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h- Stability of Functional Dynamic Equations on Time Scales by Alternative Variation of Parameters

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Abstract

In this paper, we concentrate on nonlinear functional dynamic equations of the form

$$x^{\Delta}(t) = a(t)x(t) + f\big(t, x(t)\big), \;\; t \in \mathbb{T}$$

on time scales and study h-stability, which implies uniform exponential stability, uniform Lipschitz stability, or uniform stability in particular cases. In our analysis, we use an alternative variation of parameters, which enables us to focus on a larger class of equations since the dynamic equations under the spotlight are not necessarily regressive. Also, we establish a linkage between uniform boundedness and h-stability notions for solutions of dynamic equations under sufficient conditions in addition to our stability results.

1. Introduction

The theory of time scales, which was initiated by S. Hilger in 1988 (see [1]), has taken noticeable attention in pure and applied mathematics in the last decades. The main objective of this theory is three-fold: unification, extension, and discretization of conventional calculus. Since the theory of time scales avoids the disjoint study of continuous and discrete mathematical structures, it has become a hot topic for researchers, and time scale analogs of existing theories have been reconstructed on hybrid time domains, namely time scales. Furthermore, recent studies in this field indicate that it is possible to establish a linkage between dynamical equations on time scales and other disciplines such as economics, physics, biology, or engineering sciences. We refer to readers [2]-[10] in order to polish the application potential of time scales in different fields.

Stability theory of differential and difference equations is one of the landmark topics of qualitative theory of dynamical equations in applied mathematics. Since the theory of time scales enables researchers to analyze differential and difference equations in a joint framework, researchers established a unified stability theory for dynamic equations defined on time scales for various stability types. By a quick literature review, one may easily find pioneering papers on the stability, asymptotic stability, or exponential stability of dynamic equations on time scales (see [11]-[15]). However, it should be pointed out that the stability analysis of nonlinear equations is grueling, especially when the equation is constructed on arbitrary time domains. For example, it is challenging and sometimes impossible to design a controller for a nonlinear system that ensures exponential stability. Hence, the utilization of the h-stability notion has opened a new window into stability analysis by providing a generalized approach. The concept of h-stability is first introduced by M. Pinto in [16] as an extension of notions of exponential stability and uniform Lipschitz stability. As it is discussed in [17], the following relationship holds between the well-known stability types

h -stability ⇒ uniform exponential stability ⇒ uniform Lipschitz stability ⇒ uniform stability.

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As a consequence of the above-given implications, this topic has taken prominent attention in a small duration of time, and mathematicians have studied h-stability for solutions of dynamic equations on continuous, discrete, and hybrid time domains. We refer to readers [16]-[25] as inspiring papers on this topic.

In this research, we focus on the following nonlinear abstract dynamic equation defined on an arbitrary time scale $\mathbb{T}$,

$$x^{\Delta}(t) = a(t)x(t) + f\big(t, x(t)\big), \; t \in \mathbb{T}, \qquad (1)$$

and obtain a stability analysis based on the h-stability concept. The abstract equation (1) has tremendous application potential, and its particular forms can be found in numerous papers in different fields. For example, one may easily observe that equation (1) turns into

- A single artificial effective neuron with dissipation model

$$x'(t) = -a(t)x(t) + \mathrm{b}(t)\tanh\big(x(t)\big) + I(t),$$

for all $t \in \mathbb{R}$,

- Continuous-time Lasota-Wazewska model on the survival of red blood cells

$$x'(t) = -rx(t) + \eta(t)e^{-\gamma x(t)},$$

for all $t \in \mathbb{R}$,

- Discrete-time Clark's model in population dynamics without delay

$$\Delta x_n = (\gamma - 1)x_n + F(x_n),$$

for all $t \in \mathbb{N}_0$,

under particular choices of a, f, and $\mathbb{T}$ (see [26]-[28], respectively). From our mathematical point of view, it is reasonable to study the abstract equation (1) on arbitrary time scales since the obtained stability results might be used for several real-life models under sufficient conditions. The analogy between the nonlinear abstract dynamic equation (1) and the specific nonlinear models in the applied sciences reveals the application potential of outcomes of the paper on a wide range of disciplines. Moreover, we shall highlight that the concept of regressivity is essential for the theory of dynamical equations on time scales since it is inevitable to define generalized exponential function. By regressiveness of a dynamic equation

