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## FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS

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### Abstract

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In this paper, by making use of identity proved in Shi et al (2014) for fractional integrals, several inequalities of Hermite-Hadamard type for quasi-convex functions via Riemann-Liouville fractional integrals and the well known Hölder integral inequality are obtained. Some applications for the special means are also given.

**Keywords:** Quasi-convex function, Hermite-Hadamard inequality, Riemann-Liouville fractional integral.

## QUASI-KONVEKS FONSIYONLAR İÇİN KESİRLİ HERMİTE-HADAMARD TİPLİ EŞİTSİZLİKLER

### Özet

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Bu makalede, kesirli integraller için Shi ve arkadaşları tarafından (2014)'de elde edilen özdeşlik kullanılarak, Riemann-Liouville kesirli integralleri ve literatürde iyi bilinen Hölder eşitsizliği yardımıyla quasi konveks fonksiyonlar için Hermite-Hadamard tipli eşitsizlikler elde edilmiştir. Ayrıca özel ortalamalar için bazı uygulamalar verilmiştir.

**Anahtar Kelimeler:** Quasi konveks fonksiyon, Hermite-Hadamard eşitsizliği, Riemann-Liouville kesirli integrali.

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## 1. Introduction

One of the most famous inequality for convex functions is so called Hermite-Hadamard inequality as follows:

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and  $x, y \in I$  with  $x < y$ , then

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(t) dt \leq \frac{f(x) + f(y)}{2}$$

is known as the Hermite-Hadamard inequality.

**Definition 1.1** A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ .

**Definition 1.2** A mapping  $f : I \rightarrow \mathbb{R}$  is called quasi-convex on the convex set  $I$  if for all  $x, y \in I$  and  $\lambda \in [0, 1]$

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

This class of functions strictly contains the class of convex functions defined on a convex set in a real linear space, see Eberhard & Pearce (2000) and citations therein for an overview. Recent studies have shown that quasi convex functions have quite close resemblances to convex functions see, example Dragomir & Bond (1997), Dragomir & Pearce (1998), Dragomir (1995), Pearce & Rubinov (1999) for quasi convex and even more general extensions of convex functions in the context of Hermite-Hadamard's inequalities. Apart from generalizations to theory, weakening the convexity condition can increase applicability. Thus in Pearce (2004) use is made of quasi-convexity to obtain a global extremum with rather less effort than via convexity.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.3** Let  $f \in L_1[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ . Here is  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ . In the case of

$\alpha=1$ , the fractional integral reduces to the classical integral. Some recent result and properties concerning the operator can be found Belarbi & Dahmani (2009); Dahmani (2010); Gorenflo & Mainardi (1997); Iscan (2013); Miller & Ross (1993); Sarıkaya & Ogunmez (2012); Sarıkaya et al (2013); Set (2012) .

We establish here new Hermite-Hadamard type inequalities for quasi-convex function via Riemann-Liouville fractional integral. An interesting feature of our results is that they provide new estimate, on these types of inequalities for fractional integrals.

## 2. Main Results

**Lemma 2.1** [15] Assume  $a, b \in \mathbb{R}$  with  $a < b$  and  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable function on  $(a, b)$ . If  $f' \in (L[a, b])$  then the following equality holds:

$$\begin{aligned} \Phi_{\alpha}(a, b) = & \frac{b-a}{16} \left[ \int_0^1 (1-t^{\alpha}) f' \left( t \frac{3a+b}{4} + (1-t) \frac{a+b}{2} \right) dt \right. \\ & - \int_0^1 t^{\alpha} f' \left( ta + (1-t) \frac{3a+b}{4} \right) dt \\ & + \int_0^1 (1-t^{\alpha}) f' \left( t \frac{a+3b}{4} + (1-t)b \right) dt \\ & \left. - \int_0^1 t^{\alpha} f' \left( t \frac{a+b}{2} + (1-t) \frac{a+3b}{4} \right) dt \right] \end{aligned}$$

where for  $\alpha > 0$

$$\begin{aligned} & \Phi_{\alpha}(a, b) \\ &= \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f \left( \frac{a+b}{2} \right) \right] - \frac{4^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^{\alpha}} \\ & \times \left[ J_{a^{+}}^{\alpha} f \left( \frac{3a+b}{4} \right) + J_{\left( \frac{3a+b}{4} \right)^{+}}^{\alpha} f \left( \frac{a+b}{2} \right) + J_{\left( \frac{a+b}{2} \right)^{+}}^{\alpha} f \left( \frac{a+3b}{4} \right) + J_{\left( \frac{a+3b}{4} \right)^{+}}^{\alpha} f(b) \right]. \end{aligned}$$

