OKU Fen Bilimleri Enstitüsü Dergisi
8(1): 45-56, 2025

OKU Journal of The Institute of Science and Technology, 8(1): 45-56, 2025

Osmaniye Korkut Ata Üniversitesi
Fen Bilimleri Enstitüsü
Dergisi

Osmaniye Korkut Ata University
Journal of The Institute of Science
and Technology

Sınır Noktasında Beta Türevli Sturm-Liouville Operatörlerinin Sınıflandırılması

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Araştırma Makalesi

Makale Tarihçesi:
Geliş tarihi: 24.03.2024
Kabul tarihi:22.07.2024
Online Yayınlanma: 15.01.2025

Anahtar Kelimeler:
Sınır noktası durumu
Tekil Sturm-Liouville operatörü
Beta-Sturm-Liouville problemi
Sınır noktası sınıflandırması

ÖZ

Çalışmada tekil Beta-Sturm-Liouville operatörü

$$\Omega(y) = -T_\beta(f(t)\, T_\beta\, y(t)) + g(t)y(t), \qquad [0,\infty)$$

ele alınmıştır. Bu operatör için Weyl'in sınır noktası sınıflandırmasına yönelik bir kriter verilmiştir. Bu amaçla öncelikle beta hesabının temel kavramları ve bazı teoremler verilmiştir. Everit yöntemi (1966) kullanılarak Beta-Sturm-Liouville denkleminin sınır noktası durumunda hangi koşullar altında olacağı gösterilmiştir.

Classification of Sturm-Liouville Operators with Beta-Derivatives at the Limit-Point

Research Article

Article History:
Received: 24.03.2024
Accepted: 22.07.2024
Published online: 15.01.2025

Keywords:
Limit-point case
Singular Sturm-Liouville operator
Beta-Sturm-Liouville problem
Limit-point classification

ABSTRACT

The singular Beta-Sturm-Liouville operator

$$\Omega(y) = -T_\beta(f(t)\, T_\beta\, y(t)) + g(t)y(t) \text{ on } [0,\infty)$$

was taken into consideration in this study. A criterion for Weyl's limit-point classification was given for this operator. For this purpose, firstly the basic concepts of beta calculation and some theorems were given. using Everit's method (1966), it was shown under what conditions the Beta-Sturm-Liouville equation will be in its limit-point case.

To Cite: Yalçınkaya Y. Classification of Sturm-Liouville Operators with Beta-Derivatives at the Limit-point. Osmaniye Korkut Ata Üniversitesi Fen Bilimleri Enstitüsü Dergisi 2025; 8(1): 45-56.

1. Introduction

It is known that fractional differential equations emerged as a result of the correspondence between Newton and Leibnitz towards the end of the 17th century. Today, it is used in fields such as bio-engineering, electromagnetic theory, mechanics, physics and control theory, and analytics (Miller and Ross, 1993; Oldham and Spainer, 1974; Ross, 1975; Podlubny, 1999). There are many studies in the literature about fractional derivatives, and new studies are still being conducted on the subject. Different in 1730, L. Euler first defined fractional derivatives using the gamma function, and later Reimann-Liouville and Caputo expanded this definition. Beta derivatives are a generalization of classical derivatives that arise in the context of fractional calculus. Fractional calculus deals with derivatives and integrals of non-integer order, providing a powerful mathematical tool to describe systems with memory and long-range interactions. Beta derivatives find applications in various fields, including physics, engineering, signal processing, and finance. They provide a way to model complex phenomena involving memory effects and non-local interactions, offering a more accurate description of real-world systems compared to classical derivatives. Additionally, they extend the toolbox of fractional calculus, enabling the analysis of a broader class of problems. During this period, many researchers have carried out studies in this field (Levinson, 1949; Atangana et al., 2016; Martinez et al., 2018; Atangana and Algahtani, 2021; Fadhal et al., 2022).

In this context, Sturm-Liouville operators are differential operators typically encountered in the study of ordinary differential equations. These operators involve a second-order linear differential equation of the form:

$$L[y] = dtd(p(t)dtdy) + q(t)y = \lambda w(t) \tag{1}$$

where y is a function, *p(t)*, *q(t)*, and *w(t)* are given functions, and λ is a parameter. These operators often arise in various fields of science and engineering, including physics, where they represent systems exhibiting wave-like behaviour, among others.