$$x^{\Delta}(t) = a(t)x(t), \; t \in \mathbb{T},$$

we mean $1 + \mu(t)a(t) \neq 0$ for all $t \in \mathbb{T}^k$ where μ and the set $\mathbb{T}^k$ are defined as in the next section. Even though every function is regressive when $\mathbb{T} = \mathbb{R}$, regressivity becomes a restrictive condition for classes of dynamic equations when $\mathbb{T}$ has discrete structures. This issue is pointed out by the authors of [21], and their outcomes indicate that without regressivity assumption, it is still possible to study h-stability for dynamic equations on time scales. In this manuscript, we aim to study h-stability for scalar-valued dynamic equations on time scales without assuming regressivity. Motivated by the papers [29]-[32], we use a regressive auxiliary function to invert an alternative variation of parameters to achieve this task. Thus, the regressivity condition becomes redundant for the main equation of the manuscript. Furthermore, our approach does not only provide an alternative tool for h-stability analysis but also improves the current literature since it enables us to construct comparative results regarding the h-stability and boundedness for dynamic equations on time scales.

The organization of the paper is as follows: The next section is devoted to preliminaries of time scales calculus for the readership. In Section 3, we present the main results of the manuscript, and in the last section, we provide an elaborative conclusion.

2. Time Scales Essentials

We give the following introductory information for the readers who are not familiar with time scale calculus. The following definitions, results, and examples are given due to the pioneering book [33].

A time scale denoted by $\mathbb{T}$, which inherits the standard topology on $\mathbb{R}$, is an arbitrary, nonempty, closed subset of real numbers. We define the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ by $\sigma(t) \coloneqq \inf\{s \in \mathbb{T}, \; s > t\}$, while the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined as $\rho \coloneqq \sup\{s \in \mathbb{T}, \; s < t\}$ for $t \in \mathbb{T}$. Also, the graininess (step-size) function $\mu(t): \mathbb{T} \to [0, \infty)$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right-dense if $\mu(t) = 0$ or equivalently $\sigma(t) = t$; otherwise, it is called right-scattered. In a similar fashion, a point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, or else it is called left-scattered. By the notation $[s,t)_{\mathbb{T}}$, we mean the intersection $[s,t) \cap \mathbb{T}$, and the intervals $[s,t]_{\mathbb{T}}$, $(s,t)_{\mathbb{T}}$, and $(s,t]_{\mathbb{T}}$ can be defined in the same manner. A function $f: \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right dense points and its left-sided limit exists at left dense points. Besides, C_{rd} stands for all rd-continuous functions defined on $\mathbb{T}$. The set $\mathbb{T}^k$ is given in the following way: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Delta-derivative of a function $f: \mathbb{T} \to \mathbb{C}$ at $t \in \mathbb{T}$ is given by

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}, & \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) > 0 \end{cases}$$

provided the limit exists. For $f \in C_{rd}$ and $s,t \in \mathbb{T}$ we define delta-integral as

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s),$$

where $F^{\Delta} = f$ on $\mathbb{T}^k$.

Table 1 illustrates the main characteristics of three essential time scales.

A function $f: \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$ and f is called positively regressive if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}^k$. The notations $\mathcal{R}$ and $\mathcal{R}^+$ indicate the set of all regressive functions, and the set of all positively regressive functions, respectively. For $h > 0$, we introduce $\mathbb{C}_h := \{z \in \mathbb{C}: z \neq -1/h\}, \mathbb{J}_h := \{z \in \mathbb{C}: -\pi/h < Im(z) < \pi/h\}$, and $\mathbb{C}_0 := \mathbb{J}_0 := \mathbb{C}$. For $h \geq 0$ and $z \in \mathbb{C}_h$, the cylinder transformation $\xi_h: \mathbb{C}_h \to \mathbb{J}_h$ is defined by

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \dfrac{1}{h} Log(1 + zh), & h > 0 \end{cases}$$

Then the unified exponential function $e_p(.,s)$ on a time scale $\mathbb{T}$ is defined by

$$e_p(t,s) := \exp\left\{ \int_s^t \xi_{\mu(\tau)}\left(p(\tau)\right)\Delta\tau \right\} \text{ for } s,t \in \mathbb{T}.$$

Moreover, the exponential function $e_p(.,s)$ is the unique solution to the initial value problem

$$\begin{cases} x^{\Delta}(t) = p(t)x(t), \ \ t \in \mathbb{T}^k \\ x(s) = 1 \end{cases},$$

and if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}^k$.

