The other (ii) and (iii) are likewise done.

"$\Leftarrow$":**(i)** Let $3^{\beta}p^2 \| s$ $x \equiv \pm ur\left(\mathrm{mod}\,3^{\beta-1}p^2\right)$, $y \equiv \pm us\left(\mathrm{mod}\,3^{\beta}p^2\right)$ and $ry - sx = \pm\, 3^{\beta}p^2$. In that case, $\exists b, d \in \mathbb{Z}$ is such that $x = ur + 3^{\beta-1}bp^2$ and $y = us + 3^{\beta}dp^2$ .With these equations, $ry - sx = r\left(us + 3^{\beta}dp^2\right) - s\left(ur + 3^{\beta-1}bp^2\right) = 3^{\beta}p^2$ is achieved and if both sides of the equation is divided by $3^{\beta}p^2$, $rd - {bs}/{3} = 1$ is achieved. Accordingly, for $T_0 :=$ $\begin{pmatrix} r & b/3 \\ s & d \end{pmatrix}$, $\det T_0 = 1$ and $3^{\beta}p^2 \| s$ $T_0 \in \Gamma_0(3^{\beta}p^2) \subset N_0(3^{\beta}p^2)$ is achieved.

With a similar process, the others (ii) and (iii) are likewise done.

Theorem 3.6. Let $N = 3^{\beta}p^2$ and $\beta \geq 4$. For p prime, $p > 3$ and $n \geq 3$ $(n-\text{gon})$ then, the suborbital graph $F_{u,3^{\beta}p^2}$ of $N_{PSL(2,\mathbb{R})}\left(\Gamma_0\left(3^{\beta}p^2\right)\right)$, is a forest .

Proof. Let C be a circuit in $F_{u,3^{\beta}p^2}$ of minimal length. Suppose first that C is directed, $v_1 \to v_2 \to v_3 \to \ldots \to v_k$. We may choose the vertices of C apart from ∞ in the interval $\left[\frac{u}{3^{\beta}p^2}, \frac{u+3^{\beta}p^2}{3^{\beta}p^2}\right]$ as $v_1 < v_2 < v_3 < \ldots < v_k$. If $v_1 = \frac{u+3^{\beta}p^2}{3^{\beta}p^2}$, v_k is single edged in $\left[\frac{u}{3^{\beta}p^2}, \frac{u+3^{\beta}p^2}{3^{\beta}p^2}\right]$, which $v_k \to \infty$. Because the chosen circuit is in the form of $\infty \to v_1 \to v_2 \to v_3 \to \ldots \to v_k \to \infty$. Let us assume $v_k = \frac{r}{3^{\beta}p^2}$.

Since there can no prime number exist between two adjacent vertices in $F_{u,3^{\beta}p^2}$ the circuit cannot go to the left of 1. Which means, if $v_k = \frac{r}{3^{\beta}p^2}$, then $r > 2^{\alpha}p^2$. Therefore, $r = 3^{\beta}p^2 + k$ under the condition that $\exists k > 0$. Additionally, since $r < 3^{\beta}p^2 + u$,

$k < u < 3^{\beta}p^2$ is achieved. Because $1 \equiv -ur \pmod{3^{\beta-1}p^2}$, $1 \equiv -u\left(3^{\beta}p^2 + k\right) \pmod{3^{\beta-1}p^2}$ equation is achieved. Accordingly, $1 \equiv -uk \pmod{3^{\beta-1}p^2}$ if and only if $\frac{k}{3^{\beta}p^2} \to \frac{1}{0} \in F_{u,3^{\beta}p^2}$. However, if $\frac{k}{3^{\beta}p^2} < 1$ this creates a contradiction. Therefore, such a C circuit does not exist. Thus, we can take $\infty \to \frac{u}{3^{\beta}p^2} \to v_2 \to v_3 \to ... \to v_k \to \infty$ as a C circuit. $v_k > \frac{u+1}{3^{\beta}p^2}$.