It is easy to see that

$$\Phi_1(a, b) = \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f \left( \frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x) dx.$$

**Theorem 2.1** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable on  $I^{\circ}$  such that  $f' \in L([a, b])$  with  $a, b \in I$ ,  $a < b$  and  $\alpha > 0$ . If  $|f'|^q$  is quasi-convex function on  $[a, b]$  and  $q \geq 1$ , then we have the following inequality:

$$\begin{aligned}
 & |\Phi_\alpha(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \frac{\max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\}}{\alpha+1} \right]^{\frac{1}{q}} \\
 & + \frac{\alpha}{\alpha+1} \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}.
 \end{aligned}$$

**Proof.** Using Lemma 2.1 and well known power mean inequality and the quasi-convexity of  $|f'|^q$  on  $[a, b]$ , we get

$$\begin{aligned}
 & |\Phi_\alpha(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \int_0^1 t^\alpha \left| f' \left( ta + (1-t) \frac{3a+b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{3a+b}{4} + (1-t) \frac{a+b}{2} \right) \right| dt \right. \\
 & \quad \left. + \int_0^1 t^\alpha \left| f' \left( t \frac{a+b}{2} + (1-t) \frac{a+3b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{a+3b}{4} + (1-t)b \right) \right| dt \right] \\
 & \leq \frac{b-a}{16} \left[ \left( \int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^\alpha \max \left\{ |f'(a)|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_0^1 (1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t^\alpha) \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^\alpha \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_0^1 (1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t^\alpha) \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

Substituting

$$\int_0^1 t^\alpha dt = \frac{1}{\alpha+1}$$

and

$$\int_0^1 (1-t^\alpha) dt = \frac{\alpha}{\alpha+1}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.1 is complete.

**Corollary 2.1** *In Theorem 2.1, if we choose  $q = 1$ , we have*

$$\begin{aligned} & |\Phi_\alpha(a, b)| \\ & \leq \frac{b-a}{16} \left[ \frac{\max \left\{ \left| f'(a) \right|, \left| f' \left( \frac{3a+b}{4} \right) \right| \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|, \left| f' \left( \frac{a+3b}{4} \right) \right| \right\}}{\alpha+1} \right. \\ & \quad \left. + \frac{\alpha}{\alpha+1} \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|, \left| f' \left( \frac{a+b}{2} \right) \right| \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|, |f'(b)| \right\} \right) \right]. \end{aligned}$$

**Corollary 2.2** *In Theorem 2.1, if we choose  $\alpha = 1$ , we have*

$$\begin{aligned} & |\Phi_1(a, b)| \\ & \leq \frac{b-a}{32} \left[ \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\}^{\frac{1}{q}} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\}^{\frac{1}{q}} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, |f'(b)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

**Corollary 2.3** *In Theorem 2.1, if we choose  $q = 1$  and  $\alpha = 1$ , we have*

$$\begin{aligned} |\Phi_1(a, b)| & \leq \frac{b-a}{32} \left[ \max \left\{ \left| f'(a) \right|, \left| f' \left( \frac{3a+b}{4} \right) \right| \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|, \left| f' \left( \frac{a+3b}{4} \right) \right| \right\} \right. \\ & \quad \left. + \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|, \left| f' \left( \frac{a+b}{2} \right) \right| \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|, |f'(b)| \right\} \right]. \end{aligned}$$

**Theorem 2.2** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$  such that  $f' \in L([a, b])$  with  $a, b \in I$ ,  $a < b$  and  $\alpha > 0$ . If  $|f'|^q$  is quasi-convex function on  $[a, b]$ ,  $q > 1$  and  $q \geq r \geq 0$ , then we have the following:*

$$\begin{aligned}
 & |\Phi_\alpha(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \left( \frac{q-1}{\alpha(q-r)+q-1} \right)^{1-\frac{1}{q}} \left( \frac{1}{\alpha r+1} \right)^{\frac{1}{q}} \right. \\
 & \quad \times \left( \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \\
 & \quad + \left( \frac{1}{\alpha} \frac{\Gamma \left( \frac{2q-r-1}{q-1} \right) \Gamma \left( \frac{1}{\alpha} \right)}{\Gamma \left( \frac{2q-r-1}{q-1} + \frac{1}{\alpha} \right)} \right)^{1-\frac{1}{q}} \left( \frac{\Gamma \left( 1 + \frac{1}{\alpha} \right) \Gamma(r+1)}{\Gamma \left( r + \frac{1}{\alpha} + 1 \right)} \right)^{\frac{1}{q}} \\
 & \quad \times \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \Big] 1.
 \end{aligned}$$