In Table 2, we give some examples of exponential functions on specific time scales. In the sequel, we present the following results as groundwork for the outcomes of the manuscript.

Theorem 1 (Variation of Constants [33, Theorem 2.77]). Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the regressive initial value problem

$$\begin{cases} x^{\Delta}(t) = p(t)x(t) + f(t) \\ x(t_0) = x_0 \end{cases}$$

is given by

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

Theorem 2 ([33, Theorem 6.1]). Let $x, f \in C_{rd}$ and $p \in \mathcal{R}^+$. Then

$$x^{\Delta}(t) \leq p(t)x(t) + f(t) \text{ for all } t \in \mathbb{T}$$

implies

$$x(t) \leq x(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau,$$

for all $t \in \mathbb{T}$.

Theorem 3 (Gronwall's inequality [33, Theorem 6.4]). Let $x, f \in C_{rd}$ and $p \in \mathcal{R}^+$, $p \geq 0$. Then

$$x(t) \leq f(t) + \int_{t_0}^t x(\tau)p(\tau)\Delta\tau$$

for all $t \in \mathbb{T}$ implies

$$x(t) \leq f(t) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) f(\tau) p(\tau) \Delta\tau$$

for all $t \in \mathbb{T}$.

Table 1. Three essential time scales

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$q^{\mathbb{Z}} \cup \{0\}, q > 1$
$\rho(t)$	t	$t-1$	$\frac{t}{q}$
$\sigma(t)$	t	$t+1$	qt
$\mu(t)$	0	1	$(q-1)t$
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$	$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}$
$\int_0^t f(\tau)\Delta\tau$	$\int_0^t f(\tau) d\tau$	$\sum_{\tau=0}^{t-1} f(\tau), (0 < t)$	$\int_1^t f(\tau) d_q\tau = (q-1)\sum_{\tau=0}^{n-1} q^{\tau} f(q^{\tau}), t = q^n$

Table 2. Examples of exponential functions

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$q^{\mathbb{N}_0}$	$\frac{1}{n}\mathbb{Z}$
$e_\alpha(t, t_0)$	$e^{\alpha(t-t_0)}$	$(1+\alpha)^{t-t_0}$	$(1+\alpha h)^{\frac{(t-t_0)}{h}}$	$\prod_{s \in [t_0, t)} [1 + (q-1)\alpha s]$	$\left(1 + \frac{\alpha}{n}\right)^{n(t-t_0)}$

3. Main Results

We start this section by bringing the abstract functional dynamic equation (1) into the spotlight which is defined as

$$x^{\Delta}(t) = a(t)x(t) + f\big(t, x(t)\big), t \in \mathbb{T}$$

where $a: \mathbb{T} \to \mathbb{R}$, $f: \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ belong to C_{rd} and $f(t, 0) = 0$.

Firstly, we introduce the notion of h-stability in the light of [20, Definition 2.2].

Definition 1. The nonlinear dynamic equation (1) is said to be an h-equation if there exist a positive function $h: \mathbb{T} \to \mathbb{R}$, a constant $c \geq 1$, and $\delta > 0$ such that

$$|x(t, t_0, x_0)| \leq c|x_0| \frac{h(t)}{h(t_0)}, t \geq t_0$$

if $|x_0| < \delta$. Moreover, if h is a bounded function, then (1) is called h-stable.