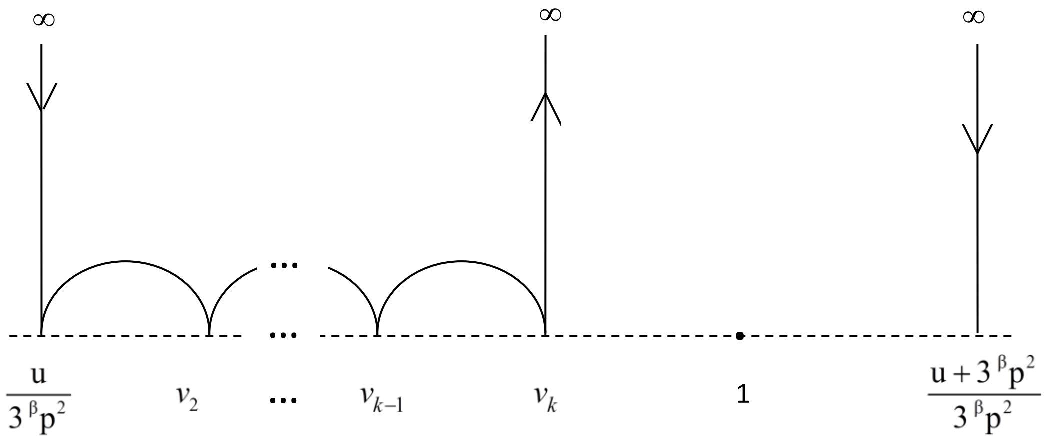

Figure 2. Path of the action

Let v be the largest rational greater than v_1 for which $v_1 \to v$ is an edge in $F_{u,3^{\beta}p^2}$.

We see that v_2 must equal v. Assume otherwise that $v_2 < v$. If v is a vertex in C, then we obtain a circuit which is a shorter length than C. If v is not a vertex in C then there are vertices v_i, v_{i+1} in C such that $v_i < v < v_{i+1}$. In this case, the edges $v_2 \to v$ and $v_i \to v_{i+1}$ cross to each other, it is a contradict the fact that no edges of $F_{u,3^{\beta}p^2}$

cross in H. Consequently, $v_2 = v$. As $v_1 < v_2$, $v_2 = \dfrac{u+\frac{m}{k_0}}{3^{\beta}p^2}$ for some positive integers m and k_0. Since $v_1 \to v_2$ is an edge in $F_{u,3^{\beta}p^2}$, then $3^{\beta}p^2 v_1 \to 3^{\beta}p^2 v_2$ is an edge $F_{u,3^{\beta}p^2}$. Thus m must be 1. From the edge $\dfrac{u}{3^{\beta}p^2} \to \dfrac{u+\frac{m}{k_0}}{3^{\beta}p^2} = \dfrac{uk_0+m}{3^{\beta}p^2k_0} \in F_{u,3^{\beta}p^2}$ is obtained. Also $3^{\beta}p^2k_0u - 3^{\beta}p^2k_0u - 3^{\beta}p^2m = -3^{\beta}p^2$ then $m=1$ and $\dfrac{u}{3^{\beta}p^2} \overset{<}{\longrightarrow} \dfrac{uk_0+1}{3^{\beta}p^2k_0}$.

We obtain $u^2 + uk_0 + 1 \equiv 0 \pmod{3^{\beta-1}p^2}$ by Theorem 3.5.

Now we define the following transformation $\varphi := \begin{pmatrix} -u & \dfrac{u^2+uk_0+1}{3^{\beta}p^2} \\ -3^{\beta}p^2 & u+k_0 \end{pmatrix}$, $\det\varphi = 1$.