**Proof.** Using Lemma 2.1 and well known Hölder inequality and the quasi-convexity of  $|f'|^q$  on  $[a, b]$ , we get

$$\begin{aligned}
 & |\Phi_\alpha(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \int_0^1 t^\alpha \left| f' \left( ta + (1-t) \frac{3a+b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{3a+b}{4} + (1-t) \frac{a+b}{2} \right) \right| dt \right. \\
 & \quad + \int_0^1 t^\alpha \left| f' \left( t \frac{a+b}{2} + (1-t) \frac{a+3b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{a+3b}{4} + (1-t)b \right) \right| dt \Big] \\
 & \leq \frac{b-a}{16} \left[ \left( \int_0^1 t^{\alpha \left( \frac{q-r}{q-1} \right)} dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^{\alpha r} \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \right. \\
 & \quad + \left( \int_0^1 (1-t^\alpha)^{\left( \frac{q-r}{q-1} \right)} dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t^\alpha)^r \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \\
 & \quad + \left( \int_0^1 t^{\alpha \left( \frac{q-r}{q-1} \right)} dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^{\alpha r} \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \\
 & \quad + \left( \int_0^1 (1-t^\alpha)^{\left( \frac{q-r}{q-1} \right)} dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t^\alpha)^r \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} dt \right)^{\frac{1}{q}} \Big]
 \end{aligned}$$

Substituting

$$\int_0^1 t^{\alpha r} dt = \frac{1}{\alpha r + 1},$$

$$\int_0^1 t^{\alpha \left(\frac{q-r}{q-1}\right)} dt = \frac{q-1}{\alpha(q-r)+q-1},$$

$$\int_0^1 (1-t^{\alpha})^{\left(\frac{q-r}{q-1}\right)} dt = \frac{1}{\alpha} \frac{\Gamma\left(\frac{2q-r-1}{q-1}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{2q-r-1}{q-1} + \frac{1}{\alpha}\right)},$$

and

$$\int_0^1 (1-t^{\alpha})^r dt = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(r+1)}{\Gamma\left(r+\frac{1}{\alpha}+1\right)}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.2 is complete.

**Corollary 2.4** *In Theorem 2.2, if we choose  $r = 0$ , we have*

$$\begin{aligned} & |\Phi_{\alpha}(a,b)| \\ & \leq \frac{b-a}{16} \left[ \left( \frac{q-1}{\alpha q + q-1} \right)^{1-\frac{1}{q}} \right. \\ & \quad \times \left( \max \left\{ \left| f'(a) \right|^q, \left| f'\left(\frac{3a+b}{4}\right) \right|^q \right\} + \max \left\{ \left| f'\left(\frac{a+b}{2}\right) \right|^q, \left| f'\left(\frac{a+3b}{4}\right) \right|^q \right\} \right)^{\frac{1}{q}} \\ & \quad + \left. \left( \frac{1}{\alpha} \frac{\Gamma\left(\frac{2q-1}{q-1}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{2q-1}{q-1} + \frac{1}{\alpha}\right)} \right)^{1-\frac{1}{q}} \right. \\ & \quad \times \left. \left( \max \left\{ \left| f'\left(\frac{3a+b}{4}\right) \right|^q, \left| f'\left(\frac{a+b}{2}\right) \right|^q \right\} + \max \left\{ \left| f'\left(\frac{a+3b}{4}\right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \right]. \end{aligned}$$



**Corollary 2.5** In Theorem 2.2, if we choose  $r = q$ , we have

$$\begin{aligned}
 & |\Phi_\alpha(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \left( \frac{1}{\alpha q + 1} \right)^{\frac{1}{q}} \right. \\
 & \quad \times \left( \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \\
 & \quad + \left( \frac{1}{\alpha} \frac{\Gamma \left( \frac{1}{\alpha} \right)}{\Gamma \left( 1 + \frac{1}{\alpha} \right)} \right)^{1-\frac{1}{q}} \left( \frac{\Gamma \left( 1 + \frac{1}{\alpha} \right) \Gamma(q+1)}{\Gamma \left( 1 + \frac{1}{\alpha} + q \right)} \right)^{\frac{1}{q}} \\
 & \quad \times \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \Big].
 \end{aligned}$$