Baskaya (2024) calculated the asymptotic expansion of the eigenparameter by considering the single boundary condition Sturm-Liouville problem with an eigenparameter. He also showed that the problem also has a symmetric single-well potential, which is an important function in quantum mechanics. In the study by Kabatas (2023), the asymptotic behaviors of the solutions of Sturm-Liouville problems related to polynomially eigenparameter dependent boundary conditions were obtained when the potential function was a real-valued $\boldsymbol{L1}$- function in the range $(\mathbf{0}, \mathbf{1})$. Additionally, asymptotic formulas are given for the derivatives of the solutions. In this study, asymptotic estimates of eigenvalues are examined for regular Sturm-Liouville problems with eigenvalue parameters at all boundary conditions with a symmetric single-well potential that is symmetrical to the midpoint of the relevant interval and does not increase in the first half-region.

The term "limit-point classification" in this context likely refers to the study of the properties and classification of the limit points of solutions to Sturm-Liouville equations when the operators are extended to include beta-derivatives. Beta-derivatives are fractional derivatives generalizing the classical derivatives and have applications in various fields such as fractional calculus and mathematical physics.

The classification of limit points is crucial in understanding the behaviour of solutions to Sturm-Liouville equations. It plays a significant role in determining the existence and uniqueness of solutions, as well as in studying their qualitative properties. This area of research involves advanced techniques from functional analysis, spectral theory, and fractional calculus, among others, and it aims to provide a deeper understanding of the behaviour of solutions to differential equations with fractional derivatives.

Zhaowen et al. (2020), the classification of the limit point state and the limit circle was examined by examining the 2α-order coherent fractional Sturm-Liouville operator, two interval criteria of the limit-point state were obtained, and examples showing the main results were presented (Zhaowen Zheng and Huixin Liu). Braeutigam (2017), Sturm-Liouville matrix operators in $L_n^2(I), I[0,\infty)$ space were examined and the conditions that enable the boundary point situation to be realized for the minimum closed symmetric operator produced by $l^k[y]\,(k\epsilon\mathbb{N})$ on these matrices and the matrix-valued distribution. They obtained the boundary point conditions for Sturm-Liouville operators with their coefficients. Mirzoev (2014) the well-developed spectral theory of second-order quasi-differential operators was applied to Sturm-Liouville operators with dispersion coefficients to form the Titchmarsch-Weyl theory for such operators. In the study, coefficients that provide limit point or limit circle situations were found. In the study, we examine the eigenvalues of self-adjoint Sturm-Liouville problems for singular problems with a singular endpoint in the limit-point situation (Zhang et al., 2014). The Sturm-Liouville problem with a conformable fractional was studied by Allahverdiev et al. (2021), who also provided a criterion for classifying Sturm-Liouville conformable fractional operators in singular cases at the limit point.

In this work, the fractional beta Sturm-Liouville operator was examined and the criterion for limit-point classification in the singular case was given for this operator. Materials and methods of (Hardy et al., 1934; Everit, 1966; Everit, 1972; Allahverdiev et al., 2021) were used to prove our results. Limit-point and limit-circle situations are active research topics in singular Sturm-Liouville problems. Therefore, the results obtained in classical Sturm-Liouville problems should be examined in Beta-Sturm-Liouville problems. Using Everit's method (1966), Theorem 2.1 (below) shows under what conditions singular Beta-Sturm-Liouville problems will be in the limit-point case.

Definition 1.1: (Atangana et al., 2016; Martinez et al., 2018) Let β be a positive number and $\beta\epsilon(0,1)$. A function $f:[0,\infty)\to\mathbb{R}$ the β derivative of f(t) of order β is given

$$T_\beta f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(t)}{\varepsilon}$$

and $(T_\beta f)(t)=\frac{d^\beta f(t)}{dt^\beta}$.