Then $\varphi \in N_0\left(3^{\beta}p^2\right)$.Since $\varphi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-u\\-3^{\beta}p^2\end{pmatrix} = \dfrac{u}{3^{\beta}p^2} = v_1$, $\varphi(\infty) = v_1$ and $\varphi\begin{pmatrix}u\\3^{\beta}p^2\end{pmatrix} = \dfrac{u+\frac{1}{k_0}}{3^{\beta}p^2} = v_2$, that is $\varphi(v_1) = v_2$. In general, $\varphi\begin{pmatrix}u+\dfrac{x}{y}\\3^{\beta}p^2\end{pmatrix} = \dfrac{u\left(k_0-\frac{x}{y}\right)+1}{3^{\beta}p^2\left(k_0-\frac{x}{y}\right)} = \dfrac{u+\dfrac{y}{k_0y-x}}{3^{\beta}p^2}$.

In addition to $\begin{pmatrix} -u & \frac{u^2+uk_0+1}{3^{\beta}p^2} \\ -3^{\beta}p^2 & u+k_0 \end{pmatrix}\begin{pmatrix} u+\frac{1}{k_0} \\ 3^{\beta}p^2 \end{pmatrix} = \frac{u+\frac{1}{k_0-\frac{1}{k_0}}}{3^{\beta}p^2}$, … continues likewise.

This means, $\frac{1}{0} \to \frac{u}{3^{\beta}p^2} \to \frac{u+\frac{1}{k_0}}{3^{\beta}p^2} \to \frac{u+\frac{1}{k_0-\frac{1}{k_0}}}{3^{\beta}p^2} \to \frac{u+\frac{1}{k_0-\frac{1}{k_0-\frac{1}{k_0}}}}{3^{\beta}p^2} \to \ldots$

is obtained. Therefore, it is clear that $\cfrac{1}{k_0-\cfrac{1}{k_0-\cfrac{1}{k_0}}} \atop \vdots$ fraction is an irrational number for $k_0 > 2$. Therefore, we can easily see that $\varphi^i(v_1) < \frac{u+1}{3^{\beta}p^2}$ for positive integers i. Therefore continues likewise $\varphi^i(v_1) < \frac{u+1}{3^{\beta}p^2}$ is obtained for $\forall i \in$ .We now that $v_{i+1} = \varphi^i(v_1) = \varphi^{i+1}(\infty)$ for $0 \le i \le k-1$. We already know that $\varphi(v_1) = v_2$. Now assume that $v_i = \varphi^{i-1}(v_1)$ for all $1 \le i \le s$. Then let us show that $v_{s+1} = \varphi^s(v_1)$. If not, then first assume that $v_{s+1} < \varphi^s(v_1)$. Then by transitive action, $v_s \to \varphi^{s-1}(v_1) \to \varphi^{s-1}(v_2) = \varphi^s(v_1)$ is an edge in $F_{u,3^{\beta}p^2}$. If $\varphi^s(v_1)$ is not a vertex in C, as $\varphi^s(v_1) < v_k$, there exist vertices v_t and v_{t+1} such that $v_t < \varphi^s(v_1) < v_{t+1}$ and therefore the edges $v_t \to v_{t+1}$ and $v_s \to \varphi^s(v_1)$ cross, a contradiction. If $\varphi^s(v_1)$ is a vertex in C , as $v_{s+1} < \varphi^s(v_1)$, $\varphi^s(v_1) = v_m$ for some $m \ge s+2$. However, in this case, we would have a circuit