**Corollary 2.6** In Theorem 2.2, if we choose  $\alpha = 1$ , we have

$$\begin{aligned}
 & |\Phi_1(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \left( \frac{q-1}{2q-r-1} \right)^{1-\frac{1}{q}} \left( \frac{1}{r+1} \right)^{\frac{1}{q}} \right. \\
 & \quad \times \left( \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \\
 & \quad + \left( \frac{\Gamma \left( \frac{2q-r-1}{q-1} \right)}{\Gamma \left( \frac{2q-r-1}{q-1} + 1 \right)} \right)^{1-\frac{1}{q}} \left( \frac{\Gamma(r+1)}{\Gamma(r+2)} \right)^{\frac{1}{q}} \\
 & \quad \times \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \Big].
 \end{aligned}$$

**Remark 2.1** If we choose  $r = 1$ , then Theorem 2.2 reduces the Theorem 2.1.

**Theorem 2.3** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$  such that  $f' \in L([a, b])$  with  $a, b \in I$ ,  $a < b$  and  $\alpha > 0$ . If  $|f'|^q$  is quasi-convex function on  $[a, b]$ ,  $q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then we have the following inequality:

$$\begin{aligned} & |\Phi_\alpha(a, b)| \\ & \leq \frac{b-a}{16} \left[ \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \right. \\ & \quad \times \left( \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \\ & \quad + \left( \frac{\Gamma \left( 1 + \frac{1}{\alpha} \right) \Gamma(p+1)}{\Gamma \left( p + \frac{1}{\alpha} + 1 \right)} \right)^{\frac{1}{p}} \\ & \quad \times \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \Big] . \end{aligned}$$

**Proof.** Using Lemma 2.1 and well known Hölder inequality and the quasi-convexity of  $|f'|^q$  on  $[a, b]$ , we get

$$\begin{aligned} & |\Phi_\alpha(a, b)| \\ & \leq \frac{b-a}{16} \left[ \int_0^1 t^\alpha \left| f' \left( ta + (1-t) \frac{3a+b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{3a+b}{4} + (1-t) \frac{a+b}{2} \right) \right| dt \right. \\ & \quad + \int_0^1 t^\alpha \left| f' \left( t \frac{a+b}{2} + (1-t) \frac{a+3b}{4} \right) \right| dt + \int_0^1 (1-t^\alpha) \left| f' \left( t \frac{a+3b}{4} + (1-t)b \right) \right| dt \Big] \\ & \leq \frac{b-a}{16} \left[ \left( \int_0^1 (t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f'(a) \right|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 (t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}} \right] . \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \\
 & + \left( \int_0^1 (t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} dt \right)^{\frac{1}{q}} \\
 & + \left( \int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left( \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}}.
 \end{aligned}$$

Substituting

$$\begin{aligned}
 \int_0^1 (t^\alpha)^p dt &= \frac{1}{\alpha p + 1}, \\
 \int_0^1 (1-t^\alpha)^p dt &= \frac{\Gamma\left(1 + \frac{1}{\alpha}\right) \Gamma(p+1)}{\Gamma\left(p + \frac{1}{\alpha} + 1\right)},
 \end{aligned}$$

into the above inequality and simplifying lead to the required inequality. The proof of Theorem 2.3 is complete.

**Corollary 2.7** *In Theorem 2.3, if we choose  $\alpha = 1$ , we have*

$$\begin{aligned}
 & |\Phi_1(a, b)| \\
 & \leq \frac{b-a}{16} \left[ \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left( \max \left\{ |f'(a)|^q, \left| f' \left( \frac{3a+b}{4} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+b}{2} \right) \right|^q, \left| f' \left( \frac{a+3b}{4} \right) \right|^q \right\} \right)^{\frac{1}{q}} \\
 & \quad + \left( \frac{\Gamma(p+1)}{\Gamma(p+2)} \right)^{\frac{1}{p}} \\
 & \quad \times \left( \max \left\{ \left| f' \left( \frac{3a+b}{4} \right) \right|^q, \left| f' \left( \frac{a+b}{2} \right) \right|^q \right\} + \max \left\{ \left| f' \left( \frac{a+3b}{4} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \Big].
 \end{aligned}$$