Definition 1.2: *Let* $f:[a,\infty)\to\mathbb{R}$ *be a function. Then the beta-integral of f is:*

$$I_\beta^a\big(f(t)\big)=\int_a^t (x+\frac{1}{\Gamma(\beta)})^{\beta-1}f(x)dx$$

where $0<\beta\le 1$ *and* $(T_\beta^b\big(f(t)\big)=\lim_{t\to b^-}(T_\beta^b\big(f(t)\big)$.

Theorem 1.3: (Martinez et al., 2018) *For $t>0$ and $0<\beta\leq 1$, let $f(t)$ and $g(t)$ be β-differentiable functions. The following properties are talked about:*

i. $T_\beta\big(\lambda f(t)+\delta g(t)\big)=\lambda T_\beta f(t)+\delta T_\beta g(t) \quad for\ all\ \lambda,\delta\epsilon\mathbb{R}$

ii. $T_\beta\big(f(t)g(t)\big)=f(t)T_\beta g(t)+g(t)T_\beta f(t)$

iii. $T_\beta\left(\frac{f(t)}{g(t)}\right)=\frac{f(t)T_\beta g(t)-g(t)T_\beta f(t)}{g^2(t)}$

iv. $T_\beta f(t)=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{df(t)}{dt}$.

Theorem 1.4: *Let $f,g:[0,b]\to\mathbb{R}$ be β-differentiable functions. Then, the following relation holds*

$$\int_0^b f(t)T_\beta g(t)d_\beta t=f(t)g(t)|_0^b-\int_0^b g(t)T_\beta f(t)\,d_\beta t. \tag{2}$$

Proof: By Theorem 1.3, we obtain

$$\begin{aligned}&\int_0^b f(t)T_\beta g(t)d_\beta t+\int_0^b g(t)T_\beta f(t)\,d_\beta t\\ &=\int_0^b f(t)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}g'(t)d_\beta t\\ &+\int_0^b g(t)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'(t)d_\beta t\\ &=f(t)g(t)|_0^b-\int_0^b g(t)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'(t)d_\beta t\\ &+\int_0^b g(t)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'(t)d_\beta t\\ &=f(b)g(b)-f(0)g(0).\end{aligned}$$

Let

$$L_\beta^2(0,b)=\left\{\begin{array}{c}f:\left(\int_0^b|f(t)|^2d_\beta t\right)^{1/2}\\ =\left(\int_0^b|f(t)|^2\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}d_\beta t\right)^{1/2}<\infty\end{array}\right\}$$

Then $L_\beta^2(0,b)$ is a Hilbert space endowed with the inner product

$$\langle f,g\rangle=\int_0^b f(t)\overline{g(t)}d_\beta t,\qquad f,g\in L_\beta^2(0,b).$$

The β -Wronskian of f and g is defined by

$$W_\beta(f,g)(t) = p(t)\big[f(t)T_\beta g(t) - g(t)T_\beta f(t)\big], \qquad t\epsilon[0,b].$$

Definition 1.5: (Martinez et al., 2018) *Two functions f,g will be said to be effectively proportional if there are constants θ_1, θ_2 not both zero, such that $\theta_1 f(t) = \theta_2 g(t)$. Basically, any function is proportional to a null function.*

Theorem 1.6: ***i)*** *If all the functions are effectively proportional and , if $\sigma_1, \sigma_2, \dots, \sigma_n$ are positive and $\sigma_1 + \sigma_2 +, \dots. + \sigma_n = 1$ then*

$$\int |f^{\sigma_1} g^{\sigma_2} \dots I^{\sigma_n}|\, d_\beta t < \left(\int |f|\, d_\beta t\right)^{\sigma_1} \left(\int |g|\, d_\beta t\right)^{\sigma_2} \dots \left(\int |I|\, d_\beta t\right)^{\sigma_n}$$

unless one of the functions is null.

ii) *If s>1 and $\frac{1}{s} + \frac{1}{s'} = 1$ then*

$$\int |f(t)g(t)|\, d_\beta t < \left(\int |f|^s\, d_\beta t\right)^{\frac{1}{s}} \left(\int |g|^{s'}\, d_\beta t\right)^{\frac{1}{s'}}$$

will result unless (a) f^s and $g^{s'}$ are functionally proportionate or (b) fg is null.

Proof: (Martinez et al., 2018), because the proofs are unchanged except for a few minor notational adjustments.