Remark 1. Since the time scale exponential function $e_p(., s)$ can be regarded as a solution of the homogeneous dynamic equation $x^{\Delta}(t) = p(t)x(t)$, then the solution of the regressive initial value problem

$$\begin{cases} x^{\Delta}(t) = p(t)x(t), \ \ t \in \mathbb{T}^k \\ x(t_0) = x_0 \end{cases} \tag{2}$$

is h-stable if there exist a positive, bounded function $h: \mathbb{T} \to \mathbb{R}$ and a constant $c \geq 1$ such that

$$\left|e_p(t, t_0)\right| \leq c\frac{h(t)}{h(t_0)}, \ t \geq t_0. \tag{3}$$

Additionally, we provide the following definition for constructing the last result of the manuscript, which establishes a linkage between boundedness and h-stability.

Definition 2 ([34]). A solution x to a dynamical equation

$$x^{\Delta}(t) = f(t, x)$$

is said to be globally uniformly bounded if for every $\delta > 0$, there exists $c := c(\delta)$ such that $|x(t_0)| \leq \delta$ implies $|x(t)| \leq c$ for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0 \geq 0$.

We present the following lemma due to [32, Lemma 3.1].

Lemma 1. The nonlinear dynamic equation (1) has a solution x if and only if

$$x(t) = x(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))\Big([a(\tau) - p(\tau)]x(\tau) + f(\tau, x(\tau))\Big)\Delta\tau, \tag{4}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$, where $p: [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is regressive.

Now, we are ready to give the first stability result of the manuscript.

Theorem 4. Consider the following initial value problem

$$\begin{cases} x^{\Delta}(t) = a(t)x(t) + f(t, x(t)),\ t \in \mathbb{T} \\ x(t_0) = x_0 \end{cases} \tag{5}$$

where $a, f \in C_{rd}$, $f(t,0) = 0$, and a is not necessarily regressive. Also, we introduce the following auxiliary regressive initial value problem

$$\begin{cases} x^{\Delta}(t) = p(t)x(t) \\ x(t_0) = x_0 \end{cases}. \tag{6}$$

Assume that the following conditions hold:

C1: Solution of (6) is h-stable.

C2: There exists a function g such that

$$|f(t,x) - f(t,y)| \leq g(t)|x - y|.$$

C3: There exists $M > 0$ such that

$$\int_{t_0}^{t} \frac{h(\tau)}{h(\sigma(\tau))}\big(|a(\tau) - p(\tau)| + g(\tau)\big)\Delta\tau \leq M.$$

Then, (5) is h-stable.

Proof. Suppose that conditions **C1-C3** are satisfied. By (4), we obtain the inequality

$$|x(t)| \leq |x_0||e_p(t,t_0)| + \int_{t_0}^{t} |e_p(t,\sigma(\tau))|\,(|a(\tau) - p(\tau)||x(\tau)| + |f(\tau, x(\tau))|)\Delta\tau.$$

Then we use the condition **C1** together with (3) and get

$$\begin{aligned} |x(t)| &\leq \mathrm{c}|x_0|\frac{h(t)}{h(t_0)} + \int_{t_0}^{t} \mathrm{c}\frac{h(t)}{h(\sigma(\tau))}(|a(\tau) - p(\tau)||x(\tau)| + |f(\tau, x(\tau))|)\Delta\tau \\ &= \mathrm{c}|x_0|\frac{h(t)}{h(t_0)} + \mathrm{c}\frac{h(t)}{h(t_0)}\int_{t_0}^{t} \frac{h(t_0)}{h(\sigma(\tau))}(|a(\tau) - p(\tau)||x(\tau)| + |f(\tau, x(\tau))|)\Delta\tau \\ &\leq \mathrm{c}|x_0|\frac{h(t)}{h(t_0)} + \mathrm{c}\frac{h(t)}{h(t_0)}\int_{t_0}^{t} \frac{h(t_0)}{h(\sigma(\tau))}(|a(\tau) - p(\tau)||x(\tau)| + g(\tau)|x(\tau)|)\Delta\tau \end{aligned}$$

by the adoption of **C2** in the last step.