### 3. Applications

We shall consider the means for two positive numbers  $a > 0$  and  $b > 0$ . We take

1. Arithmetic mean:

$$A(a, b) = \frac{a+b}{2}.$$

2. Harmonic mean:

$$H(a, b) = \frac{2ab}{a+b}.$$

3. Generalized log-mean:

$$L_p(a, b) = \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, & p \neq 0, -1 \text{ and } a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \text{ and } a \neq b, \\ I(a, b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

Now, using the results of Section 2, we give some applications to special means of numbers.

**Theorem 3.1** Let  $b > a > 0$ ,  $q \geq 1$  and  $p \in \mathbb{R}$ .

1. If  $p > 1$  and  $q(p-1) \leq 1$ , or  $p < 0$  and  $p \neq -1$  then

$$\begin{aligned} & \left| \frac{A(a^p, b^p) + [A(a, b)]^p}{2} - [L_p(a, b)]^p \right| \\ & \leq |p|^q \frac{b-a}{32} \left[ \max \left\{ a^{q(p-1)}, A(a, A(a, b))^{q(p-1)} \right\}^{\frac{1}{q}} + \max \left\{ A(a, b)^{q(p-1)}, A(A(a, b), b)^{q(p-1)} \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \max \left\{ A(a, A(a, b))^{q(p-1)}, A(a, b)^{q(p-1)} \right\}^{\frac{1}{q}} + \max \left\{ A(A(a, b), b)^{q(p-1)}, b^{q(p-1)} \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

2. If  $p = -1$ , then

$$\begin{aligned} & \left| \frac{1}{2} \left[ \frac{1}{H(a, b)} + \frac{1}{A(a, b)} \right] - \frac{1}{L(a, b)} \right| \\ & \leq \frac{b-a}{32} \left[ \max \left\{ \frac{1}{a^{2q}}, \frac{1}{A(a, A(a, b))^{2q}} \right\}^{\frac{1}{q}} + \max \left\{ \frac{1}{A(a, b)^{2q}}, \frac{1}{A(A(a, b), b)^{2q}} \right\}^{\frac{1}{q}} \right] \end{aligned}$$

$$+ \max \left\{ \frac{1}{A(a, A(a, b))^{2q}}, \frac{1}{A(a, b)^{2q}} \right\}^{\frac{1}{q}} + \max \left\{ \frac{1}{A(A(a, b), b)^{2q}}, \frac{1}{b^{2q}} \right\}^{\frac{1}{q}} \Big].$$

3. If  $q = 1$  and  $p \geq 2$ , or  $q = 1$  and  $-1 \neq p < 0$ , then

$$\begin{aligned} & \left| \frac{A(a^p, b^p) + [A(a, b)]^p}{2} - [L_p(a, b)]^p \right| \\ & \leq |p| \frac{(b-a)}{32} [\max \{a^{(p-1)}, A(a, A(a, b))^{(p-1)}\} + \max \{A(a, b)^{(p-1)}, A(A(a, b), b)^{(p-1)}\} \\ & \quad + \max \{A(a, A(a, b))^{(p-1)}, A(a, b)^{(p-1)}\} + \max \{A(A(a, b), b)^{(p-1)}, b^{(p-1)}\}]. \end{aligned}$$

4. If  $p = -1$  and  $q = 1$ , then

$$\begin{aligned} & \left| \frac{1}{2} \left[ \frac{1}{H(a, b)} + \frac{1}{A(a, b)} \right] - \frac{1}{L(a, b)} \right| \\ & \leq \frac{b-a}{32} [\max \left\{ \frac{1}{a^2}, \frac{1}{A(a, A(a, b))^2} \right\} + \max \left\{ \frac{1}{A(a, b)^2}, \frac{1}{A(A(a, b), b)^2} \right\} \\ & \quad + \max \left\{ \frac{1}{A(a, A(a, b))^2}, \frac{1}{A(a, b)^2} \right\} + \max \left\{ \frac{1}{A(A(a, b), b)^2}, \frac{1}{b^2} \right\}]. \end{aligned}$$

**Proof.** The assertion follows from Corollary 2.2 applied to the quasi-convex mapping  $f(x) = x^p$ ,  $p \in \mathbb{R}$ .

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