Theorem 1.7: *If $\rho > 1$, $f \in L^\rho_\beta(0,a)$ and*

$$F(t) = \int_0^t |f(x)|\, d_\beta x,$$

then we have

$$F(t) = O\left[\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^{\frac{\beta}{\rho'}}}{\beta}\right],$$

for small t.

Proof: Theorem 1.6 indicates that

$$F^{\rho} \leq \int_{0}^{t} |f(x)|^{\rho}\, d_{\beta}x \left(\int_{0}^{t} d_{\beta}x\right)^{\rho-1} = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta(\rho-1)}}{\beta} \int_{0}^{t} |f(x)|^{\rho}\, d_{\beta}x$$

and the second factor have a tendency toward 0.

Theorem 1.8: *If* $\rho > 1$ *and* $f \in L_{\beta}^{\rho}(0,a)$ *then* $F(t) = O\left[\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\frac{\beta}{\rho'}}}{\beta}\right]$ *both for small and for large* t.

Proof: Theorem 1.7 yields the proof for small t. We demonstrate the outcame for large t. Select τ such that

$$\int_{\tau}^{\infty} |f(x)|^{\rho}\, d_{\beta}x < \varepsilon^{\rho}$$

and suppose $t > \tau$. Then, we obtain

$$\begin{aligned}
\left(F(t) - F(\tau)\right)^{\rho} &= \left(\int_{\tau}^{\infty} |f|\, d_{\beta}x\right)^{\rho} \\
&\leq \frac{\left(t+\frac{1}{\Gamma(\beta)}-\tau\right)^{\beta(\rho-1)}}{\beta} \int_{\tau}^{t} |f(x)|^{\rho}\, d_{\beta}x \\
&< \frac{\varepsilon^{\rho}\left(t+\frac{1}{\Gamma(\beta)}-\tau\right)^{\beta(\rho-1)}}{\beta},
\end{aligned}$$

$$F(t) < F(\tau) + \varepsilon \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\frac{\beta}{\rho'}}}{\beta}$$

$$< 2\varepsilon \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\frac{\beta}{\rho'}}}{\beta}$$

for sufficiently large t.

2. Main Results and Discussion

The fractional singular beta-Sturm-Liouville expression

$$\Omega(y) = -T_\beta\left(f(t)T_\beta y(t)\right) + g(t)y(t) \quad on\ [0,\infty) \tag{3}$$

given below, including real valued f and g functions, will be examined:

i. $\quad g \in L^2_\beta[0,b]\ for\ all\ b > 0,$

ii. $\quad f\ is\ absolutely\ continuous\ on\ [0,b] for\ all\ b > 0,$ (4)

iii. $\quad f(t) > 0\ for\ all\ t \in [0,\infty).$

The set $\mathcal{D}$ is defined by: $y\epsilon\mathcal{D}$ if

i. $\quad y \in L^2_\beta[0,\infty),$

ii. $\quad T_\beta y\ is\ absolutely\ continuous\ on\ [0,b]\ for\ all\ b > 0,$ (5)

iii. $\quad \Omega(y) \in L^2_\beta[0,\infty),$

iv. $\quad y(0) = 0.$

By Theorem 1.4, for $z_1, z_2 \in \mathcal{D}$ The Green's formula that we have is

$$\int_0^\infty \Omega(z_1)(t)\overline{z_2(t)}\, d_\beta t - \int_0^\infty z_1(t)\overline{\Omega(z_2)(t)}\, d_\beta t = [z_1, z_2](\infty) - [z_1, z_2](0), \tag{6}$$

where

$$[z_1, z_2](t) = f(t)\left\{z_1(t)\overline{T_\beta z_2(t)} - T_\beta z_1(t)\overline{z_2(t)}\right\}, \quad t \in [0,\infty). \tag{7}$$

Theorem 2.1: *If the function g is bounded below on [0,∞) and*

$$\int_0^\infty \{f(t)\}^{-\frac{1}{2}}\, d_\beta t < \infty, \tag{8}$$

then the differential operator Ω defined by (3) is in the limit-point case at infinity.