Next, we set

$$z(t) = \int_{t_0}^{t} \frac{h(t_0)}{h(\sigma(\tau))}(|a(\tau) - p(\tau)||x(\tau)| + g(\tau)|x(\tau)|)\Delta\tau$$

and observe

$$\begin{aligned} z^{\Delta}(t) &= \frac{h(t_0)}{h(\sigma(t))}(|a(t) - p(t)||x(t)| + g(t)|x(t)|) \\ &\leq \frac{h(t_0)}{h(\sigma(t))}\left[|a(t) - p(t)|\left(\mathrm{c}|x_0|\frac{h(t)}{h(t_0)} + \mathrm{c}\frac{h(t)}{h(t_0)}z(t)\right) + g(t)\left(\mathrm{c}|x_0|\frac{h(t)}{h(t_0)} + \mathrm{c}\frac{h(t)}{h(t_0)}z(t)\right)\right] \\ &= \mathrm{c}|x_0|\frac{h(t)}{h(\sigma(t))}\big(|a(t) - p(t)| + g(t)\big) \end{aligned}$$

$$+\left(c\frac{h(t)}{h(\sigma(t))}\big(|a(t)-p(t)|+g(t)\big)\right)z(t).$$

Consequentially, we deduce the inequality

$$z^{\Delta}(t) \leq \varphi(t)z(t)+\psi(t),$$

where

$$\varphi(t)=c\frac{h(t)}{h(\sigma(t))}\big(|a(t)-p(t)|+g(t)\big), \qquad (7)$$

and

$$\psi(t)=|x_0|\varphi(t). \qquad (8)$$

One may easily observe that $\varphi \in \mathcal{R}^{+}$, and then Theorem 2 implies

$$z(t) \leq z(t_0)e_{\varphi}(t,t_0)+\int_{t_0}^{t} e_{\varphi}\big(t,\sigma(\tau)\big)\psi(\tau)\,\Delta\tau$$
$$=\int_{t_0}^{t} e_{\varphi}\big(t,\sigma(\tau)\big)\psi(\tau)\,\Delta\tau \qquad (9)$$

since $z(t_0)=0$. If we write the inequality (9) explicitly, then we have

$$z(t) \leq \int_{t_0}^{t} c|x_0|\frac{h(\tau)}{h(\sigma(\tau))}\big(|a(\tau)-p(\tau)|+\ g(\tau)\big)$$
$$\exp\left(\int_{\sigma(\tau)}^{t} \xi_{\mu(s)}\big(\varphi(s)\big)\Delta s\right)\Delta\tau$$

where

$$\xi_{\mu(t)}\big(\varphi(t)\big)=\frac{1}{\mu(t)}Log\left(1+c\mu(t)\frac{h(t)}{h(\sigma(t))}\big(|a(t)-p(t)|+g(t)\big)\right)$$

when $\mu>0$, and

$$\xi_{\mu(t)}\big(\varphi(t)\big)=c\frac{h(t)}{h(\sigma(t))}\big(|a(t)-p(t)|+g(t)\big)$$

when $\mu=0$.

Here we get

$$z(t) \leq \int_{t_0}^{t} c|x_0|\frac{h(\tau)}{h(\sigma(\tau))}\big(|a(\tau)-p(\tau)|+g(\tau)\big)$$
$$\exp\left(\int_{\sigma(\tau)}^{t} c\frac{h(s)}{h(\sigma(s))}\big(|a(s)-p(s)|+g(s)\big)\Delta s\right)\Delta\tau$$

which yields to

$$|x(t)| \leq c|x_0|\frac{h(t)}{h(t_0)}$$
$$+c^2|x_0|\frac{h(t)}{h(t_0)}\int_{t_0}^{t}\frac{h(\tau)}{h(\sigma(\tau))}\big(|a(\tau)-p(\tau)|+$$
$$g(\tau)\big)\exp\left(\int_{\sigma(\tau)}^{t} c\frac{h(s)}{h(\sigma(s))}\big(|a(s)-p(s)|+\right.$$
$$\left. g(s)\big)\Delta s\right)\Delta\tau.$$

By using **C3**, we write

$$|x(t)| \leq c|x_0|\frac{h(t)}{h(t_0)}+c^2|x_0|\frac{h(t)}{h(t_0)}Me^{cM}$$
$$=|x_0|\frac{h(t)}{h(t_0)}\big(c+c^2Me^{cM}\big)$$

which shows x is h-stable. The proof is complete. □

Next, we present an inequality that is crucial for establishing a comparative stability result.