$\infty \to v_1 \to \ldots \to v_s \to v_m \to v_k \to \infty$ which is of a shorter length, again a contradiction. Now suppose finally that $v_{s+1} > \varphi^s(v_1)$. Then from the above $v_{s+1} > \varphi^s(v_1) > \varphi^{s-2}(v_1)$. Since $v_1 < v_2$ and φ is strictly increasing $\varphi(v_1) < \varphi(v_2)$. Moreover $\varphi(\varphi(v_1)) = \varphi^2(v_1) = \varphi(v_2)$ and $\varphi(v_1) < \varphi^2(v_1)$. Since $\varphi(v_1) < \varphi^2(v_1)$ and φ is strictly increasing $\varphi(\varphi(v_1)) < \varphi(\varphi^2(v_1))$ and $\varphi^2(v_1) < \varphi^3(v_1)$. With a similar process $\varphi^{s-2}(v_1) < \varphi^{s-1}(v_1) < \varphi^s(v_1)$ is obtained. As $\varphi^{s-2}(v_1) < \varphi^{s-1}(v_1) < \varphi^s(v_1)$ and $\varphi^{-(s-1)}(\varphi^{s-2}(v_1)) = \infty$, $\varphi^{-(s-1)}(v_{s+1}) > \varphi^{-(s-1)}(\varphi^s(v_1)) = \varphi(v_1) = v_2$. Hence by transitive action, $v_1 = \varphi^{-(s-1)}(v_s) \to \varphi^{-(s-1)}(v_{s+1})$ is an edge in $F_{u,3^\beta p^2}$, which is contradiction to the choice of v_2. Consequently $v_{i+1} = \varphi^i(v_1)$ for $0 \le i \le k-1$. Thus, $v_k < \frac{u+1}{3^\beta p^2}$ a contradiction.Finally, assume that there is an anti-directed circuit C as minimal length, of the form $\infty \to v_1 = \frac{u}{3^\beta p^2} \ldots \to v_t \leftarrow v_{t+i} \ldots v_k \to \infty$ for some $t \ge 1$. We know from the above that $v_i = \varphi^i(\infty)$ for $i \le t$. Let v be the largest rational greater than $\frac{u}{3^\beta p^2}$ such that $v_1 \leftarrow v$ is an edge $F_{u,3^\beta p^2}$. Then $v = \frac{u + \frac{1}{k_0}}{3^\beta p^2}$ for some integer k_0. And also t must be greater than 1, otherwise $v < v_s = \frac{u + \frac{1}{k_0}}{3^\beta p^2}$ for some $s \ge 3$ and then we would circuit $\infty \to v_1 \to v_s \to \ldots v_k \to \infty$ of a shorter length, a contradiction. Hence we must have $v_1 \to v_2 = \frac{u + \frac{1}{k_0}}{3^\beta p^2}$. Let $w = \varphi^{t+1}(\infty)$. Since, by transitive action,

$v_t \to \varphi^{t-1}(v_1) \to \varphi^{t-1}(v_2)$ is an edge $F_{u,3^\beta p^2}$. Therefore, the inequality $v_{t+1} < w$ must be true. For if $v_{t+1} > w$ then, as $\varphi^{-(t-1)}(\varphi^{t-2}(v_1)) = \infty$ and $\varphi^{t-2}(v_1) < \varphi^{t}(v_1) = \varphi^{t}(\varphi(\infty)) = \varphi^{t+1}(\infty)$, $w < v_{t+1}$, $\varphi^{-(t-1)}(v_{t+1}) > \varphi^{-(t-1)}(w) = v_2$ and $\varphi^{-(t-1)}(v_t) = v_1 \leftarrow \varphi^{-(t-1)}(v_{t+1}) > v_2$ is an edge in $F_{u,3^\beta p^2}$, which contradicts the choice of v_2. However, if $v_{t+1} < w$ then we would have $w = v_s$ for some $s \geq t+2$ and therefore we would have the circuit $\infty \to v_1 \to \ldots \to v_t \to v_s \to \ldots v_k \to \infty$ of a shorter length, which again gives a conradiction. This shows that C must be directed. Hence the proof of the theorem is completed.

4. RESULTS

In this paper, the following results are obtained:

The number of all the orbits of $\Gamma_0(3^\beta p^2)$ on $\hat{\mathbb{Q}}$ calculated for $\beta \geq 4$.

Edge conditions is achieved of $F_{u,3^\beta p^2}$ suborbital graphs of $N_{PSL(2,\mathbb{R})}(\Gamma_0(3^\beta p^2))$ on $\hat{\mathbb{Q}}$.

And also it is shown to contain no circuit of $F_{u,3^\beta p^2}$ suborbital graphs of $N_{PSL(2,\mathbb{R})}(\Gamma_0(3^\beta p^2))$ on $\hat{\mathbb{Q}}$.

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