Proof: It is known, (Everit, 1966 and Titchmarsh, 1962) that Ω is the limit point at infinity if and only if

$$[z_1, z_2](\infty) = 0 \tag{9}$$

for all $z_1, z_2 \in \mathcal{D}$. Therefore, it suffices to demonstrate that, for every $z_1, z_2 \in \mathcal{D}$ (where z_1 and z_2 are real-valued functions)

$$\lim_{b\to\infty} f(b)z_1(b)\,T_\beta z_2(b) = 0. \tag{10}$$

We can presume, without losing generality, that there exists a positive constant ξ such that

$$g(t) \geq \xi > 0 \; for \; all \; t\epsilon[0,\infty). \tag{11}$$

From (11), for all $b > 0$ we get

$$G(b) = \int_0^b g(t)\, d_\beta t \geq \; \xi \frac{\left(b + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta} \tag{12}$$

implies that

$$[G(b)]^{-\frac{1}{2}} \leq \left[\xi \frac{\left(b + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right]^{-\frac{1}{2}} \quad (b > 0). \tag{13}$$

The result of integration by parts (2) is

$$\int_0^b \{f(t)T_\beta z_1(t)T_\beta z_2(t) + g(t)z_1(t)z_2(t)d_\beta t\} \; = f(t)z_1(t)T_\beta z_2(t)|_0^b + \int_0^b \Omega(z_2(t)\, z_1(t))d_\beta t \tag{14}$$

for all $z_1, z_2 \in \mathcal{D}$ and for all $b > 0$.

If we take $z_1 = z_2 \in \mathcal{D}$ in (14), then we get

$$\int_0^b \left\{f(t)\left(T_\beta z_1(t)\right)^2 + g(t)z_1^2(t)\right\} d_\beta t = f(b)z_1(b)T_\beta z_1(b) + \int_0^b \Omega(z_1(t)\, z_1(t))d_\beta t.$$

It follows from (4) and (11) that the left-hand integrand is non-negative. $L_\beta^2[0,\infty)$ is the integrand on the right in equation (5). If $f(t)\left(T_\beta z_1(t)\right)^2 + g(t)z_1^2(t) \notin L_\beta^1[0,\infty)$, then $f(b)z_1(b)T_\beta z_1(b) \to \infty$, as $b \to \infty$. This is not possible because, for every large b, $T_\beta z_1(b)$ and $z_1(b)$ would have the same sign, and z_1 could not then belong to $L_\beta^2[0,\infty)$. This defies the hypothesis that $z_1 \in \mathcal{D} \subset L_\beta^2[0,\infty)$ Therefore, we conclude that

$$f(t)^{\frac{1}{2}}T_\beta z_1(t) \in L_\beta^2[0,\infty), \qquad g(t)^{\frac{1}{2}}z_1(t) \in L_\beta^2[0,\infty) \tag{15}$$

for all $z_1 \in \mathcal{D}$. From (14), we deduce that, for all $z_1, z_2 \in \mathcal{D}$

$$\lim_{b\to\infty} f(b)z_1(b)T_\beta z_2(b) \tag{16}$$

exists and is finite.

If $\Phi \in L_\beta^2[0,\infty)$ then $\Phi \in L_\beta^1[0,b]$, it follows from Theorem 1.8 that

$$\lim_{b\to\infty}\left(\frac{\left(b+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)^{-\frac{1}{2}\beta}\int_0^b |\Phi(t)|\, d_\beta t = 0. \tag{17}$$

For all $b \geq 0$ define

$$F(b) = \int_0^b \{f(t)\}^{-\frac{1}{2}}\, d_\beta t. \tag{18}$$

By (8), we get

$$\lim_{b\to\infty} F(b) = M, where\ 0 < M < \infty. \tag{19}$$

From (2), we conclude that

$$\int_0^b f(t)^{\frac{1}{2}} T_\beta z_2(t) d_\beta t = F(b) f(b)\, T_\beta z_2(b) + \int_0^b F(t)\, \Omega\big(z_2(t)\big) d_\beta t - \int_0^b F(t)\, g(t) z_2(t) d_\beta t.$$