Lemma 2 ([21, Lemma 3.25])**.** Suppose that $m \in C_{rd}(\mathbb{T}\times\mathbb{R}^{+},\mathbb{R})$ is non-decreasing in the second argument x for each fixed $t \geq t_0$ with the property

$$x(t)-\int_{t_0}^{t} m\big(\tau,x(\tau)\big)\Delta\tau \leq y(t)-\int_{t_0}^{t} m\big(\tau,y(\tau)\big)\Delta\tau,$$

for $t \geq t_0 \in \mathbb{T}$ and $x,y \in C_{rd}(\mathbb{T},\mathbb{R}^{+})$. If $x(t_0)<y(t_0)$, then $x(t)<y(t)$ for all $t \geq t_0 \in \mathbb{T}$.

Theorem 5. Suppose that there exists a function $k \in C_{rd}(\mathbb{T}\times\mathbb{R}^{+},\mathbb{R}^{+})$ so that

$$|f(t,x)| \leq k(t,|x|), \qquad (10)$$

where k is increasing with respect to its second argument. We set

$$m(t,x) = |a(t) - p(t)|x(t) + k\big(t, x(t)\big), \qquad (11)$$

and also assume $p \in \mathcal{R}^+$. Consider the following auxiliary equation

$$\begin{cases} u^{\Delta}(t) = p(t)u(t) + m\big(t, u(t)\big) \\ u(t_0) = u_0 \end{cases}. \qquad (12)$$

If (12) is h-stable, then (5) is also h-stable whenever $u_0 = |x_0|$.

Proof. Suppose that the inequality (10) holds, and (12) is h-stable. We fix $u_0 = |x_0|$ and observe that the function m given in (11) is increasing with respect to its second term. By (4), (10), and (11), we have

$$\begin{aligned} |x(t)| &\le |x_0| e_p(t, t_0) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big)\big(|a(\tau) - p(\tau)||x(\tau)| + \big|f\big(\tau, x(\tau)\big)\big|\big)\,\Delta\tau \\ &\le |x_0| e_p(t, t_0) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big)\big(|a(\tau) - p(\tau)||x(\tau)| + k(\tau, |x|)\big)\,\Delta\tau \\ &\le |x_0| e_p(t, t_0) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) m(\tau, |x(\tau)|)\,\Delta\tau, \end{aligned}$$

which yields to

$$\begin{aligned} &|x(t)| - \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) m(\tau, |x(\tau)|)\,\Delta\tau \\ &\le |x_0| e_p(t, t_0) \\ &= u(t) - \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) m(\tau, |u(\tau)|)\,\Delta\tau. \end{aligned}$$

Subsequently, we have $|x(t)| < u(t)$ due to Lemma 2. Hence,

$$|x(t)| < u(t) \le c u_0 \frac{h(t)}{h(t_0)} = c|x_0| \frac{h(t)}{h(t_0)},$$

and this proves our assertion. ◻

The following result focuses on boundedness and h-stability.

Theorem 6. Suppose that the auxiliary regressive initial value problem given in (6)

$$\begin{cases} x^{\Delta}(t) = p(t)x(t) \\ x(t_0) = x_0 \end{cases}$$

is h-stable with an increasing function h. Also, consider the dynamic equation given in (5), which is

$$\begin{cases} x^{\Delta}(t) = a(t)x(t) + f\big(t, x(t)\big) \\ x(t_0) = x_0 \end{cases}$$

with condition **C2** of Theorem 4. Then, the solution of the nonlinear equation (5) is globally uniformly bounded whenever

$$e_{\kappa}(t, t_0) \le \omega \qquad (13)$$

where

$$\kappa(t) = |a(t) - p(t)| + g(t). \qquad (14)$$

Proof. Assume that (6) is h-stable with an increasing function h, and **C2** and (13) hold. In the light of (4) and **C2**, one may easily obtain the inequality

$$|x(t)| \le |x_0| |e_p(t, t_0)| + \int_{t_0}^{t} \big|e_p\big(t, \sigma(\tau)\big)\big| (|a(\tau) - p(\tau)||x(\tau)| + \big|f\big(\tau, x(\tau)\big)\big|)\Delta\tau$$