If the result multiply by $\{G(b)\}^{-\frac{1}{2}}$ take into account each of the individual terms. It follows from (13), (15) and (17) that

$$\{G(b)\}^{-\frac{1}{2}}\int_0^b f(t)^{\frac{1}{2}} T_\beta z_2(t) d_\beta t = O\left(\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)^{-\frac{\beta}{2}}\int_0^b f(t)^{\frac{1}{2}} |T_\beta z_2(t)| d_\beta t\right)$$

$$= o(1)\ as\ b \to \infty.$$

Similarly, using (13), (15), and (17) as $b \to \infty$ we get

$$\{G(b)\}^{-\frac{1}{2}}\int_0^b F(t)\, \Omega\big(z_2(t)\big) d_\beta t = O\left(M\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)^{-\frac{\beta}{2}}\int_0^b |\Omega z_2(t)| d_\beta t\right) = o(1).$$

Let $b' > 0$ be fixed. Then, for all $b > b'$ we have

$$\{G(b)\}^{-\frac{1}{2}}\int_0^b F(t)\, g(t) z_2(t) d_\beta = \{G(b)\}^{-\frac{1}{2}}\left\{\int_0^{b'} F(t)\, g(t) z_2(t) d_\beta t + \int_{b'}^{b} F(t)\, g(t) z_2(t) d_\beta t\right\}$$

$$= O\left(b^{-\frac{1}{2}}\right) + O\left(M\left\{G(b)^{-1}\int_{b'}^{b} g(t) d_\beta t \int_{b'}^{b} g(t) z_2^2(t) d_\beta t\right\}^{\frac{1}{2}}\right)$$

$$= o(1) + O\left(M\left\{\int_{b'}^{b} g(t) z_2^2(t) d_\beta t\right\}^{\frac{1}{2}}\right).$$

The left-hand side approaches zero as b→∞, as (15) implies.

Hence by (19), we get, for all $z_2 \in \mathcal{D}$

$$\lim_{b\to\infty}\{G(b)\}^{-\frac{1}{2}} f(b) T_\beta z_2(b) = 0. \tag{20}$$

Let us consider $\{G(b)\}^{\frac{1}{2}} z_1$ where $z_1 \in \mathcal{D}$; assume that

$$\liminf_{b\to\infty}\{G(b)\}^{\frac{1}{2}} z_1(t) > 0. \tag{21}$$

There is a constant μ, $0 < \mu < \infty$, such that for all $t > b_0$ (say) we have $|z_1(t)|^2 \geq \mu^2\{G(t)\}^{-1}$.

Let η be a positive constant depending on b_0 and $\xi > 0$. If the equation is multiplied by the positive number *g(t)* and integrate over $[b_0, b]$ and (12) is used,

$$\int_{b_0}^{b} g(t)|z_1^2(t)| d_\beta t \geq \mu^2 \int_{b_0}^{b} \frac{g(t)}{G(t)} d_\beta t = \mu^2\left[\ln G\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)\right]\Big|_{b_0}^{b} \geq \eta\mu^2 \ln\xi\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right)$$

obtained. This implies that $g^{\frac{1}{2}} z_1 \notin L_\beta^2[0,\infty)$. This runs counter to (15). Then there is a series $\{b_i,\ i \geq 1\}$, for which

$$\lim_{i\to\infty}\{G(b_i)\}^{\frac{1}{2}} z_1(b_i) = 0. \tag{22}$$

And such that $b_i \to \infty$ as $i \to \infty$ From (20) and (22), we deduce that for each pair $z_1, z_2 \in \mathcal{D}$ there exists a squence $\{b_i,\ i \geq 1\}$ such that

$$\lim_{i\to\infty} f(\,b_i)z_1(b_i)T_\beta z_2(b_i) = 0.$$

Now that (16) has been satisfied, (10) holds for any for $z_1, z_2 \in \mathcal{D}$. The proof is complete.

3. Conclusion

A criterion for the limit-point classification of singular cases of Beta-Sturm-Liouville operators was given in the work. To start, the fundamental ideas of beta computation and a few theorems applied in the research were presented. Next, we were provided a criterion for the Beta-Sturm-Liouville operator's limit-point categorization according to Weyl.

Acknowledgement

I am grateful to the editor for his interest and the referees for their review and insightful comments on this work.

Conflict of interest

The writer says they have no competing interests.

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