$$\le |x_0| |e_p(t, t_0)| + \int_{t_0}^{t} \big|e_p\big(t, \sigma(\tau)\big)\big| (|a(\tau) - p(\tau)||x(\tau)| + g(\tau)|x(\tau)|)\Delta\tau$$

$$\le \mathrm{c}|x_0| \frac{h(t)}{h(t_0)} + \int_{t_0}^{t} c \frac{h(t)}{h\big(\sigma(\tau)\big)} (|a(\tau) - p(\tau)||x(\tau)| + g(\tau)|x(\tau)|)\Delta\tau.$$

Hereby monotonicity of h, we get

$$h^{-1}(t)|x(t)| \leq c|x_0|h^{-1}(t_0) + c\int_{t_0}^{t} h^{-1}(\sigma(\tau))(|a(\tau)-p(\tau)| |x(\tau)| + g(\tau)|x(\tau)|)\Delta\tau$$

$$\leq c|x_0|h^{-1}(t_0) + c\int_{t_0}^{t} h^{-1}(\tau)|x(\tau)|(|a(\tau)-p(\tau)| + g(\tau))\Delta\tau. \tag{15}$$

Then, we set $u(t) = h^{-1}(t)|x(t)|$ and rewrite (15) as follows:

$$u(t) \leq cu(t_0) + c\int_{t_0}^{t} u(\tau)\big(|a(\tau)-p(\tau)| + g(\tau)\big)\Delta\tau.$$

Here, Theorem 3 implies $u(t) \leq cu(t_0)e_\kappa(t,t_0)$, where κ is as in (14). Then we have

$$|x(t)| \leq c\frac{h(t)}{h(t_0)}e_\kappa(t,t_0)|x_0| \leq c^*\frac{h(t)}{h(t_0)}|x_0|,$$

for $c^* = c\omega \geq 1$. The proof is complete. □

4. Concluding Comments

This study focuses on functional dynamic equations of the form (1) on time scales and provides a detailed analysis regarding h-stability. In the setup of the paper, an alternative variation of parameters formula is used via an auxiliary regressive function p. This approach does not only elicit a new point of view but also relaxes a compulsory condition, namely regressivity, from the dynamic equation of interest. Therefore, contrary to Theorem 1, we do not assume the regressiveness of the main equation for the inversion of the solution.

This study consists of three main results. In Theorem 4 and Theorem 5, we propose sufficient conditions for h-stability of (1) via an h-stable and regressive auxiliary dynamic equation; for instance, see **C1** of Theorem 4. Note that one may easily write the following identity for the generalized exponential function

$$|e_p(t,t_0)| = |e_p(t,\theta)||e_p(\theta,t_0)| = \frac{|e_p(t,\theta)|}{|e_p(t_0,\theta)|}$$

for $t_0 \leq \theta \in \mathbb{T}$, by utilizing [33, Theorem 2.36]. Then, h-stability of the linear equation in (2) is straightforward by setting $h(t) = |e_p(t,\theta)|$, if $e_p(t,\theta)$ is bounded. By [32, Remark 3.8] (see also [35, Example 1]), we have $e_p(t,\theta) \to 0$ as $t \to \infty$ for any negative-valued function p satisfying $|p(t)| \leq \eta$ for all $t \in \mathbb{T}$ where $\sup\mathbb{T} = \infty$, $\eta > 0$ and $-\eta \in \mathcal{R}^+$. This indicates $e_p(t,\theta)$ is bounded. Thus, we shall point out that the h-stability assumption we made for the auxiliary system is a checkable condition. Moreover, the additional conditions introduced in Theorem 4 and 5 are foreseeable since we convert (1) to an integral equation. In the last main result of the manuscript, namely Theorem 6, the connection between uniform boundedness and h-stability notions is highlighted similarly to Theorem 4 and 5.

The outcomes of the manuscript are not only a unification but also a significant extension for the established literature since they allow us to consider h-stability of functional dynamical equations on general domains not restricted to $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

Contributions of the authors

All authors contributed equally to the study.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The authors declare that this study complies with Research and Publication Ethics.

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